The geometry of Globally Hyperbolic Spacetimes-with-timelike-boundary

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Based in joint work with L.A. AKÉ & J.L. FLORES, Rev. Mat. Iberoam. (2021)

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EXCELENCIA MARÍA DE MAEZTU

Overview

Ambient: $(\overline{M}, g)$ glob. hyp. s.t. with-timelike-boundary ∂M

- $\rightsquigarrow$ intrisinsically ∂M identifiable to naked singularities

Systematic studies:

- D. Solís' thesis '06, arXiv:1803.01171
  (background for P. Chrusciel, G. Galloway, D. Solís '09)
- Aké's thesis'18
  (which includes L. Aké, JL. Flores, M.S. '21)

Overview

Focus on geometric properties:

- Development of causality/ causal ladder
- Determination of properties of M from ∂M
- Global orthogonal splitting (*with orthogonality of* ∂M)

Overview

Background for topics including:

- Mixed PDE problems (Dirac, Klein Gordon...)
- Wave equation
  - Snell law at ∂M
  - Bounds for quantization
- Initial *boundary* value problem (IBVP)
  - Asymptotic value problem at conformal infinity
  - Finite distance (Numerics)

Overview

Aims:

1. Spacetimes $\overline{M}$ with timelike bd. ∂M
    1. General properties
    2. Causality (in $\bar{M}$ and inherited by ∂M, M)
    3. Convexity (of ∂M, M)
    4. Relation with the causal bd.
2. Splitting for glob. hyperb. $\overline{M}$
    - Statement
    - Involved techniques
    - Sketch of proof
3. Notes on the PDE viewpoint

1. Spacetimes $\overline{M}$ with timelike bd ∂M

- Connected $(n+1)$-manifold with boundary $\overline{M} = M \cup \partial M$
  - M interior -manifold
  - ∂M boundary n-manifold (possibly non-connected)

Note: $\overline{M}$ closure of open subset in $\tilde{M}$ (with no bd.)

In particular, $\overline{M} \subset \overline{M}^d$ "double mfold"

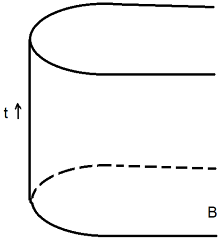
- Metric g on $\overline{M}$
  Lorentzian signature $(-,\underbrace{+,\ldots,+}_{n})$
  - Smooth (natural sense)
  - Extensible (globally) to some $(\tilde{M},\tilde{g})$ (with no bd.)
  - BUT the natural extension of g to $\overline{M}^d$ is only C^0
    (additional conditions required for smoothness)

- Timelike boundary:
  - g Lorentz restricted to the mfd ∂M $(-,\underbrace{+,\ldots,+}_{n-1})$
  - *M and ∂M consistently time-oriented*
    (in particular, $\exists T$ timelike v. on $\overline{M}$ tangent to ∂M)

Two examples/ standard applications

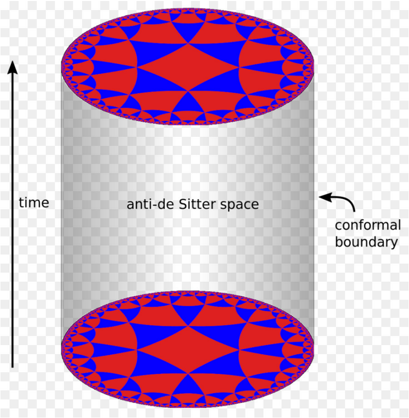
1. Cut-off of some $\tilde{M}$ with no bd.
   - artificial bd. for numerics, quantization...

F. ex: $\mathbb{R} \times \bar{B}$ in $\mathbb{L}^{n+1} = (\mathbb{R} \times \mathbb{R}^n, \langle \cdot, \cdot \rangle)$
$\bar{B} \subset \mathbb{R}^n$: closure open subset with smooth topol. bd.
(tipycally compact $\bar{B}$ "finite distance" problem)



2 Conformal completion (and choice of conformal factor)

F. ex: (universal) anti-de Sitter



Source: Wikipedia (M-theory) Creative Commons

- Anti-de Sitter $(\mathbb{R}^{n+1}, g_{AdS})$

$$g_{AdS} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, g_{\mathbb{S}^{n-1}}, \qquad t \in \mathbb{R}, \rho > 0$$

  - Standard static "gauge" (extensible to $\rho = 0$)
  - Warped product with base hyperbolic $\mathbb{H}^n$ (totally geodesic)

- Conformal choice $(\rho > 0)$

$$\frac{g_{AdS}}{\sinh^2\rho} = -\cosh^2 y\, dt^2 + dy^2 + g_{\mathbb{S}^{n-1}}, \qquad t \in \mathbb{R}, y \in (0, \infty)$$

$$dy = -d\rho/\sinh\rho \rightsquigarrow y = -\ln(\tanh(\rho/2))$$

- Conformal boundary $y = 0$
  $ESU^n = (\mathbb{R} \times \mathbb{S}^{n-1}, -dt^2 + g_{\mathbb{S}^{n-1}})$

In this example (and other natural conformal choices):

- ∂M: timelike hypersurface
  - Intrinsically: isometric to ESU^n
  - Extrinsically: totally geodesic
- Weyl tensor $\equiv 0$

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In general

- Intrinsic/extrinsic properties in ∂M imposed
  $\rightsquigarrow$ **definition of asymptotic behaviours**
  Stresses the role of causality
- Implicit smooth extensibility of **Weyl (obstruction to ∂M)**

Causality: regularity of curves

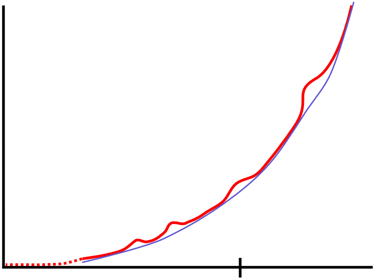
Mimicking the case $\partial M = \emptyset$:

- future/past directed timelike, causal curves γ
- $\ll$, $\leq$
- $I^{\pm}(p), J^{\pm}(p)$

but...

Caution

- Regularity for causal curves above: H^1 / locally Lipschitz
- Case $\partial M = \emptyset$ enough piecewise smooth curves
  Case $\partial M \neq \emptyset$: $J_{ps}(p) \subsetneq J_{H^1}(p)$ (even if ∂M is C^∞)



$\bar{H}$ hypograph (including closed lower half-space)
$\exists$ Riemannian minimizers C^1 no piecewise C^2
$\overline{M} = \mathbb{R} \times \bar{H}$, $g = -dt^2 + dx^2 + dy^2$

Causality: ladder

Lower levels of the causal ladder: as when $\partial M = \emptyset$

Proposition

In any spacetime with timelike boundary:

- *strongly causal $\Rightarrow$ distinguishing*
  $(p \neq q \Rightarrow I^+(p) \neq I^+(q), I^-(p) \neq I^-(q))$
- *(future or past) distinguishing $\Rightarrow$ causal $\Rightarrow$ chronological.*

Stable causality: consistency

Proposition

For $(\overline{M}, g)$ with timelike boundary, they are equivalent:

1. *Stability of causality:* $\exists$ *causal* g' *with* $g < g'$.
2. $\exists$ *time function* t
   (continuous and strictly increasing on f.d. causal curves)
3. $\exists$ *temporal function* τ
   (τ is smooth with timelike past-directed $\nabla\tau$).

(Stably causal $\Rightarrow$ strongly causal)

Proofs.

1 $\Rightarrow$ 2. As in Hawking'73

2 $\Rightarrow$ 3. As in Bernal & S.'05, S.'05 or posterior approaches on cones (no restriction for $\nabla\tau$ on ∂M required)

3 $\Rightarrow$ 1 (and the last assertion). Standard.

Higher levels of the causal ladder: as when $\partial M = \emptyset$:

- Caus. cont.: $I^{\pm}(p)$ characterizes p (distinguishing) + varies continuously (reflecting)
- Two highest:
  - caus. simple: causal + $J^{\pm}(p)$ closed
  - glob. hyp.: causal + $J^{+}(p) \cap J^{-}(q)$ compact

  (including simplification strong causality $\rightsquigarrow$ causality)
- Stably caus. $\Leftarrow$ Caus. contin. $\Leftarrow$ Caus. simple $\Leftarrow$ Glob. hyp.

- Globally hyp. $\Leftrightarrow \exists$ Cauchy subset ($\rightsquigarrow$ top. hyp. with bd.)

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  1. **Lemma:** an achronal set $A \subset \overline{M}$ with $A \cap \mathrm{edge}(A) = \emptyset$ is a locally Lipschitz hypersurf. (with boundary) transverse to ∂M. Moreover, if $\mathrm{edge}(A) = \emptyset$ then A is closed (as a subset of $\overline{M}$)
  2. **Prop.** Let $F \neq \emptyset, \overline{M}$ be a future set (i.e $I^+(F) = F$).
     - Its topological bd is an achronal closed locally Lipschitz hypers. transverse to ∂M.

     In particular, so is any Cauchy subset.

Causality: properties inherited by M, ∂M

Proposition

1. $(\overline{M}, g)$ *causally continuous* $\Rightarrow$
   $(M, g|_M)$ *causally continuous,* $(\partial M, g|_{\partial M})$ *stably causal*.
2. $(\overline{M}, g)$ *globally hyperbolic* $\Rightarrow$
   $(\partial M, g|_{\partial M})$ *glob. hyperbolic*

(∂M possibly non-connected)

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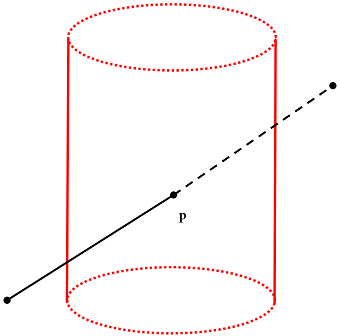
- No more implications, except consequences of above:
  - $\overline{M}$ causally simple $\Longrightarrow$ ∂M stably causal
  - $\overline{M}$ glob. hyp. or caus. simple $\Longrightarrow$ M caus. continuous
- Or directly related: $(\overline{M}, g)$ reflecting $\Rightarrow$ (M, g) reflecting (Galloway, Liang'23).

Example 1: **Globally hyp.** $\overline{M}$ **but non-causally simple** M

$\overline{M} = \mathbb{L}^3 - \{\text{Open cylinder }\}$ (Solís'06)

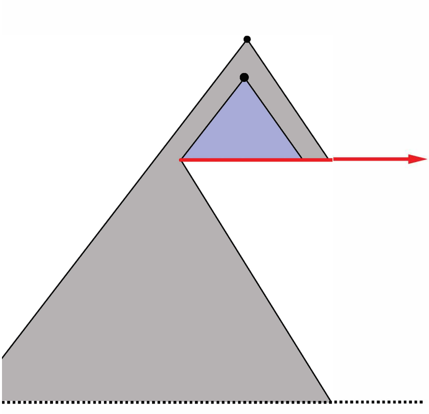


($\exists$ lightlike geod. in $\overline{M}$ tangent to ∂M at $!p$)

Example 2: **Caus. cont.** $\overline{M}$ but **non-causally continuous** ∂M

$M = \{$ open half $y > 0$ in $\mathbb{L}^3\}$,

$\partial M = \mathbb{L}^2 - \{$half x-axis$\} \subset \{y = 0\}$



Moreover: **caus. simple** $\overline{M}$ but non-causally continuous ∂M

1. Start with closed (solid) cylinder $\overline{C} = \mathbb{R} \times \overline{D} \subset \mathbb{L}^3$, where $\overline{D} = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ (closed unit disk)
2. Remove the arc $A = \{(0, \cos\theta, \sin\theta) : 0 \leq \theta \leq \pi/4\}$ and consider the spacetime $\overline{M} = \overline{C} \setminus A$ (clearly ∂M is not causally continuous)
3. $\overline{M} := \overline{C} \setminus A$ is causally simple!

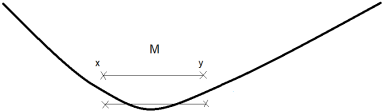
Question: Let $\bar{M}$ be a GH-ST-TB

- M is always causally continuous
- M is never glob. hyperb.
- When is it causally simple?

Convexity: Riemannian (and Finslerian)

$(\bar{M}, g_R)$ complete Riemannian mfd smooth boundary ∂M. Easily,

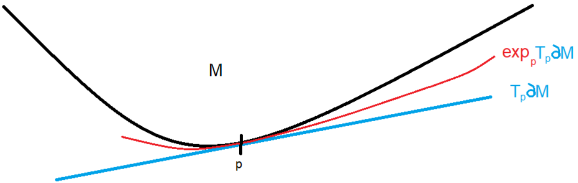
- M convex [$p, q \in M$ connected by a minimizing geod. in M] $\Longrightarrow \partial M$ infinitesimally convex [2nd f.f. $II \geq 0$]



- The converse also holds (even for Finsler metrics)
  - Bishop '74: when g_R is C^4
  - Bartolo, Caponio, Germinario, S. '11: for $C^{1,1}$ (and Finsler)

Basic ideas

1. Local: Infinit. convex. $II \geq 0 \implies$ Local convex.
(locally $\exp_p(T_p\partial M)$ does not reach M)



2. Global: Local convex. $+$ $\bar{M}$ complete $\implies$ convex M

Basic ideas

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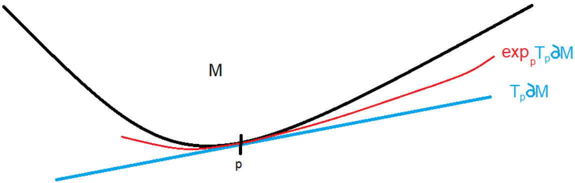

2. Global: Local convex. + $\bar{M}$ complete $\Longrightarrow$ convex M

- Note: completeness of $\bar{M}$ weakened in the Finslerian setting: compactness of forward closed balls $\cap$ backward ones (weaker then forward or backward completeness)

Convexity: Lorentzian (Herrera, S. '23)

Spacetimes-with-timelike-boundary:

- logically independent convexities for ∂M:
  timelike/null/spacelike $II(v,v) \geq 0$ for resp. v
  $\rightsquigarrow$ null convexity conformally invariant
- ∂M timelike/null/spacelike infinit. convex
  $\Longleftrightarrow$ Locally convex exp. for timelike/null/spacelike geod.
  (see also null case in Hintz, Uhlmann '19)

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- ∂M timelike/null/spacelike infinit. convex
  $\iff$ Locally convex exp. for timelike/null/spacelike geod.
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Proposition

For $(\bar{M}, g)$ GH-ST-TB (or just caus. simple):
M is causally simple $\iff$ ∂M is null convex

(Lorentzian glob. hyp. $\bar{M} \rightsquigarrow$ Riemannian completeness $\bar{M}$)

Convex for static/stationary spacetimes

Corollary

For a standard static s.t. $(\mathbb{R} \times S, g = -\Lambda(x)dt^2 + g_0)$*:*
causally simple $\Longleftrightarrow$ $(S, g_R := g_0/\Lambda)$ *convex*
Moreover, for $\bar{D} \subset S$ *complete (open with bd) are equivalent:*

1. *The "cylinder"* $\mathbb{R} \times D$ *is causally simple*
2. ∂D *is infinitesimally* g_R*-convex*
3. $\mathbb{R} \times \partial D$ is *null-convex*

Standard stationary: $(\mathbb{R} \times S, g = -\Lambda dt^2 + \omega \otimes dt + dt \otimes \omega + g_0)$

- Finsler metrics codifying causality (Caponio, Javaloyes, S. '11)
  $F = (\sqrt{\omega^2 + g_0} - \omega)/\Lambda$ Finsler metric
  (play the role of $g_R = g_0/\Lambda$ but $F(v) \neq F(-v)$)

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  (play the role of $g_R = g_0/\Lambda$ but $F(v) \neq F(-v)$)

Proposition (Caponio, Germinario S. '16)

For a standard stationary s.t.:
causally simple $\Longleftrightarrow$ (S, F) is convex
Moreover, for $\bar{D} \subset S$ complete (or just with forward $\cap$ backward closed F-balls in $\bar{D}$ compact) equivalent:

- *The "cylinder" $\mathbb{R} \times D$ is causally simple*
- *∂D is infinitesimally F-convex*
- *$\mathbb{R} \times \partial D$ is null-convex*

Convexity in asymptotically flat s.t.

For any asymptotically flat s.t. and D large ball:

- $\mathbb{R} \times \bar{D}$ is a GH-ST-TB
- $\mathbb{R} \times \partial D$ is null-convex

$\Rightarrow$ $\mathbb{R} \times D$ is causally simple

Convexity in asymptotically flat s.t.

For any asymptotically flat s.t. and D large ball:

- $\mathbb{R} \times \bar{D}$ is a GH-ST-TB
- $\mathbb{R} \times \partial D$ is null-convex

$\Rightarrow$ $\mathbb{R} \times D$ is causally simple

- **Caution** *not* time-convex

  for physical conformal choices: pebble tossed straight

  - will reach a maximum radius
  - ... and fall down

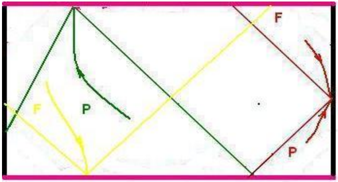
  $\rightsquigarrow$ timelike geodesic violating (local) time-convexity!

Causal boundary and naked singularities

- Timelike bd ∂M: is a "conformal" one
- Causal boundary $\partial_c M$: intrinsic
  (valid for all strongly causal s.t. M without bd.)

Relation between ∂M and $\partial_c M$?

- **TIP:** $I^-(\gamma)$, γ timelike $\uparrow$ inext, **TIF**: $I^+(\gamma)$, $\gamma \downarrow$
- **Causal bd:** TIP's + TIF's, some of them identified
  - Formally , pairs (P, F)
  - P TIP or $\emptyset$; F TIF or $\emptyset$
  - S-relation (Szabados) $P \sim_s F$
- **Naked singularities**: each TIP identified with some TIF
  - Formally pairs (P, F) with $P \neq \emptyset \neq F$
  - Agrees with classical notion of naked:
    $\exists$ Inextens. future-directed (resp past-directed) causal γ lying in the past (resp. future) of some $p \in M$
  - $\partial_{naked} M \subset \partial_c M$



Proposition

For $M \subset \overline{M}$ with timelike bd.:

- $\partial M \subset \partial_{naked} M$
  Indeed, each $p \in \partial M$ gives the naked singularity (P, F) with $P = I^{-}(p) \cap M$, $F = I^{+}(p) \cap M$
- *If, additionally, $\overline{M}$ is GH-ST-TB:*
  $\partial M = \partial_{naked} M$

2. Splitting for globally hyperbolic $\overline{M}$

Statement

Theorem. Any $(\overline{M}, g)$ globally hyperbolic with-timelike-boundary:

(A) Admits an "adapted" Cauchy temp. τ
(with $\nabla\tau$ tangent to ∂M on bd.)

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(A) Admits an "adapted" Cauchy temp. τ
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(B) Then, $\overline{M}$ splits as $\mathbb{R} \times \bar{\Sigma}$, $g = -\Lambda d\tau^2 + g_\tau$,

- τ becomes the natural projection $\mathbb{R} \times \bar{\Sigma} \to \mathbb{R}$
- $\Lambda : \mathbb{R} \times \bar{\Sigma} \to \mathbb{R}$ is positive (*lapse*)
- $\bar{\Sigma}$ (n)-manifold with boundary
- g_τ Riemannian metric on each slice $\{\tau\} \times \bar{\Sigma}$
(positive semidefinite metric tensor with radical$=$ Span(∂_τ))
- τ-slices are spacelike Cauchy *hypersurfaces-with-bd*

As a consequence:

- Interior $M = \mathbb{R} \times \Sigma$
  $\rightsquigarrow$ class of causally continuous s.t. which splits
- Boundary ∂M (global. hyp. no bd., possibly non-connected):
  - admits $\tau|_{\partial M}$ as Cauchy temporal
  - orthogonal to the Cauchy τ slices

Moreover, any glob hyp. s.t.-with-timelike-bd $\overline{M}$ can be isometrically embedded

- $\hookrightarrow$ as the closure of an open set in a glob. hyp. s.t. $\tilde{M}$ (without boundary)
- $\hookrightarrow$ as a submanifold in $\mathbb{L}^N$ for some $N \in \mathbb{N}$ applying to $\tilde{M}$ Nash-type Müller & S.'11 (for $\partial M = \emptyset$)

Involved techniques

Previous techniques for $\partial M = \emptyset$

- Starting point: Geroch '70 topological splitting
- Smoothness + orth. splitting: AN Bernal & MS '03, '05 (extended to other problems in AN Bernal & MS '06, '07; O Müller & MS '11; O Müller '16)
- New approaches:
  1. A. Fathi, A. Siconolfi '12:
     weak-KAM theory
  2. Sullivan'76 - D. Monclair'14
     Applicability of attractors, chain recurrency and Conley Th.
  3. P. Chrusciel, J.D.E. Grant & E. Minguzzi '16:
     Seifert's approach to smoothability
  4. P. Bernard & S. Suhr '18:
     Conley theory, most general cones

The orthogonal splitting (part (B) of Th.) follows from τ (part (A))

The orthogonal splitting (part (B) of Th.) follows from τ (part (A))

1. Take τ:
   - *temporal* (smooth, with $\nabla\tau$ past-directed timelike)
   - *Cauchy* (with Cauchy slices $\tau =$ constant)
   - *adapted* (with $\nabla\tau$ tangent to ∂M on ∂M)

   and put $\bar{\Sigma} := \tau^{-1}(0)$
2. Splitting obtained flowing $\bar{\Sigma}$ with $X = -\nabla\tau/|\nabla\tau|^2$.
3. Consistency: X is tangent to ∂M (integral curves starting at ∂M remain in ∂M).

Strategy to find τ:

1. Find any Cauchy temporal τ_0 on $\overline{M}$ (non-adapted)
   - Any of the techniques of the case $\partial M = \emptyset$ work
2. Use τ_0 to prove C^0-stability glob. hyp. in $\overline{M}$ (or stability in the interval topology for conformal classes)
   - Intuitively clear
3. Using stability, reduce to the case of a very simple ∂M...
4. ... where the techniques for the case $\partial M = \emptyset$ are appliable (in particular: Bernal, S.'05 + O Müller '16)

Steps to construct (adapted) τ:

1. $\exists g' > g$ such that g' is globally hyperbolic.
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   τ g^*-Cauchy temporal $\Rightarrow$ τ g-Cauchy temporal

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   there exist $g' > g$ such that g' is globally hyperbolic.
2. Lemma: if g^* satisfies $g \leq g^* \leq g'$ then:
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3. Find such a g^* such that:
   1. $\partial M^{\perp_g} = \partial M^{\perp_{g^*}}$
   2. the natural extension of g^* to the double $\overline{M}^d$ is smooth:
      - $\exists$ a natural isometry $i : \overline{M}^d \to \overline{M}^d$ "reflecting" on ∂M
        (in particular, ∂M is g^*-totally geodesic)

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   Proof: technical, modify g using

   - a global tubular neighborhood E of ∂M and
   - a choice $\hat{g}_0$ on ∂M s.t. $g|_{\partial M} < \hat{g}_0 < g'|_{\partial M}$

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Very particular case of O. Müller, LMP'16:

- Let g^d be a globally hyperbolic metric invariant by the action of a compact Lie group G. Then, it admits a Cauchy temporal function τ^d invariant by G.

In our case $G = \{1, i\}$

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4. Find a g^*-Cauchy temporal τ^d on all $\overline{M}^d$ invariant by i.
5. $\tau := \tau^d|_{\overline{M}}$ works!

3. Notes on the PDE viewpoint

GH-ST-TB: Arena for field eqns

Some examples:

- K. O. Friedrichs '58: general study of symmetric positive linear differential equations, including **Cauchy mixed problems**
- N. Ginoux, S. Murro '22: systematic study of Friedrich systems in GH-ST-TB
  - Symmetric hyperbolic: Dirac, energy momentum, geometric wave op.
  - Symmetric positive: Klein-Gordon, reaction-difussion

(also Drago et al. '20 Grosse & Murro '20)

Especially, GH-ST-TB ambient for wave eqn and quantization

- Lupo's thesis '15 Ch. 5 "Notes towards a theory of spacetimes with timelike boundaries and boundary value problems"
  $\rightsquigarrow$ (local) uniqueness for wave eq with Dirichlet bd conditions
- P. Hintz, G. Uhlmann '19, with J. Zhai '23: inverse boundary problem for semilinear wave eqn. including Snell's
- Fundamental solutions, quantization :
  - C. Dappiaggi, N. Drago, H. Ferreira'20 Maxwell eqn.
  - C. Dappiaggi, N. Drago, R. Longhi '19 waves in static s.t.
  - C. Dappiaggi, A. Marta '22 Klein-Gordon asym. anti de Sitter
  - M. Benini, C. Dappiaggi, A. Schenkel '18 algebraic QFT
- W. Janssen '22. Quantization of fields: glob. hyperb.
  $\rightsquigarrow$ semi-glob. hyperb. (finite union in time of glob hyp pieces)
  $\rightsquigarrow$ GH-ST-TB

GH-ST-TB evolved from Einstein eqn: IBVP

Model: classic Cauchy initial value approach for Einstein eqns.:

- Choquet-Bruhat '52 — Choquet-Bruhat & Geroch '69: **Well posedness**, involves a **(harmonic) gauge**

⇝ Initial boundary value problem on $(\mathbb{R} \times \partial\Sigma) \cup \Sigma$

(For vacuum, following Z. An & M. Andersson'22)

- **Cauchy value problem**

Let Σ, (n) manifold $\partial\Sigma = \emptyset$ with initial data (γ, κ)
(Riem. γ, 2 cov sym tensor κ; Gauss, Codazzi $Ricc = 0$)

- (V, g) vacuum development:
  glob. hyperb. $Ricc = 0$ s-t containing the data (Σ, γ, κ)
- *Equivalent* vacuum developments $(V_1, g_1), (V_2, g_2)$:
  if they contain a *common subdevelopment*:
  $\exists$ vacuum development (V', g) and isometric embeddings
  $\psi_i : V' \to V_i$ such that $\psi_i^* g_i = g, (i = 1, 2)$
- $\mathcal{V}$: space of equivalence classes of vacuum developments
  $\mathcal{I}$: moduli space of initial data up to Diff(Σ)

Main classic result: $\exists$ canonic bijective map $D : \mathcal{V} \to \mathcal{I}$.

- D yields a parametrization of solutions
- To obtain D^{-1}: solve PDE (using a gauge)

- **Initial boundary value problem (IBVP)**

Initial data (γ, κ) on some $\bar{\Sigma}$, $\partial\Sigma \neq \emptyset$
Corner boundary : $(\mathbb{R} \times \partial\Sigma) \cup \bar{\Sigma} \rightsquigarrow ([0,1) \times \partial\Sigma) \cup \Sigma$

Ideally seeked th.: $\exists$ *canonic bijective* $D : \mathcal{V} \to \mathcal{I} \times_c \mathcal{B}$.

- $\mathcal{B}$ boundary data on $\partial M \equiv [0,1) \times \partial\Sigma$ up to some diffeom.
  - Data: geometry of ∂M
    intrinsic (Lorentzian metric) and/or extrinsic (2nd ff II)
  - Diffeom.: Ideally $\mathrm{Diff}_0(\bar{M})$ (diff. M identity on ∂M)
- $\times_c$ compatibility at the corner $\{0\} \times \partial\Sigma$.

Basic open problem (for appropiate geometric bd data):

Is there a choice of gauge for which IBVP is well posed?

IBVP, asymptotic case

Asympt. AdS (vacuum cosmol. const) $Ricc = \lambda g$, $\lambda < 0$

- AdS: $(\Sigma, \gamma) = \mathbb{H}^n, \kappa = 0, B =$ ESU^n

Some results:

- Friedrich '95 well posed IBVP (dim $3+1$ or even)
  - Geometric data: conformal class of the metric at infinity
  - Reduction to a finite maximally dissipative flux
  - Mixed problem: tractable using O. Guès '90
  - Riemannian background: Graham, Lee'90

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  - Riemannian background: Graham, Lee'90
- A. Enciso, N. Kamran '19 approach valid also for odd dim
  - Make use of key algebraic similarities Lorentz/ Riemann
  - Improve regularity of initial data (polyhomogeneity)
    $\rightsquigarrow$ no problem log terms in Fefferman-Graham expansion
- G. Holzegel, J. Luk, J. Smulevici, C. Warnick'20 ,
  stability of AdS under optimally dissipative flux on ∂M
  $+$ (conjectured) non-stability for flux$= 0$
  (reflecting Dirichlet/ Neuman)

Some results: IBVP, finite distance

Primary application to **numerical relativity**:

- Simulation isolated astrophys. systems (neutron stars, BH's) $\rightsquigarrow$ introduce as ∂M the boundary of the computational grid

Especially complicated (even Riemanian counterparts)

Some results: IBVP, finite distance

$\bar{\Sigma}$ compact, $Ric = 0$

- Friedrich-Nagy '99 (MR: "rarely studied problem")
  well-posed under gauge-dependent conditions
  - Σ: first-order frame formalism, orthonormal tetrad, associated metric connection, Weyl components
  - Bd: mean curv. + gauge-dependent Weyl

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- Kreiss-Reula-Sarbach-Winicour '09 '07:
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  ⇝ closer to classic Cauchy problem
  - Gauge-dependent conditions on bd and Σ
  - Kreiss-Winicour '14: identifies local geometric meaning (intrinsic or extrinsic) of pieces of the data

  An-Andersson '22 (arxiv): progress on harmonic gauge and uniqueness ⇝ construction of unique maximal globally hyperbolic vacuum development

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  An-Andersson '22 (arxiv): progress on harmonic gauge and uniqueness $\rightsquigarrow$ construction of unique maximal globally hyperbolic vacuum development
- Fournadavlos-Smulevici '21 '23:
  well posedness for totally geodesic/ umbilic ∂M.

Thank you
for your attention

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