Recent Developments in Mathematical General Relativity II: Black Holes, Inside and Out

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Part I: Black Hole Interiors

Quick Review: Strong Cosmic Censorship and the Kerr Cauchy Horizons I

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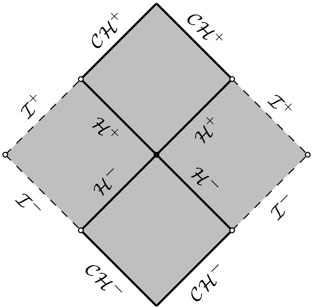

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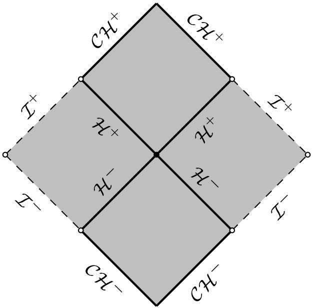

The key "problem" is that the spacetime ends its unique development in a smooth Cauchy horizon.

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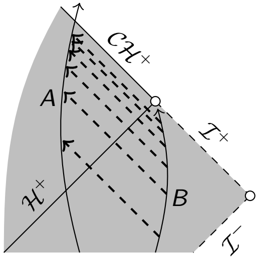

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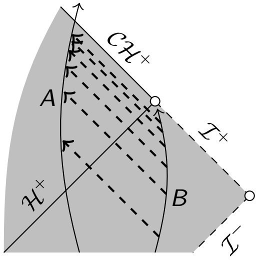

The hope/expectation is that for generic perturbations of Kerr, the Cauchy horizon becomes singular.

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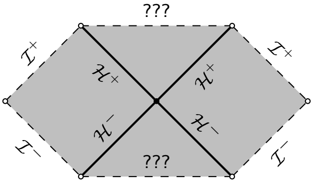

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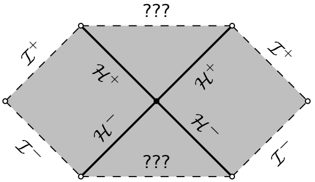

Unfortunately(?) this does not happen!

Aside: The Characteristic Initial Value Problem

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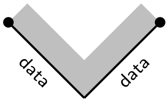
Theorem (Rendall 1990, Luk 2012)

Suppose metric components are given along two transversally intersecting intersecting null hypersurfaces in such a way that the null constraint equations are satisfied. Then there exists a unique solution to the Einstein vacuum equations in a future neighborhood of the union of the two hypersurfaces.

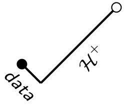


(Remember that these Penrose diagrams have a precise meaning!)

The Dafermos–Luk Theorem I

Theorem (Dafermos–Luk 2017)

Consider a characteristic initial value problem of the following form:



where along $\mathcal{H}^+$ the data decays to the data along the event horizon of a rotating Kerr solution at an integrable polynomial rate.

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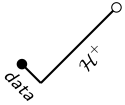
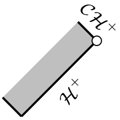


where along $\mathcal{H}^+$ *the data decays to the data along the event horizon of a rotating Kerr solution at an integrable polynomial rate.*

Then there will be at least a small piece of a Cauchy horizon to which the metric extends in C^0 *(!), and the spacetime will asymptotically be close to a Kerr spacetime in the* C^0 *(!) topology:*



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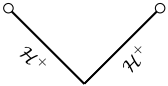
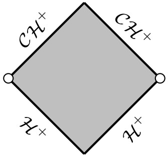


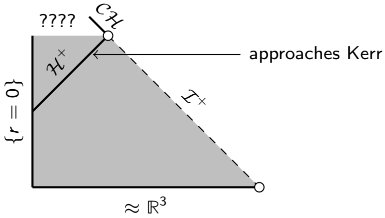
where along $\mathcal{H}^+$ the data is a small perturbation of the data along the Kerr event horizons and moreover decays to the Kerr data at an integrable polynomial rate.

Then the entire Cauchy horizon persists and the metric extends to the boundary in C^0(!), and the spacetime will be everywhere close to a Kerr spacetime in the C^0(!) topology:



Remarks on the Theorems

- The point of the first theorem is that it can be applied in the case of gravitational collapse of one ended data:



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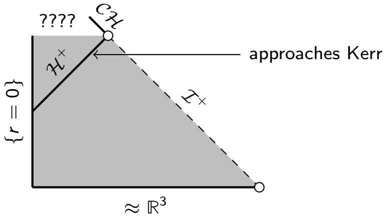

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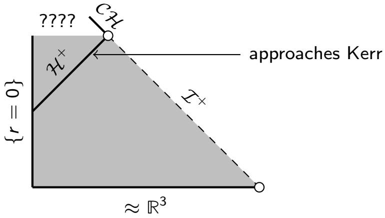

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- We have stability in C^0, but what about higher order regularity? It is useful to consider spherically symmetric model problems...

The Einstein-Maxwell-Real Scalar Field System

The Einstein-Maxwell-real scalar field system is the following set of equations for a Lorentzian manifold $(\mathcal{M}, g)$, function $\phi : \mathcal{M} \to \mathbb{R}$, and 2-form $F_{\mu\nu} \in \wedge^2(\mathcal{M})$:

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \mathbf{T}^{(\mathrm{sf})}_{\mu\nu} + \mathbf{T}^{(\mathrm{em})}_{\mu\nu},$$

$$\mathbf{T}^{(\mathrm{sf})}_{\mu\nu} \doteq \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left|\nabla\phi\right|^2_g, \qquad \mathbf{T}^{(\mathrm{em})}_{\mu\nu} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} \left|F\right|^2_g,$$

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The Kerr solution is not spherically symmetric; however, there exists the spherically symmetric Reissner–Nordström solution to the Einstein-Maxwell-real scalar field system which has the same Penrose diagram as the Kerr black hole and the same smooth Cauchy horizon phenomenon. This system thus provides a spherically symmetric warm-up for the general problem.

Luk-Oh's Result

Theorem (Luk-Oh 2019)

Strong cosmic censorship holds for solutions to the spherically symmetric Einstein-Maxwell-real scalar field system arising from asymptotically flat Cauchy hypersurfaces diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$*. More specifically, the maximal developments are generically inextendible as a* C^2 *solution.*

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- In view of (Siberski 22) in certain settings of the above theorem, C^2 may be replaced by $C^{0,1}$. It remains an open problem to characterize the regularity in a sharp way.
- It is interesting to extend this result to the case of solutions arising from Cauchy hypersurfaces diffeomorphic to $\mathbb{R}^3$. (One must in fact modify the matter model, e.g., consider a charged scalar field.) There are significant additional difficulties in this case.

Prior Work in the Physics Literature and "Weak Null Singularities"

- In the context of certain spherically symmetric matter models (simpler than Einstein-Maxwell-scalar field), some earlier heuristic works in the physics literature (Hiscock 1981, Poisson–Israel 1989, 1991, Burko–Ori 1998) had anticipated the possibility of the Cauchy horizon remaining and becoming a type of "weak null singularity."

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- Weak null singularities for vacuum were first constructed in Luk 2018. (We'll discuss the nonlinear and geometric aspects of these more in the final lecture.)

Instability Results for Linearized Perturbations around Kerr

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The proof uses a scattering theory method introduced by Luk–Oh–S. (2023) in the context of the spherically symmetric Einstein-Maxwell-scalar field system.

Aside: Importance of Decay Rates

- For all of these results the exact decay rates along the event horizon are very important. Polynomial decay at integrable rates is associated with obtaining C^0 stability of the Cauchy horizon, and (any) polynomial lower bound is associated (conjecturally) with a weak null singularity.

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- Particularly striking is the heuristic and numerical analysis of Dias–Reall–Santos who studied linearized perturbations of a Reissner–Nordström–de Sitter black hole, and found that for any large k, one can find such a black hole so that linearized perturbations extend to the Cauchy horizon in a C^k fashion! (A way to "save" strong cosmic censorhsip by considering rougher perturbations has been put forth in Dafermos–S. 2018.)

Part II: Black Hole Uniqueness and Stability Conjectures

The sub-extremal Kerr spacetimes

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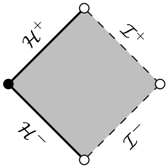

In the extremal case $|a| = M$, the leftmost corner of the diagram is not actually in the spacetime.

The Uniqueness Conjecture

Conjecture (Black Hole Uniqueness, Informal Version)

Suppose that $(\mathcal{M}, g)$ *solves the Einstein vacuum equations, arises from asymptotically flat initial data, is stationary, and has a complete null infinity. Then the "domain of outer communication" is isometric to a Kerr black hole.*

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- More precise statements of the conjecture include some additional technical assumptions but we suppress them here.

The Final State Conjecture

The black hole uniqueness conjecture informs the following conjecture:

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Generic solutions to the Einstein vacuum equations arising from asymptotically flat data have a domain of outer communication which eventually settles down to a finite number of Kerr black holes moving away from each other.

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- This is a very bold conjecture. While it is difficult to think of a likely way it would fail, we don't really have much direct evidence for the far reaching aspects of the conjecture.
- If we take this conjecture as given, then it is a consequence of the Dafermos–Luk theorem and the correspondoning heuristics that weak null singularities should be ubiquitous.

An important first step towards weak cosmic censorship (and the final state conjecture) is the black hole stability conjecture.

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Small perturbations of a Cauchy hypersurface in a maximal sub-extremal Kerr spacetime lead to spacetimes with a complete null infinity and whose domain of outer communication asymptote to a nearby Kerr solution.

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- We emphasize that it is only the domain of outer communication that one expects to be stable. In the interior we have seen that we expect the formation of weak null singularities.

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- If the stationary field is not normal to the horizon then Hawking 1972 showed there must exist an axisymmetric Killing vector field along the event horizon. In the analytic category one then expects to be able to extend the Killing vector field and apply the Carter and Robinson theory. (This was carried out in Chruściel–Costa 2008).

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- Klainerman and Ionescu, and later Alexakis, initiated a program to remove the need for analyticity from the above. Among other results, Alexakis–Ionescu–Klainerman (2010) showed that any stationary black hole solution which is close to a Kerr solution must be a Kerr solution.

Progress Towards the Black Hole Stability Conjecture: Nonlinear Results

- In a combination of works (2019-2022), Klainerman, Szeftel, Giorgi, and Shen have established the stability of Kerr spacetimes with $|a| \ll M$, which corresponds to a very slowly rotating black hole.

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- *If one takes the Teukolsky equation analysis as given*, then Andersson, Bäckdahl, Blue and Ma (2019, 2022) have an approach to linear stability in the full sub-extremal range $|a| < M$ using the "outgoing radiation gauge." Häfner, Hintz, and Vasy established decay statements for linearized gravity for $|a| \ll M$ using a modified wave coordinate gauge.