Euler's proof of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

He starts with the following observation: if P is a polynomial of degree n in x, having n different non-zero zeros $a_1, \ldots, a_n$ and such that $P(0) = 1$, then $P(x) = Q(x)$, where

$$Q(x) := \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\ldots\left(1 - \frac{x}{a_n}\right).$$

To see this, observe that $Q(a_1) = \left(1 - \frac{a_1}{a_1}\right)\left(1 - \frac{a_1}{a_2}\right)\ldots\left(1 - \frac{a_1}{a_n}\right) = 0$. Similarly $Q(a_2) = \ldots = Q(a_n) = 0$. So P and Q have the same zeros and the same degrees, whence

$$P(x) = cQ(x), \quad \text{for some } c \text{ in } \mathbb{R}.$$

Since $P(0) = 1$ and $Q(0) = 1$ it follows that $P(x) = Q(x)$, hence

$$P(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\ldots\left(1 - \frac{x}{a_n}\right).$$

Next consider $f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$. Then $f(0) = 1$ and the zeros of f are $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Now Euler assumes that for the "infinite polynomial" $f(x)$ a similar result holds as the one above for $P(x)$. So he concludes that

$$(*) \qquad \begin{aligned} 1 - \tfrac{x^2}{3!} + \tfrac{x^4}{5!} + \cdots &= \left(1 - \tfrac{x}{\pi}\right)\left(1 + \tfrac{x}{\pi}\right)\left(1 - \tfrac{x}{2\pi}\right)\left(1 + \tfrac{x}{2\pi}\right)\ldots \\ &= \left(1 - \tfrac{x^2}{\pi^2}\right)\left(1 - \tfrac{x^2}{2^2\pi^2}\right)\left(1 - \tfrac{x^3}{3^2\pi^2}\right)\ldots \end{aligned}$$

Now expand the righthand side in powers of x. Then the coefficient of x^2 is equal to

$$-\left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Since the coefficient of x^2 in the lefthand side is equal to $-\frac{1}{3!} = -\frac{1}{6}$, equating of the coefficients of x^2 leads to

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

hence

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \ !$$