Holmes theorem

In chapter 7 we described a remarkable property of the Lo Shu, namely that the numbers "formed by its rows" satisfy the equality

$$(*) \qquad 816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 293^2.$$

Also we remarked that such a property holds for any magic square of order three, if interpreted correctly. More precisely this means the following

Theorem Let

a_1	b_1	c_1
a_2	b_2	c_2
a_3	b_3	c_3

be a magic square. Then

$$(**) \qquad \begin{cases} [a_1x^2 + b_1x + c_1]^2 + [a_2x^2 + b_2x + c_2]^2 + [a_3x^2 + b_3x + c_3]^2 = \\ [c_1x^2 + b_1x + a_1]^2 + [c_2x^2 + b_2x + a_2]^2 + [c_3x^2 + b_3x + a_3]^2 \end{cases}$$

Before we prove this result, let us show how it implies the equation $(*)$. In this case we have

$$a_1 = 8, b_1 = 1, c_1 = 6, a_2 = 3, b_2 = 5, c_2 = 7, a_3 = 4, b_3 = 9, c_3 = 2.$$

Now one only needs to observe that

$$\begin{aligned} 816 &= 8.10^2 + 1.10 + 6, \quad & 618 &= 6.10^2 + 1.10 + 8 \\ 357 &= 3.10^2 + 5.10 + 7, \quad & 753 &= 7.10^2 + 5.10 + 3 \\ &\text{etc.} \end{aligned}$$

Then $(*)$ follows from $(**)$ by making the substitution $x = 10$.

To prove the theorem we will show that the coefficient of x^4 (the highest x-power) in the lefthand side of $(**)$ is equal to the coefficient of x^4 in the righthand side of $(**)$ and in the same way we show that similar results hold for the coefficients of x^3, x^2, x and x^0. First of all equating the coefficients of x^4 gives the following equation

$$a_1^2 + a_2^2 + a_3^2 = c_1^2 + c_2^2 + c_3^2.$$

In order to simplify this and the other equations which we find by comparing the coefficients of $x^3, x^2, \ldots$, we introduce the notion of the inner product of two vectors: if

$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $w = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ are two column vectors in $\mathbb{R}^3$, then we define the *innerproduct* $\langle v, w\rangle$ by the formula

$$\langle v, w\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

So for example $\langle v, v\rangle = x_1^2 + x_2^2 + x_3^2$.
Now let us return to the magic square

a_1	b_1	c_1
a_2	b_2	c_2
a_3	b_3	c_3

We denote its first column by v_1, the second by b_2 and the third by v_3.
Then the equation $a_1^2 + a_2^2 + a_3^2 = c_1^2 + c_2^2 + c_3^2$ can be written as

$$\langle v_1, v_1\rangle = \langle v_3, v_3\rangle.$$

More generally one can check that the equality (∗∗) can be written in the form

$$\begin{aligned} &\langle v_1, v_1\rangle x^4 + 2\langle v_1, v_2\rangle x^3 + (\langle v_2, v_2\rangle + 2\langle v_2, v_3\rangle)x^2 + 2\langle v_2, v_3\rangle x + \langle v_3, v_3\rangle = \\ &\langle v_3, v_3\rangle x^4 + 2\langle v_3, v_2\rangle x^3 + (\langle v_2, v_2\rangle + 2\langle v_2, v_1\rangle)x^2 + 2\langle v_2, v_1\rangle x + \langle v_1, v_1\rangle. \end{aligned}$$

It follows that the theorem is proved if we can show that

$$\langle v_1, v_1\rangle = \langle v_3, v_3\rangle$$

and

$$\langle v_1, v_2\rangle = \langle v_3, v_2\rangle.$$

To show these equalities we finally use Lucas' formula, which asserts that every magic square of order three is of the form

a+b	a-b-c	a+c
a-b+c	a	a+b-c
a-c	a+b+c	a-b

So $v_1 = \begin{pmatrix} a+b \\ a-b+c \\ a-c \end{pmatrix}$, $v_2 = \begin{pmatrix} a-b-c \\ a \\ a+b+c \end{pmatrix}$ and $v_3 = \begin{pmatrix} a+c \\ a+b-c \\ a-b \end{pmatrix}$. To see that $\langle v_1, v_1\rangle = \langle v_3, v_3\rangle$ one has to check that

$$(a+b)^2 + (a-b+c)^2 + (a-c)^2 = (a+c)^2 + (a+b-c)^2 + (a-b)^2.$$

This can be done easily by expanding both sides of this equation.
Similarly one can check that $\langle v_1, v_2 \rangle = \langle v_3, v_2 \rangle$ by verifying that

$$(a+b)(a-b-c) + (a-b+c)a + (a-c)(a+b+c) =$$
$$(a+c)(a-b-c) + (a+b-c)a + (a-b)(a+b+c).$$

These easy verifications are left to the reader. This completes the proof of the theorem.