The ζ-function and the Riemann Hypothesis

In the same paper Euler not only computed the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but also for all even numbers $s = 2, 4, 6, \ldots, 26$ he computed the value of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

He found $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}, \ldots$
Furthermore it is not difficult to see that for each $s > 1$ the value of $\zeta(s)$ is finite (compare $\zeta(s)$ with $\int_1^{\infty} \frac{1}{x^s} dx$).

It was the great mathematician Bernhard Riemann (1826-1866) who extended the ζ-function for all *complex* numbers s with $Re(s) > 1$ and then showed that this function could be extended analyticely to the whole complex plane, minus $s = 1$. The ζ-function turned out to have zeros at $s = -2$, $s = -4$, $s = -6, \ldots$, but it also has other zeros. All the zeros Riemann computed turned out to lay on the line $Re(s) = \frac{1}{2}$.
This led him to formulate, what is nowadays known as, the most famous unsolved problem in Mathematics.

Riemann Hypothesis *All (non-trivial) zeros of the ζ-function lay on the line $Re(s) = \frac{1}{2}$.*

The Riemann Hypothesis is one of the seven so-called *Millennium Problems* of the Clay Mathematics Institute (`http://www.claymath.org/millennium/`).
The first to solve each one of these problem receives one million dollars from the CMI.
In order, for almost all readers, to get a formulation of the Riemann Hypothesis which does not need complex numbers and highbrow mathematics, we give the following easy to understand reformulation of the Riemann Hypothesis.

Riemann Hypothesis for Dummies

A *prime number* p is positive integer greater than 1 which has no other divisors than p and 1. So for example $3, 5, 7$ are prime numbers, but 9 and 15 are not (since $9 = 3 \times 3$ and $15 = 3 \times 5$).
Prime numbers can be considered as the atoms of all positive integers in the sense that each positive integer greater than 1 is a product of a finite number of prime numbers.
For example $14 = 2 \times 7$, $50 = 2 \times 5 \times 5$, $62 = 2 \times 31$ etc. In the number 50 the prime 5 appears more than once. In the sequel such numbers as 50 will be neglected. More precisely we only consider so-called *square-free* numbers, i.e. numbers in which all prime numbers are different. So for example $15 = 3 \times 5$ and $70 = 2 \times 5 \times 7$ are square free, while $40 = 2 \times 2 \times 2 \times 5$ and $75 = 2 \times 5 \times 5$ are not square free.

Now we call a square free number *P-even* (Prime-even) if it is a product of an even number of different prime numbers. Similarly we call a square free number *P-odd* if it is

a product of an odd number of different primes. So for example 70 is P-odd, while 15 is P-even.
Finally we define the so-called *Mertens function* $M(x)$ for all real numbers $x \geq 1$, as follows

$$M(x) \quad = \quad \text{the difference between the number of } P\text{-even integers } \leq x \text{ and the number of } P\text{-odd integers } \leq x.$$

For example $M(34) = 9 - 10 = -1$, since the P-even integers ≤ 34 are 6 $(= 2 \times 3)$, 10 $(= 2 \times 5)$, 14 $(= 2 \times 7)$, 22 $(= 2 \times 11)$, 26 $(= 2 \times 13)$, 34 $(= 2 \times 17)$, 15 $(= 3 \times 5)$, 21 $(= 3 \times 7)$, 33 $(= 3 \times 11)$, while the P-odd integers ≤ 34 are $2, 3, 7, 11, 13, 17, 23, 29, 31$ and 30 $(= 2 \times 3 \times 5)$.

Now the Riemann Hypothesis is equivalent to the following statement:

For every $\varepsilon > 0$ there exists a positive number C_ε such that

$$|M(x)| = C_\varepsilon x^{\frac{1}{2}+\varepsilon}.$$