A survey of some estimates of eigenvalues and condition numbers for certain preconditioned matrices

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1 Introduction

The rate of convergence of preconditioned iterative methods such as the Chebyshev iterative method and (generalized) conjugate gradient methods can be estimated when the condition number of the preconditioned matrix is known. The Chebyshev method requires even that the extreme eigenvalues are known, or estimated from below and above, respectively. The rate of convergence of the conjugate gradient method depends in fact more precisely on the distribution of the eigenvalues (for more information about this, see [8]).

In the present paper we survey some recently derived eigenvalue and condition number estimates. For the case of explicit preconditioners, i.e. using approximations of the inverse of the given matrix, see [7], and the references quoted therein. In the case of implicit preconditioners on factorized form we survey some recent results in [6], [12] and [24] to estimate condition numbers and to give two-sided bounds of individual eigenvalues. The results extend some results found previously in [6], [12] and [29]. The techniques used are purely algebraic and we shall consider mostly block matrix factorizations. The bounds are of three major types:

(i) two-sided bounds, either depending on the eigenvalues of $X^{-1}K$ (see [12] and [24]) or on the eigenvalues of A (see [6]). Here X is the block diagonal matrix arising in the incomplete factorization of A and K is an auxiliary matrix to be defined later; in special cases $K = D_A$, the blockdiagonal part of A.

(ii) upper bounds of the largest eigenvalue, derived from the above results, in the form $\frac{1}{2-\beta}$, where $2 > \beta \geq \lambda_{\max}(X^{-1}K)$ (see [6], [12] and [29]) or in the form

$\frac{1}{1-\tau}$, where $0 \leq \tau < 1$ and depends on certain relations between the action of the preconditioner and of the given matrix on a given positive vector, see [14].

(iii) upper bounds of the largest eigenvalue, which depend on m, typically as cm, for some constant c, where m is the number of diagonal blocks in the block matrix partitioning of A.

Finally, one can combine the techniques used in (ii) and (iii). Clearly, in each application, the minimum of the above bounds gives the best estimate.

A frequently used technique in the construction of preconditioners is the use of small perturbations of the given matrix during the factorization, when required for the preconditioner to satisfy certain conditions for eigenvalue bounds to hold. For an early use of such bounds, see [1], and for later analysis, see [15]. One can use some graph theoretic means to estimate the perturbation parameters and condition numbers resulting from them (see [7] and the references therein), but this will not be included here.

For many problems the condition number depends on some problem parameters. For difference equations for elliptic equations, for example, this parameter is usually the meshwidth (h) of the difference grid and we want to estimate the order of the condition number w.r.t. this parameter (as $h \to 0$). For certain preconditioners one can reduce the condition number by an order of magnitude, for difference equations for second order problems from $O(h^{-2})$ to $O(h^{-1})$, $h \to 0$. By deriving a lower bound, one can show, (see [7]) that this order is also best possible, if the sparsity of the matrices involved in the preconditioner is of the same order as for the given matrix. However, using certain recursive constructions of the preconditioner one can show that the order can be further significantly reduced, while letting the sparsity grow only slowly, for instance as $O(\log m)$; but this topic is outside the scope of the present paper.

Besides the dependence on h, there is a dependence of the coefficients in the differential equation and on the shape of the domain. Discontinuities in the coefficients can influence the condition number for certain orderings of the meshpoints. The condition number may also depend on the shape of the domain or aspect ratio of elements used in the discretization. For further information about such dependence, see [24] and [30].

The following notations are used in this paper: unless otherwise stated when A and B are symmetric, $A \geq B$ means that $A - B$ is positive semidefinite. $\lambda_i(A)$ denotes the ith eigenvalue of a symmetric matrix A where the eigenvalues are numbered in a nondecreasing order. $\lambda_{\max}(A)$ denotes the maximal eigenvalue of A and $\lambda_{\min}(A)$

denotes the smallest eigenvalue. A is s.p.d. and s.p.s.d. denotes that A is symmetric and positive definite and symmetric and positive semidefinite, respectively. A is partitioned in block matrices (which may be scalars) consistent with a given partitioning of vectors and split as

$$A = D_A + L_A + L_A^T,$$

where D_A, L_A is the block diagonal part and lower block triangular part of A, respectively. We let $m \times m$ be the number of blocks in A, and n be the order of A.

2 Upper and lower bounds of eigenvalues

Let A be a symmetric matrix. Consider a block incomplete preconditioner in the form $C = (X + L)X^{-1}(X + L^T)$, where X is a block diagonal and s.p.d. matrix and L is a block lower triangular matrix. Incidently, the assumption that L is a block lower triangular matrix is more of concern for practical implementations. For the proofs, it is used only in Theorems 2.5, 3.3 and 3.6.

In [6] it is shown that $1/(2 - \beta)$ is an upper bound of the spectral radius of the preconditioned matrix $C^{-1}A$ if $A \leq \beta X + L + L^T$ where $\beta < 2$. Here we extend this result and generalize it to each eigenvalue of $C^{-1}A$ and also give a lower bound. This presentation is based on the results in [24]. To this end, we need first the following basic result.

Theorem 2.1 *Let A be a symmetric matrix and let $C = (X + L)X^{-1}(X + L^T)$ where X is s.p.d. Assume that $\alpha_i \leq \lambda_i(X^{-1}K) \leq \beta_i$, where α_i, β_i are constants and $K = A - L - L^T$. Then*

$$\lambda_i(\widehat{M}(D(\alpha))) \leq \lambda_i(C^{-1}A) \leq \lambda_i(\widehat{M}(D(\beta))),$$

where $D(\sigma) = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_i = \alpha_i$, $\sigma_i = \beta_i$, respectively,

$$\widehat{M}(D) = (I + \widehat{L})^{-1} + (I + \widehat{L}^T)^{-1} + (I + \widehat{L})^{-1}(D - 2I)(I + \widehat{L}^T)^{-1},$$

and $\widehat{L} = PX^{-\frac{1}{2}}LX^{-\frac{1}{2}}P^T$. Here P is an orthogonal matrix such that

$$PX^{-\frac{1}{2}}KX^{-\frac{1}{2}}P^T = \mathrm{diag}(\lambda_1(X^{-1}K), \ldots, \lambda_n(X^{-1}K)).$$

Proof. We prove the second inequality only. The first follows in a similar way. Note then first that, by assumption, $\widetilde{K} \leq D(\beta)$ where $\widetilde{K} = PX^{-\frac{1}{2}}KX^{-\frac{1}{2}}P^T$. Next, a computation using a similarity transformation of $C^{-1}A$ shows that

$$\begin{aligned}
& PX^{-\frac{1}{2}}(X+L^T)(C^{-1}A)(X+L^T)^{-1}X^{\frac{1}{2}}P^T \\
= \;& PX^{\frac{1}{2}}(X+L)^{-1}A(X+L^T)^{-1}X^{\frac{1}{2}}P^T \\
= \;& PX^{\frac{1}{2}}(X+L)^{-1}(K+L+L^T)(X+L^T)^{-1}X^{\frac{1}{2}}P^T \\
= \;& PX^{\frac{1}{2}}(X+L)^{-1}X^{\frac{1}{2}}P^T(\widetilde{K}+\widehat{L}+\widehat{L}^T)PX^{\frac{1}{2}}(X+L^T)^{-1}X^{\frac{1}{2}}P^T \\
\leq \;& (I+\widehat{L})^{-1}(D(\beta)-2I+I+\widehat{L}+I+\widehat{L}^T)(I+\widehat{L}^T)^{-1},
\end{aligned} \tag{2.1}$$

which implies the conclusion. □
This theorem yields immediately the next result, see [13], [12] and [24] for earlier appearances of this result.

Theorem 2.2 *Assume that the conditions of Theorem* 2.1 *hold. Then*

(a)
$$\lambda_{\min}(M(\alpha_i)) \leq \lambda_i(C^{-1}A) \leq \lambda_{\max}(M(\beta_i)),$$

where $M(\sigma) = (I+\widetilde{L})^{-1}+(I+\widetilde{L}^T)^{-1}+(\sigma-2)(I+\widetilde{L})^{-1}(I+\widetilde{L}^T)^{-1}$ *and* $\widetilde{L} = X^{-\frac{1}{2}}LX^{-\frac{1}{2}}$.
(b) *If* $\beta_i < 2$, *then*

$$\lambda_i(C^{-1}A) \leq \frac{1}{2-\beta_i},$$

and, in particular, if $\lambda_i(X^{-1}K) < 2$, *then*

$$\lambda_i(C^{-1}A) \leq (2-\lambda_i(X^{-1}K))^{-1}.$$

Proof. The result appeared originally in [24]. We give here an alternative proof by use of Theorem 2.1.
(a) Let $\widehat{L}$ be as defined in Theorem 2.1. It is readily seen that

$$\lambda_i((I+\widehat{L})^{-1}(D(\beta)-\beta_i I)(I+\widehat{L}^T)^{-1}) = 0.$$

Therefore, applying Theorem 2.1 and Weyl's theorem (cf. Parlett, 1980, p.192) shows that

$$\begin{aligned}
\lambda_i(C^{-1}A) &\leq \lambda_i(\widehat{M}(\beta_i I) + (I+\widehat{L})^{-1}(D(\beta)-\beta_i I)(I+\widehat{L}^T)^{-1}) \\
&\leq \lambda_{\max}(\widehat{M}(\beta_i I)) + \lambda_i((I+\widehat{L})^{-1}(D(\beta)-\beta_i I)(I+\widehat{L}^T)^{-1}) \\
&= \lambda_{\max}(\widehat{M}(\beta_i I)) = \lambda_{\max}(M(\beta_i)).
\end{aligned} \tag{2.2}$$

The lower bound follows in a similar way.
(b) An elementary computation shows that

$$\begin{aligned}
M(\sigma) &= (I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1} + (\sigma-2)(I+\widetilde{L})^{-1}(I+\widetilde{L}^T)^{-1} \\
&= \tfrac{1}{2-\sigma}I - (2-\sigma)[(I+\widetilde{L})^{-1} - \tfrac{1}{2-\sigma}I][(I+\widetilde{L}^T)^{-1} - \tfrac{1}{2-\sigma}I],
\end{aligned} \tag{2.3}$$

which, together with part (a), completes the proof. □

The result implies in particular that if $2 > \beta \geq \lambda_{\max}(X^{-1}K)$, then

$$\lambda_{\max}(C^{-1}A) \leq \frac{1}{2-\beta}. \tag{2.4}$$

If $L = L_A$ is non-positive and X is a block-diagonal Stieltjes matrix the above condition takes the form $2 > \beta \geq \lambda_{\max}(X^{-1}D_A)$ and the result (2.4) can then also be found in [6] and [29].

The following theorem shows some cases where Theorem 2.2 is applicable to the estimate of maximum eigenvalues of $C^{-1}A$, see [12] for further details. Note first that if a matrix X is a symmetric Z-matrix and $Xv \geq 0$ for some vector $v > 0$, then X is symmetric and positive semidefinite. (This is readily shown noting first that $D^{-\frac{1}{2}}XD^{-\frac{1}{2}} + \varepsilon I$ is a diagonally dominant Z-matrix for any $\varepsilon > 0$ and then letting $\varepsilon \to 0$.)

Theorem 2.3 a) *Let X be symmetric. If $\beta X - D_A$ is a Z-matrix and the entries of $L + L^T$ are not larger than the corresponding entries of $L_A + L_A^T$, then $\beta X - K$ is a Z-matrix. If, in addition, $\beta Xv - Kv \geq 0$ for some positive vector v, then $\beta X - K$ is s.p.s.d.*
b) *Suppose X is s.p.d. and $\alpha X - K + \gamma I$ is s.p.s.d. Then $(\alpha + \gamma/\lambda_{\min}(X))X - K$ is s.p.s.d. if $\gamma \geq 0$ and $(\alpha + \gamma/\lambda_{\max}(X))X - K$ is s.p.s.d. if $\gamma \leq 0$.*
c) *$\lambda_{\max}(X^{-1}K)X - K$ and $K - \lambda_{\min}(X^{-1}K)X$ are s.p.s.d. if X is s.p.d.*

Proof. a) Noting that $\beta X - K = (\beta X - D_A) + (L + L^T - L_A - L_A^T)$, it follows from the assumptions that $\beta X - K$ is a Z-matrix. An application of the above remark shows that $\beta X - K$ is s.p.s.d. Hence $\lambda_{\max}(X^{-1}K) \leq \beta$.
Parts b) and c) are straightforward. □

Some alternative lower and upper bounds
Consider the matrix pencil $\lambda C - A$. This can also be written in the form

$$\lambda C - A = \left(\frac{\lambda}{\mu} - 1\right) A + \frac{\lambda}{\mu} R_\mu,$$

where $R_\mu = \mu C - A$ and where we let $0 < \lambda < \mu$. As has been shown in [6], and in a somewhat extended form in [7], this decomposition for the matrix pencil $\lambda C - A$ can be used to show the following alternative bounds, which relate eigenvalues of the generalized eigenvalue problem directly to those of A.

Theorem 2.4 (a) *Let A be symmetric positive semidefinite and let C be s.p.d. Let μ_1, μ_2 be sufficiently large positive numbers such that*

$$\lambda_{\max}(\mu_1 C - A) \geq 0$$

and

$$\lambda_{\min}(\mu_2 C - A) \geq 0$$

hold. Then, for all positive eigenvalues of A, the following lower and upper bounds of the eigenvalues of $C^{-1}A$ hold:

$$\frac{\mu_1 \lambda_i(A)}{\lambda_i(A) + \lambda_{\max}(\mu_1 C - A)} \leq \lambda_i(C^{-1}A) \leq \frac{\mu_2 \lambda_i(A)}{\lambda_i(A) + \lambda_{\min}(\mu_2 C - A)}.$$

(b) *Let A be symmetric positive semidefinite and diagonally scaled such that $A = dI + L_A + L_A^T$ for some positive scalar d. Let $C = (X+L)X^{-1}(X+L^T)$ and assume that X is a Stieltjes matrix, $2x - d > 0$, where x is a positive scalar such that $xv \leq Xv$ for some positive vector v and that the off-diagonal entries of $L + L^T$ are not larger than the corresponding entries of A. Then*

$$\lambda_i(C^{-1}A) \leq \begin{cases} \frac{4x\lambda_i(A)}{(2x-d+\lambda_i(A))^2}, & \text{for all} \quad \lambda_i(A) \leq 2x - d \\ \frac{x}{2x-d}, & \text{for all} \quad \lambda_i(A) \geq 2x - d \end{cases}$$

Proof. See [7].
This theorem will be applied later for classes of matrices A with a certain eigenvalue distribution and a particular class of parameter dependent incomplete factorization methods, to show that a major part of the spectrum of $C^{-1}A$, namely the set of smallest eigenvalues, essentially equals the corresponding eigenvalues of A multiplied with a constant factor. For completeness we give below also an alternative upper eigenvalue bound proposed in [14] and extended in [32]. For a convenience of the readers we provide a short proof of the result for the case that X is diagonal and L is non-positive. For a proof in the more general case where X is block diagonal, see [29].

Theorem 2.5 *Let A be symmetric and positive semidefinite and let $X + L$ be an M-matrix, where X is nonsingular and diagonal and L is lower triangular. Assume that there exists a positive vector v such that $Av \geq 0$ and that*
(i) *the off-diagonal entries of $L + L^T$ are not larger than the corresponding entries of A.*
(ii) *Let $C = (X+L)X^{-1}(X+L^T)$ and assume that*

$$Cv \geq (1 - \tau_0)Av, \ \textit{for some } \tau_0, 0 \leq \tau_0 < 1.$$

(iii) $\tau_1 = \max_i\{-(X^{-1}L^T v)_i/v_i) < 1$
where v_i, $i = 1, 2, \dots, n$ *are the components of* v.
Then

$$\lambda_{\max}(C^{-1}A) \leq \frac{1}{1-\tau}.$$

where $\tau = \max(\tau_0, \tau_1)$.

Proof. Let $B = (X_0 + L)X^{-1}(X_0 + L^T)$, where X_0 is a diagonal matrix such that $X_0 v = -L^T v$. Let $A = D_A + L_A + L_A^T$, where D_A is the diagonal and L_A the lower triangular part of A. Then

$$\begin{aligned} &C-(1-\tau)A-B \\ &\quad = X+L+L^T+LX^{-1}L^T-(1-\tau)A-X_0X^{-1}X_0-X_0X^{-1}L^T-LX^{-1}X_0-LX^{-1}L^T \\ &\quad = X-X_0X^{-1}X_0-(1-\tau)D_A+(1-\tau)(L+L^T-L_A-L_A^T) \\ &\qquad +L(\tau I-X^{-1}X_0)+(\tau I-X_0X^{-1})L^T. \end{aligned}$$

Note next that $\tau v - X^{-1}X_0 v = \tau v + X^{-1}L^T v \geq \tau_1 v + X^{-1}L^T v \geq 0$. Hence, $\tau_1 v - X^{-1}X_0 v \geq 0$, and since $\tau I - X^{-1}X_0$ is a diagonal matrix, $\tau I - X^{-1}X_0 \geq 0$. Similarly $\tau I - X_0 X^{-1} \geq 0$, which, together with assumption (i) shows that the off-diagonal entries of $C - (1-\tau)A - B$ are non-positive. Since $Bv = 0$ and $Av \geq 0$, (ii) shows that $(C - (1-\tau)A - B)v \geq Cv - (1-\tau_0)Av \geq 0$, that is, $C - (1-\tau)A - B$ is positive semidefinite. Since B is positive semidefinite, $C - (1-\tau)A$ is positive semidefinite. □

By a more detailed analysis of these bounds it follows that the upper condition number bounds depends on $O(m)$; more precisely m is an upper bound if $(L-L^T)v \leq Cv$, (see [27],[30]) or else $O(m_1 + m)$, where $m_1 = n/m$ (see [17], [30]); compare also the result in the next section. The version developed in [32] captures both point and block methods.

3 An upper bound which depends on the number of blocks

We improve here some previous upper eigenvalue bounds in [12] and show bounds which depend on the number of matrix blocks. To this end we need the following results.

Theorem 3.1 *Let $A = (A_{ij})$ be an $m \times m$ block symmetric positive semidefinite matrix. Then*

$$\rho(A) \leq \sum_{i=1}^{m} \rho(A_{ii})$$

where $\rho(A) = \max_i |\lambda_i(A)|$.

Proof. Consider first the case $m = 2$. Let $B = \begin{bmatrix} \rho(A_{11})I_1 & A_{12} \\ A_{21} & \rho(A_{22})I_2 \end{bmatrix}$ which clearly is s.p.s.d. and note that $\rho(A) \leq \rho(B)$. It is not difficult to show that

$$\frac{1}{2}(\rho(A_{11}) + \rho(A_{22}) - (\rho(A_{11}) - \rho(A_{22}))^2 + 4\|A_{12}\|_2^2)^{\frac{1}{2}} = \lambda_{\min}(B) \geq 0$$

and, hence,

$$\begin{aligned} \rho(B) &= \tfrac{1}{2}(\rho(A_{11}) + \rho(A_{22}) + (\rho(A_{11}) - \rho(A_{22}))^2 + 4\|A_{12}\|_2^2)^{\frac{1}{2}} \\ &\leq \rho(A_{11}) + \rho(A_{22}). \end{aligned}$$

Next, by induction, we find that the statement holds for any s.p.s.d. matrix, $A = (A_{ij})_{i,j=1}^{m}$. $\square$

This theorem generalizes the elementary inequality $\rho(A) \leq \operatorname{tr}(A)$, which holds for symmetric positive semidefinite matrices. The estimate has been generalized in various ways, see [22], where also further references can be found.

As an application, let $\widetilde{L}$ be a strictly lower block triangular matrix partitioned in $m \times m$ blocks. Then $(I+\widetilde{L})^{-1}$ and $(I+\widetilde{L}^T)^{-1}$ are both block triangular with identity diagonal part so the diagonal blocks of $(I+\widetilde{L})^{-1}+(I+\widetilde{L}^T)^{-1}$ equal twice the identity. Hence, provided $(I+\widetilde{L})^{-1}+(I+\widetilde{L}^T)^{-1}$ is positive semidefinite, Theorem 3.1 shows that

$$\lambda_{\max}((I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1}) \leq 2m. \tag{3.1}$$

Theorem 3.2 *Let $C = (X+L)X^{-1}(X+L^T)$, where X is s.p.d. If $\alpha C \leq \beta X + L + L^T$ with $\beta \leq 2$, where α and β are constants, then $(I+\widetilde{L})^{-1}+(I+\widetilde{L}^T)^{-1} - \alpha I$ is positive semidefinite.*

Proof. Let $A = \alpha C$ and $P = I$. Then $K = A - L - L^T = \alpha C - L - L^T \leq \beta X$ and the proof of Theorem 2.1 shows that

$$\alpha I = C^{-\frac{1}{2}} A C^{-\frac{1}{2}} \leq \widehat{M}(D(\beta)) \leq (I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1}. \qquad \square$$

We can now derive an improved upper bound.

Theorem 3.3 *Let A be a symmetric matrix and $C = (X+L)X^{-1}(X+L^T)$, where $X = \mathrm{blockdiag}(X_1, X_2, \ldots, X_m)$ is s.p.d. If $\alpha C \le \beta X + L + L^T$ and $A \le \sigma X + L + L^T$, where $\beta < 2$, σ and β are constants, then*

$$\lambda_{\max}(C^{-1}A) \le \begin{cases} \frac{(\sigma-\beta)(2-\alpha)}{2-\beta} m + \alpha, & \text{if } \sigma > 2 \\ (\sigma-1)(2-\alpha)(m-1)+\sigma, & \text{if } 1 \le \sigma \le 2 \\ 1, & \text{if } \sigma \le 1. \end{cases}$$

Proof. Under the stated assumptions it follows that, for any $x > 0$,

$$xA + \alpha C - (x\sigma + \beta)X - (1+x)(L+L^T) \le 0,$$

so

$$\frac{x}{1+x}A + \frac{\alpha}{1+x}C - \frac{x\sigma+\beta}{1+x}X - (L+L^T) \le 0.$$

Let $B = \frac{x}{1+x}A + \frac{\alpha}{1+x}C$. It follows then from Theorem 2.2 (a) that

$$\begin{aligned} &\lambda_{\max}(C^{-1}B) = \tfrac{x}{1+x}\lambda_{\max}(C^{-1}A) + \tfrac{\alpha}{1+x} \\ \le\ &\lambda_{\max}\left((I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1} + \left(\tfrac{x\sigma+\beta}{1+x} - 2\right)(I+\widetilde{L})^{-1}(I+\widetilde{L}^T)^{-1}\right). \end{aligned}$$

If $\sigma > 2$, we choose $x_0 = (2-\beta)/(\sigma-2)$ and apply Theorem 3.2 and (3.1) to show that

$$\begin{aligned} \lambda_{\max}(C^{-1}A) &\le \tfrac{x_0+1}{x_0}\lambda_{\max}((I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1}) - \tfrac{\alpha}{x_0} \\ &= \tfrac{x_0+1}{x_0}\lambda_{\max}((I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1} - \alpha I) + \alpha \\ &\le \tfrac{(\sigma-\beta)(2-\alpha)}{2-\beta} m + \alpha. \end{aligned}$$

Let $\widetilde{M} = (I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1}$ and note that an elementary computation shows that

$$(I+\widetilde{L})^{-1}(I+\widetilde{L}^T)^{-1} = (\widetilde{M}-I) + (I+\widetilde{L})^{-1}\widetilde{L}\widetilde{L}^T(I+\widetilde{L}^T)^{-1}. \tag{3.2}$$

Then, if $1 < \sigma \le 2$, a computation using (3.2) shows that

$$\begin{aligned} &\tfrac{x}{1+x}\lambda_{\max}(C^{-1}A) + \tfrac{\alpha}{1+x} \\ \le\ &\lambda_{\max}[(I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1} + (y-2)(I+\widetilde{L})^{-1}(I+\widetilde{L}^T)^{-1}] \\ \le\ &\lambda_{\max}[(2-y+(y-1)\alpha)I + (y-1)(\widetilde{M}-\alpha I) \\ &\quad +(y-2)(I+\widetilde{L})^{-1}\widetilde{L}\widetilde{L}^T(I+\widetilde{L}^T)^{-1}] \\ \le\ &\lambda_{\max}[(2-y+(y-1)\alpha)I + (y-1)(\widetilde{M}-\alpha I)], \end{aligned}$$

where $y = \frac{x\sigma+\beta}{1+x}$, i.e. $y \le 2$.

Letting $x \to \infty$, i.e. $y \to \sigma$, and using Theorem 3.2 and (3.1) once more, yields

$$\begin{aligned} &\lambda_{\max}(C^{-1}A) \\ \le\ &\lim_{x\to\infty}\{\tfrac{1+x}{x}\lambda_{\max}[(2-y+(y-1)\alpha)I + (y-1)(\widetilde{M}-\alpha I)] + \tfrac{\alpha}{x}\} \\ \le\ &\lambda_{\max}((2-\sigma+(\sigma-1)\alpha)I + (\sigma-1)(\widetilde{M}-\alpha I)) \\ \le\ &2-\sigma+(\sigma-1)\alpha + (\sigma-1)(2-\alpha)m \\ =\ &(\sigma-1)(2-\alpha)(m-1) + \sigma. \end{aligned}$$

Finally, if $\sigma \leq 1$, Theorem 2.2 (b) shows that

$$\lambda_{\max}(C^{-1}A) \leq 1,$$

which completes the proof. $\square$

If $A \geq 0$ and $A \leq \sigma X + L + L^T$ it holds that $\sigma X + L + L^T \geq 0$. Hence, if $1 \leq \sigma \leq 2$ it is readily seen that there are two constants, $\alpha \geq 0$ and $\beta < 2$ such that $\alpha C \leq \beta X + L + L^T$. Applying Theorem 3.3 shows that

$$\begin{aligned} \lambda_{\max}(C^{-1}A) &\leq (\sigma - 1)(2-\alpha)(m-1) + \sigma \\ &\leq 2(\sigma - 1)m + 2 - \sigma, \end{aligned}$$

which latter is the bound derived in [12] using less powerful techniques.

In the particular, and also commonly occuring case (see applications to follow) where $C \leq A$, the next, even smaller bound, holds.

Theorem 3.4 *Let* $C = (X+L)X^{-1}(X+L^T)$, *where* $X =$blockdiag$(X_1, X_2, \ldots, X_m)$, *and assume that* $C \leq A$. *If there exists a positive constant* $\sigma \leq 2$ *such that* $A \leq \sigma X + L + L^T$, *then*

$$\mathrm{cond}(C^{-1}A) \leq (\sigma - 1)m + 1. \tag{3.3}$$

Proof. As will be seen later, Theorem 3.4 is a corollary of the more general Theorem 3.6 to follow. $\square$

Note that the results above show that under the stated conditions we have an upper bound of the eigenvalues of $O(m)$, where $m \times m$ is the number of blocks used in the partitioning of A.

Up till now, the stated results on bounds of the maximum eigenvalue emphasize how the bounds of maximum eigenvalue of $X^{-1}K$ influence upper bounds of the maximum eigenvalue of the preconditioned matrix $C^{-1}A$. In practice, however, a few eigenvalues of $X^{-1}K$ can be essentially larger than 2, for instance, when an incomplete factorization is applied to elliptic equations, see [24]. In this case all previous results can have troubles to obtain an $O(h^{-1})$ type bound for the condition number. Substantial improvements on estimates of upper bounds of the maximum eigenvalue of the preconditioned matrix involving bounds of every eigenvalue of $X^{-1}K$ appeared recently in [24] by combining the techniques used in Section 2 and 3. As it turn out, the new bounds can be significantly smaller than the previous bounds; for further details and examples, see [24]. Again for convenience of the readers, we state the results and provide some slightly shortened proofs.

Let the order of matrices A_{ij} in the block partitioning of A be $n_i \times n_j$.

Theorem 3.5 *Let $C = (X+L)X^{-1}(X+L^T)$ where X is s.p.d. and $X = \text{blockdiag}$ $(X_1, \ldots, X_m)$. Let $A = K + L + L^T$ be a symmetric matrix, and assume that $K \le G$, where $G = \text{blockdiag}(G_1, \ldots, G_n)$ and X_i, G_i are matrices of order n_i. Let $N = [N_{ij}]$ be the block matrix partitioning of $N = (I+\widetilde{L}^T)^{-1}(I+\widetilde{L})^{-1}$ where $\widetilde{L} = X^{-\frac{1}{2}} L X^{-\frac{1}{2}}$. Then*

a) $\lambda_{\max}(C^{-1}A) \le \min\limits_{\mu<2} \left(\frac{1}{2-\mu} + \sum\limits_{i \in S(\mu)} (\rho(X_i^{-1}G_i) - \mu \right) \rho(N_{ii}),$

b) $\lambda_{\max}(C^{-1}A) \le 2 \left(\sum\limits_{i=1}^{m} \rho(N_{ii}) \right)^{\frac{1}{2}} + \sum\limits_{i \in S(2)} (\rho(X_i^{-1}G_i) - 2)\rho(N_{ii}),$

where $S(\mu) = \{i; \rho(X_i^{-1}G_i) > \mu\}$.

Proof. a) Using (2.3) it follows that for $\mu < 2$,

$$\begin{array}{rl} & X^{-\frac{1}{2}}(X+L^T)(C^{-1}A)(X+L^T)^{-1}X^{\frac{1}{2}} \\ = & (I+\widetilde{L})^{-1} + (I+\widetilde{L}^T)^{-1} + (\mu-2)(I+\widetilde{L})^{-1}(I+\widetilde{L}^T)^{-1} \\ & +(I+\widetilde{L})^{-1}(\widetilde{K}-\mu I)(I+\widetilde{L}^T)^{-1} \\ \le & M(\mu) + (I+\widetilde{L})^{-1}(\widetilde{G}-\mu I)(I+\widetilde{L}^T)^{-1} \\ \le & \frac{1}{2-\mu} I + (I+\widetilde{L})^{-1}(\widetilde{G}-\mu I)(I+\widetilde{L}^T)^{-1}, \end{array} \tag{3.4}$$

where $\widetilde{K} = X^{-\frac{1}{2}} K X^{-\frac{1}{2}}$, $\widetilde{G} = X^{-\frac{1}{2}} G X^{-\frac{1}{2}}$. Note now that $\widetilde{G} = \text{blockdiag}\,(X_i^{-\frac{1}{2}} G_i X_i^{-\frac{1}{2}})$, and for $i \in S(\mu)$, applying Theorem 3.1,

$$\begin{array}{rl} & \lambda_{\max}((I+\widetilde{L})^{-1}R(\mu)(I+\widetilde{L}^T)^{-1}) \\ = & \lambda_{\max}(R(\mu)^{\frac{1}{2}}(I+\widetilde{L}^T)^{-1}(I+\widetilde{L})^{-1}R(\mu)^{\frac{1}{2}}) \\ \le & \sum\limits_{i \in S(\mu)} (\rho(X_i^{-1}G_i) - \mu)\rho(N_{ii}), \end{array} \tag{3.5}$$

which together with (3.4) show the desired result.
b) Let $\rho_i = \rho(X_i^{-1}G_i)$ and define r_i, $i = 1, 2, \ldots, m$ by

$$r_i = \begin{cases} 2, & \text{if } \rho_i \le 2 \\ \rho_i, & \text{if } \rho_i > 2 \end{cases} \tag{3.6}$$

Then part a) shows that

$$\begin{array}{rcl} \lambda_{\max}(C^{-1}A) & \le & \min\limits_{\mu<2} \left(\frac{1}{2-\mu} + \sum\limits_{i=1}^{m} (r_i - \mu)\rho(N_{ii}) \right) \\ & \le & 2 \left(\sum\limits_{i=1}^{m} \rho(N_{ii}) \right)^{\frac{1}{2}} + \sum\limits_{i \in S(2)} (\rho_i - 2)\rho(N_{ii}). \end{array}$$

The minimum of the right hand side is taken for $\mu = 2 - \left(\sum_{i=1}^{m} \rho(N_{ii})\right)^{-\frac{1}{2}}$, which implies the upper bound in part b). $\square$
For applications of these bounds for second order elliptic difference equations, see Section 4.

Finally, we give an alternative bound to that in Theorem 3.3, which in many instances can give a sharper bound. In this case we assume that L is strictly lower block triangular.

Theorem 3.6 *Let $A \leq G + L + L^T$ be a symmetric positive semidefinite matrix, where $G = \mathrm{blockdiag}(G_1, G_2, \ldots, G_m)$, and let $C = (X + L)X^{-1}(X + L^T)$ where $X = \mathrm{blockdiag}(X_1, \ldots, X_m)$ is s.p.d. and G_i and X_i have the same order n_i. Partition $T = I + X^{-\frac{1}{2}}LX^{-\frac{1}{2}}$ into a $m \times m$ block matrix $T = [T_{ij}]_{m\times m}$ consistently with the partitioning of G. If $\alpha C \leq A$, where α is a nonnegative constant, then,*
a) $\lambda_{\max}(C^{-1}A) \leq \sum_{\rho_i \leq 2} \rho_i + \sum_{\rho_i > 2} \left(2 - (\rho_i - 2) \sum_{k=i}^{m} \rho(T_{ki}T_{ki}^T)\right) - \alpha(m-1)$, *where* $\rho_i = \rho(X_i^{-1}G_i)$
b) *if, in addition,* $\rho_i \leq 2$, $i = 1, 2, \ldots, m$, *then*

$$\lambda_{\max}(C^{-1}A) \leq \sum_{i=1}^{m} \rho_i - \alpha(m-1)$$

and, if $C \leq A$, then

$$\lambda_{\max}(C^{-1}A) \leq m + 1.$$

Proof. Define two block diagonal matrices,

$$\begin{aligned} P &= \mathrm{blockdiag}(\rho_1 I_{n_1}, \ldots, \rho_m I_{n_m}) \\ R &= \mathrm{blockdiag}(r_1 I_{n_1}, \ldots, r_m I_{n_m}), \end{aligned}$$

where r_i is defined by (3.6). It follows from (3.4) that

$$\begin{aligned} \lambda_i(C^{-1}A) \;\leq\; & \lambda_i((I + \widetilde{L})^{-1} + (I + \widetilde{L}^T)^{-1} + \\ & (I + \widetilde{L})^{-1}(P - 2I)(I + \widetilde{L}^T)^{-1}) \\ \leq\; & \lambda_i((I + \widetilde{L})^{-1} + (I + \widetilde{L}^T)^{-1} + (I + \widetilde{L})^{-1}(P - R)(I + \widetilde{L}^T)^{-1} \\ & +(I + \widetilde{L})^{-1}(R - 2I)(I + \widetilde{L}^T)^{-1}). \end{aligned} \tag{3.7}$$

Further, for any matrix D a simple computation shows that

$$\begin{aligned}
& (I+\widetilde{L})^{-1}D(I+\widetilde{L}^T)^{-1} \\
= \;& (I-(I+\widetilde{L})^{-1}\widetilde{L})D(I-\widetilde{L}^T(I+\widetilde{L}^T)^{-1}) \\
= \;& D-(I+\widetilde{L})^{-1}\widetilde{L}D-D\widetilde{L}^T(I+\widetilde{L}^T)^{-1}+(I+\widetilde{L})^{-1}\widetilde{L}D\widetilde{L}^T(I+\widetilde{L}^T)^{-1} \\
= \;& D-(I+\widetilde{L})^{-1}(I+\widetilde{L}-I)D-D(I+\widetilde{L}^T-I)(I+\widetilde{L}^T)^{-1} \\
& +(I+\widetilde{L})^{-1}\widetilde{L}D\widetilde{L}^T(I+\widetilde{L}^T)^{-1} \\
= \;& (I+\widetilde{L})^{-1}D+D(I+\widetilde{L}^T)^{-1}-D+(I+\widetilde{L})^{-1}\widetilde{L}D\widetilde{L}^T(L+\widetilde{L}^T)^{-1}.
\end{aligned}$$

Therefore, for the ith eigenvalue of the matrix $C^{-1}A$ inequality (3.7) becomes

$$\begin{aligned}
\lambda_i(C^{-1}A) \;\le\;& \lambda_i((I+\widetilde{L})^{-1}(P-R+I)+(P-R+I)(I+\widetilde{L}^T)^{-1}+R-P \\
& +(I+\widetilde{L})^{-1}\widetilde{L}(P-R)\widetilde{L}^T(I+\widetilde{L}^T)^{-1}+(I+\widetilde{L})^{-1}(R-2I)(I+\widetilde{L}^T)^{-1}).
\end{aligned} \tag{3.8}$$

Since $(I+\widetilde{L})^{-1}\widetilde{L}(P-R)\widetilde{L}^T(I+\widetilde{L}^T)^{-1}$ is negative semidefinite, applying Weyl's theorem to (3.8) we have

$$\begin{aligned}
\lambda_i(C^{-1}A) \;\le\;& \lambda_i((I+\widetilde{L})^{-1}(P-R+I)+(P-R+I)(I+\widetilde{L}^T)^{-1} \\
& +R-P+(I+\widetilde{L})^{-1}(R-2I)(I+\widetilde{L}^T)^{-1}).
\end{aligned}$$

In particular, the assumption $\alpha C\le A$ implies

$$\alpha\le\lambda_{\min}(C^{-1}A)\le\lambda_{\min}(Q)$$

where

$$\begin{aligned}
Q \;=\;& (I+\widetilde{L})^{-1}(P-R+I)+(P-R+I)(I+\widetilde{L}^T)^{-1} \\
& +R-P+(I+\widetilde{L})^{-1}(R-2I)(I+\widetilde{L}^T)^{-1},
\end{aligned}$$

which shows that $Q-\alpha I$ is positive semidefifnite. Applying (3.8) to the maximum eigenvalue yields then that

$$\lambda_{\max}C^{-1}A)\le\lambda_{\max}((Q-\alpha I)+(I+\widetilde{L})^{-1}(R-2I)(I+\widetilde{L}^T)^{-1})+\alpha \tag{3.9}$$

Furthermore, it is straightforward to show that the (k,k) block of $(I+\widetilde{L})^{-1}(R-2I)(I+\widetilde{L}^T)^{-1}$ is given by

$$\sum_{i=1}^{k}(r_i-2)T_{ki}T_{ki}^T.$$

Hence, using Theorem 3.1 to (3.9) yields

$$\begin{aligned}
\lambda_{\max}(C^{-1}A) \;\le\;& \sum_{i=1}^{m}(\rho_i-r_i)+2m-\alpha m+\sum_{k=1}^{m}\sum_{i=1}^{k}(r_i-2)\rho(T_{ki}T_{ki}^T)+\alpha \\
=\;& \sum_{\rho_i\le 2}\rho_i+\sum_{\rho_i>2}2+\sum_{i=1}^{m}(r_i-2)\sum_{k=i}^{m}\rho(T_{ki}T_{ki}^T)-(m-1)\alpha \\
=\;& \sum_{\rho_i\le 2}\rho_i+\sum_{\rho_i>2}((\rho_i-2)\sum_{k=i}^{m}\rho(T_{ki}T_{ki}^T)+2)-(m-1)\alpha,
\end{aligned}$$

which completes the proof of part a).
Part b) is a straightforward application of part a). □

Part b) of the above theorem gives an alternative proof to Theorem 3.3 of the condition number bound $m+1$, when $C \leq A$ (i.e. for $\alpha = 1$) and $1 \leq \sigma \leq 2$. Theorem 3.6 is in general sharper, because it exploits the possibility that several of the numbers ρ_i could be less than 2. Further, note that Theorem 3.4 follows directly from Theorem 3.6 b), because if the conditions of Theorem 3.4 hold, then $\rho_i \leq \sigma$ and $\alpha = 1$.

4 Applications to elliptic equations

We shall now consider some illustrative applications of the previous results. We consider in particular the use of perturbations of the given matrix during the construction of the preconditioner which enables one to easily show some bounds of the condition number but which, however, may depend on the perturbations. We also consider estimates of the smallest eigenvalues for a particular class of matrices which arise from difference approximations of second order elliptic problems.

Bounds for M-matrices
We consider first bounds of the condition number of $C^{-1}A$ for M-matrices. For early results on the existence of incomplete factorization methods for M-matrices, see [31] for pointwise methods and [10], [19], [4] and [18] for blockwise methods. For existence of such factorizations for block H-matrices, see [5]. For modified methods, where $Cv = Av$ for some positive vector v, the value 1 becomes a lower bound for the eigenvalues when we deal with M-matrices. How to implement such a method will be discussed now.

Let then $A = D_A + L_A + L_A^T$ be a splitting of a given M-matrix A partitioned in $m \times m$ blocks and let

$$C = (X + L)X^{-1}(X + L^T)$$

be a preconditioner to A, where $L = L_A$ and $X = \text{blockdiag}(X_1, X_2, \ldots, X_m)$, where the diagonal block matrices X_i are defined by

$$X_1 = Z[(D_A)_{1,1}],\ X_i = Z[(D_A)_{i,i} - (LY[X^{-1}]L^T)_{i,i}] - D_i', i = 2, 3, \ldots, m. \qquad (4.1)$$

Here $Y[B]$ denotes a sparse approximation of B, which is assumed to be nonnegative, such that $Y[B] \leq B$, entrywise. Similarly, $Z[M]$ denotes a sparse approximation such that $Z[M] \geq M$, entrywise, where M is a Z-matrix. Normally, $Z(M)$ just corresponds to deletion of some proper off-diagonal entries. $Z[M]$ leaves the diagonal of M unaffected. Finally, the modification matrix D_i' is diagonal and determined such that

$$X_i v_i = (D_A)_{i,i} v_i - (LX^{-1}L^T)_{i,i} v_i, \tag{4.2}$$

where v_i is the ith block of an a priori chosen positive vector v. Hence

$$\begin{aligned} D_i' v_i &= Z[(D_A)_{i,i} - (LY[X^{-1}]L^T)_{i,i}] v_i \\ &\quad -\{(D_A)_{i,i} - (LX^{-1}L^T)_{i,i}\} v_i. \end{aligned}$$

Note that since all entries of L have one sign, the assumptions made on the sparsity operators $Z[\cdot]$ and $Y[\cdot]$ show that D_i' is nonnegative. It can also be seen that X_i is a Stieltjes matrix. Next, note that

$$A - C = D_A - X - LX^{-1}L^T.$$

By the assumptions made, $A - C$ is a Z-matrix and by (4.2), $Av = Cv$. Hence $A - C$ is positive semidefinite, which shows that

$$\lambda_{\min}(C^{-1}A) \geq 1.$$

This means that the condition $\lambda_{\max}(\mu_1 C - A) = 0$ for the lower bound in Theorem 2.4 holds with $\mu_1 = 1$. We consider now the condition

$$(\beta X - K)v \geq 0 \tag{4.3}$$

where $\beta = 2 - \frac{1}{\mu_2}$ and μ_2 is positive. Recall that $K = A - L - L^T$ so, in the present case, $K = D_A$. By Theorem 2.2, μ_2 becomes an upper bound of the largest eigenvalue of $C^{-1}A$. If (4.3) holds only for $\mu_2 = \infty$, then Theorem 3.4 shows that $\lambda_{\max}(C^{-1}A) \leq m + 1$. In general we have $\lambda_{\max}(C^{-1}A) \leq \min\{\mu_2, (1 - \frac{1}{\mu_2})m + 1\}$. By constructing X_i as in (4.1) there is in general no guarantee that (4.3) holds for $\beta < 2$, i.e., that

$$(2 - \frac{1}{\mu_2})Xv \geq D_A v,$$

for some $\mu_2 < \infty$. It turns out to hold for model type elliptic difference equations and for certain orderings of the node points for more general types of such problems. However, it can be violated for other orderings. One way to overcome this problem is to use the method of perturbations: During the incomplete factorization, i.e., during the computation of the sequence X_i in (4.1) we add a nonnegative diagonal matrix to $(D_A)_{i,i}$ or to X_i, to make $X_i v_i$ sufficiently big so that

$$(2 - \frac{1}{\mu_2})X_i v_i = (D_A)_{i,i} v_i$$

holds for some positive μ_2. This corresponds to a perturbation Δ of A and a method to estimate the effect of this on the lower eigenvalue bound of $C^{-1}A$ has been discussed in [7] and [9]. For more robust methods of perturbations, which may have less influence on the smallest eigenvalue, see [9], [11], [14], [27]. For early references to perturbation techniques, see [1] and [2].

Note that if there is no need for a perturbation Δ, Theorem 3.4 and the above show that

$$\mathrm{cond}(C^{-1}A) \leq \min(\mu_2, (1 - \frac{1}{\mu_2})m + 1),$$

where m is the number of diagonal blocks in A.

If we use diagonal approximations $Y[X^{-1}]$ of X^{-1}, then the above method resembles a pointwise incomplete factorization method, in the sense that only pointwise diagonal entries change during the incomplete factorization. Nevertheless, it is not a true pointwise method, because one must still perform a blockwise forward and backward elimination. For this method we have then also the general upper bound $O(m)$. Such bounds can be of particular interest when an elliptic second order problem is solved on an oblong domain, like a rectangle with $N_1 \times N_2$ meshpoints, where we assume that $N_1 \gg N_2$. If we number the points such that the order of the matrix blocks in the main diagonal is N_1, then there are $m = N_2$ blocks in the main diagonal and the above shows that

$$\mathrm{cond}(C^{-1}A) \leq O(N_2),$$

which hence does not depend on N_1. This should be compared with the true pointwise method where the condition number $O(N_1 + m)$ holds, for details, see [30] and [17].

The above method can be generalized also to cases where L is different from L_A. It can then be implemented as follows.

Given a s.p.s.d. matrix A we consider approximate factorizations in the form

$$C = (X + L)X^{-1}(X + L^T),$$

where X is a blockdiagonal matrix and L is block lower triangular. There are various ways such a preconditioner can be computed. Here we choose a strategy to get the preconditioner to satisfy a given upper eigenvalue bound of $C^{-1}A$. Theorem 2.4 shows that

$$\lambda_{\max}(C^{-1}A) \leq \mu$$

for any a priori chosen upper bound μ ($\mu > \frac{1}{2}$), for which

$$\lambda_{\min}(\mu C - A) \geq 0. \tag{4.4}$$

As has been shown in Theorem 2.3, (4.4) holds if

(i) X and $(2-\frac{1}{\mu})X - D_A$ are Z-matrices

(ii) the block off-diagonal entries of $L + L^T$ are not larger than the corresponding entries of A

(iii) $((2-\frac{1}{\mu})X - K)v \geq 0$ for some positive vector v, where $K = A - L - L^T$.

Condition (ii) holds always when $L = L_A$ and condition (i) holds, in particular, if X and D_A are diagonal matrices. In more general cases we let simply

$$L_{i,j} = A_{i,j}$$

to make (ii) hold, if an entry (of) $L_{i,j}$ computed by a (block) matrix incomplete factorization, for instance, happens to be larger than the corresponding entry (of) $A_{i,j}$. Similarly, we let the off-diagonal entries of X be modified if required for (i) to hold. It remains to make $(2-\frac{1}{\mu})X - K$ positive semi-definite, and this holds if (iii) is valid.

This can simply be made to hold by computing the diagonal entries of X (block) row by (block) row during the incomplete factorization so that

$$(2-\frac{1}{\mu})(Xv)_i = (Kv)_i,\ \ i = 1, 2, \ldots, m,$$

holds. Again, this is equivalent to perturbing A with a corresponding diagonal matrix.

However, as it turns out perturbation techniques simply based on the above spectral bound can induce a disastrous decrease of the smallest eigenvalues, and therefore also in general of the rate of convergence of preconditioned conjugate gradient iterations. In particular, for matrices A such that the condition $(L_A^T - L_A)e \leq Ae$, where $e = (1, 1, \ldots, 1)^T$, is violated for a few scalar points, (say $O(h^{-1})$ points along a thin strip of the grid), the resulting condition number will be $O(h^{-2})$ for difference equations for second order elliptic problems, see [26] and [33] for further details. Other, more robust methods, include the dynamic perturbation method in [9], where a criterion based on the difference $(L_A^T - L_A)e$ was used, the methods in [33] and the relaxation method in [11]. See also [25] for techniques aimed at better robustness.

The block SSOR method

It is illustrative to consider now a still simpler type of incomplete factorization method, namely the BLOCK-SSOR method. Previous analysis of this method can be found in [35], [7], [4] and [9].Here $L = L_A$ and one lets

$$X_i = \frac{1}{\omega_i}(D_A)_{i,i}, \quad i = 1, \ldots, m, \tag{4.5}$$

where $0 < \omega_i < 2$ and $\omega_1 = 1$. Assume that D_A is a Z-matrix, $D_A v > 0$, for some positive vector v, and that L_A is non-positive.

In this case the two conditions to be satisfied are

$$(i) \qquad \{(A - C)v\}_i = (1 - \frac{1}{\omega_i})(D_A)_{i,i}v_i - \omega_{i-1}(LD_A^{-1}L^T)_{i,i}v_i \geq 0, \quad i \geq 2,$$

and

$$(ii) \qquad ((2 - \frac{1}{\mu_2})X - D_A)_{i,i}v_i = [(2 - \frac{1}{\mu_2})\omega_i^{-1} - 1](D_A)_{i,i}v_i \geq 0.$$

The latter holds with

$$\mu_2 = 1/(2 - \max_i \omega_i).$$

As has been shown in [9] for certain matrices A it is possible to choose the sequence ω_i, $\omega_i < 2$ such that the first condition, (i) holds. Otherwise, if it does not hold, one can use a method of perturbations of D_A as referred to above.

A sufficient condition for (i) to hold is

$$1 - \frac{1}{\omega_i} - \omega_{i-1}\rho([\overline{L}\,\overline{L}^T]_{i,i}) \geq 0, \quad i = 2, \ldots, m,$$

where $\rho(\cdot)$ denotes the spectral radius and $\overline{L} = D_A^{-\frac{1}{2}}LD_A^{-\frac{1}{2}}$.
Case (1): Assume first that

$$\rho([\overline{L}\,\overline{L}^T]_{i,i}) \leq \frac{1}{4}.$$

Then, letting

$$\omega_1 = 1, \ 1 - \frac{1}{\omega_i} - \omega_{i-1}/4 = 0,$$

that is, letting $\omega_i = \frac{2i}{i+1}$, $i = 1, 2, \ldots, m$, yields

$$\lambda_{\max}(C^{-1}A) \leq \mu_2 = 1/(2 - \omega_m) = \frac{m+1}{2}.$$

Case (2): Let $\rho_i = \rho([\overline{L}\,\overline{L}^T]_{i,i})$ and assume next that

$$\varepsilon \leq \rho_{i+1} \leq \rho_i \leq \rho_0 \leq \frac{1}{4}(1+\varepsilon)^2,$$

for some positive number ε. Let then

$$\omega_1 = 1, \ \omega_i^{-1} = 1 + \varepsilon - \rho_i\omega_{i-1}, \quad i = 2, 3, \ldots.$$

This choice corresponds to perturbing D_A by εD_A. Then it can be seen that

$$1 \leq \omega_i \leq \omega_{i+1} \leq \omega_0 = 2/[1+\varepsilon+\sqrt{(1+\varepsilon)^2 - 4\rho_0}], \quad i = 2, 3, \ldots.$$

Hence, in this case,

$$\mu_2 = \frac{1}{2} + \frac{1}{2}(\varepsilon + \sqrt{(1+\varepsilon)^2 - 4\rho_0})^{-1},$$

and Theorem 3.4 shows that

$$\lambda_{\max}(C^{-1}A) \leq \min(\mu_2, m+1).$$

However, with the above choice of the sequence ω_i, condition (i) is not satisfied but only

$$\{(\widetilde{A} - C)v\}_i \geq 0, \quad i \geq 2,$$

when $\widetilde{A} = A + \varepsilon D_A$.
We have then

$$\lambda_{\min}(C^{-1}A) \geq \lambda_{\min}(\widetilde{A}^{-1}A) = \frac{1}{1+\varepsilon\mu_0},$$

where $\mu_0 = \lambda_{\max}(A^{-1}D_A)$. Note that in practical applications, μ_0 is a large number. Hence ε must be small to balance μ_0. Now the above show that

$$\mathrm{cond}(C^{-1}A) \leq \min\{(1+\varepsilon\mu_0)(m+1), (1+\varepsilon\mu_0)[1 + (\varepsilon + \sqrt{(1+\varepsilon)^2 - 4\rho_0})^{-1}]\}.$$

Assume next that $\rho_0 = \frac{1}{4}$. Then it follows that

$$\mathrm{cond}(C^{-1}A) \leq \min\{\frac{m+1}{2}, \frac{1}{2}(1+\varepsilon\mu_0)[1 + (\varepsilon + \sqrt{(1+\varepsilon)^2 - 4\rho_0})^{-1}]\},$$

and we can choose the perturbation parameter ε to minimize the upper bound. It can be seen that as $\mu_0 \to \infty$ the (asymptotically) optimal value of ε minimizes $(1+\varepsilon\mu_0)\frac{1}{\sqrt{2\varepsilon}}$, that is, $\varepsilon = \mu_0^{-1}$.

The upper bound takes (asymptotically) the form

$$\mathrm{cond}(C^{-1}A) \leq \max\{\frac{m+1}{2}, 1 + \sqrt{\frac{\mu_0}{2}}\}.$$

Note that $2\mu_0$ is the condition number which holds if we use a Jacobi block diagonal preconditioning of A. For the standard five-point second order elliptic difference equation, one finds $\mu_0 = \frac{2}{(\pi h)^2}$, so

$$\mathrm{cond}(C^{-1}A) \leq \max\{\frac{m+1}{2}, 1 + \frac{1}{\pi h}\}.$$

As can be seen from the above, the condition $\rho[\overline{L}\,\overline{L}^T]_{ii}) \leq \frac{1}{4}$ implies the existence of a $\beta = 2 - \frac{1}{\mu_2}$ for some positive μ_2, such that $(\beta X - D_A)v \geq 0$. As it turns out, for elliptic difference equations, this condition is essentially satisfied only for constant coefficient model type problems. The slightly weaker condition $\rho_i \leq \frac{1}{2}(1+\varepsilon)^2$ for some small ε holds only for sufficiently smooth variable coefficients. However, by the application of Theorem 3.5 and 3.6 in [24] one can obtain some significant improvements for the more general incomplete factorization method. These results will be referred to later in this section.

Bounds of condition numbers of generalized SSOR preconditioned matrices
We consider now condition number bounds for the generalized SSOR preconditioned matrix. Here, the matrices X_i are not just a multiple of $(D_A)_{ii}$, as in (4.5), but are computed by recursion. Early presentations of this method for pointwise factorization methods can be found in [20] and [1]. Let then A be a blocktridiagonal matrix,

$$A = \text{blocktridiag}(A_{i,i-1}, A_{i,i}, A_{i,i+1}),\ \ i = 1, 2, \ldots, m,$$

which we assume in addition to be symmetric and positive definite. Let A be split as

$$A = D_A + L + L^T,$$

where D_A is the blockdiagonal part and L is the lower blocktriangular part of A. Consider the generalized SSOR preconditioned matrix

$$C = (D + L)D^{-1}(D + L^T),$$

where D is nonsingular and diagonal or blockdiagonal, $D = \text{diag}(D_1, \ldots, D_m)$, partitioned as D_A. In practice, D_i will be sparse matrices, such as bandmatrices.

We shall derive lower bounds of the condition number of $C^{-1}A$. In particular, we want to estimate the order of the condition number with respect to some problem parameter.

The previously mentioned method to compute the matrices D_i takes now the following form:

$$D_i = A_{i,i} - A_{i,i-1}X_{i-1}A_{i-1,i} - D_i',\ \ i = 1, 2, \ldots, m. \tag{4.6}$$

Here $X_0 = 0$ and $D_0' = 0$ and $X_{i-1},\ i \geq 2$ is a sparse approximation to D_{i-1}^{-1} such as a bandmatrix and D_i' is a diagonal matrix such that

$$D_i v = (A_{i,i} - A_{i,i-1}D_{i-1}^{-1}A_{i-1,i})v, \tag{4.7}$$

for some positive vector v. Frequently, $v = e$. (4.7) and (4.8) show that

$$D'_i v = A_{i,i-1}(X_{i-1} - D_{i-1}^{-1})A_{i-1,i}v.$$

Hence D'_i compensates for the error of the approximation X_{i-1} of D_{i-1}^{-1}, and in such a way that $Cv = Av$, i.e. C and A have the same action on the vector v.
(4.4) can be generalized to allow perturbations of A and takes then the form

$$D_i = (A_{ii} + \Delta_i) - A_{i,i-1}X_{i-1}A_{i-1,i} - D'_i,\ i = 1, 2, \ldots, m$$

where Δ_i is a diagonal matrix which contains small (nonnegative) perturbations of $\mathrm{diag}(A_{i,i})$. Such perturbed methods have been considered in [9], for instance. More generally, one can use a relaxed method [11], where D'_i is replaced with $\omega D'_i$ and ω, $\omega \leq 1$ is a relaxation parameter. Frequently one finds that the methods becomes more robust with respect to various problem parameters if $\omega = 1 - \delta$, and δ is a small number.

An important application of generalized SSOR methods is for second order elliptic difference equations where the problem parameter h is the meshwidth. Consider then the model equation on a rectangular domain with $(m_1 + 1) \times (m + 1)$ mesh points and $h = \frac{1}{m_1+1}$, where the matrices

$$A_{ii} = \mathrm{tridiag}(-a, 2(a + b), -a),\ A_{i,i-1} = A_{i-1,i} = b\ \mathrm{diag}(1, 1, \ldots, 1),\ i = 1, 2, \ldots, m.$$

Then the smallest eigenvalue of $A_{i,i}$ is $\lambda_1 = 2b + a(2\sin\frac{\pi h}{2})^2$. If $v = v_1$ is the corresponding eigenvector, then (4.5) shows that this becomes also an eigenvector of D_i and the corresponding eigenvalues of D_i become

$$\lambda_1^{(1)} = \lambda_1,\ \lambda_1^{(i)} = \lambda_1 - b^2\lambda_1^{(i-1)^{-1}},\ i = 2, 3, \ldots, m.$$

Therefore $\lambda_1^{(i)}$ converges monotonically to the lower bound,

$$\lambda_1^{(i)} \to b + 2a(\sin\frac{\pi h}{2})^2 + 2\sin\frac{\pi h}{2}\sqrt{ab + (a\sin\frac{\pi h}{2})^2}.$$

Hence

$$\lambda_1^{(m)} \geq b + \pi h\sqrt{ab} + 2a\left(\sin\frac{\pi h}{2}\right)^2.$$

On the other hand, for the choice $v = e$, it can be seen that the sequence $\{\lambda_1^{(i)}\}_{i\geq 1}$ is bounded below by $\{\widehat{\lambda}^{(i)}\}_{i\geq 1}$, where

$$\widehat{\lambda}_1^{(1)} = 2b,\ \widehat{\lambda}_1^{(i)} = 2b - b^2\widehat{\lambda}_1^{(i-1)^{-1}},\ i = 2, \ldots, m,$$

that is, $\widehat{\lambda}_1^{(i)} = b(1+\frac{1}{i})$, and $\widehat{\lambda}_1^{(m)} = b(1+\frac{1}{m})$.

Theorem 2.2 shows now that for any μ such that

$$\left(2-\frac{1}{\mu}\right)\lambda_1^{(m)} - \lambda_1 \geq 0,$$

we have $\lambda_{\max}(C^{-1}A) \leq \mu$. Hence we can let

$$\mu = \frac{\lambda_1^{(m)}}{2\lambda_1^{(m)} - \lambda_1} \simeq \begin{cases} \frac{1}{2}(1+\frac{\sqrt{b/a}}{\pi h}), & h \to 0, \quad \text{for the vector } v_1 \\ \frac{m+1}{2}, & h \to 0, \quad \text{for the vector } e \end{cases}$$

(we have $h = \frac{1}{n+1}$). The bound, which holds when the vector v_1 is used, is the asymptotically smallest we have derived for the model elliptic difference equation if $b \leq a$. It can be seen that the second bound holds also for modification with the eigenvector $v = v_1$. Hence, in this case

$$\mu \leq \min\{1+\frac{\sqrt{b/a}}{\pi h}, \frac{m+1}{2}\}.$$

Note that the bound $\frac{m+1}{2}$ holds independent on the order (n) of the matrix A and on the values of the coefficients a and b.
Using a technique considered in [23], this can be improved to

$$\rho(C^{-1}A) \lesssim \frac{1}{4}m.$$

In [24], similar results and also extensions to more general elliptic equations with variable coefficients, even permitting certain discontinuities in the coefficients, can be found. We state some of these results here without full details. Consider an elliptic equation in a two dimensional simply connected domain Ω,

$$-\frac{\partial}{\partial x}\left(a_1(x,y)\frac{\partial}{\partial x}u(x,y)\right) - \frac{\partial}{\partial y}\left(a_2(x,y)\frac{\partial}{\partial y}u(x,y)\right) = f(x,y) \text{ on } \Omega,$$

with boundary conditions

$$u(x,y) = g(x,y) \text{ on } \Gamma_0,$$

$$\frac{\partial u(x,y)}{\partial n} = h(x,y) \text{ on } \Gamma_1 = \partial\Omega - \Gamma_0$$

for the following problems:

$$\begin{aligned} &\text{Problem 1:} && \Gamma_0 = \partial\Omega, \\ &\text{Problem 2:} && \Gamma_1 = \{(x,y), 0 < x < a, y = 0\}, \\ &\text{Problem 3:} && \Gamma_0 = \{(x,y), 0 < y < b, x = a\}, \end{aligned}$$

where $\Omega = (0,a)\times(0,b)$, the coefficients $a_1(x,y)$ and $a_2(x,y)$ are positive.

Assume that there exists an integrable function $F(x)$ over $(0,a)$ such that

$$\frac{a_1(x_1,y)}{a_1(x_2,y)} \le \frac{F(x_1)}{F(x_2)} \tag{4.8}$$

For any $x_1 \le x_2$ in $(0,a)$ and $1/F(x)$ is also integrable over $(0,a)$, discretize the equation by a central difference scheme with a grid of meshsize h in x direction with the columnwise order and compute a block incomplete factorization by a modified block incomplete factorization. For $x \in (0,a)$, $E_x \subset (0,a)$ denotes the maximum subinterval including x such that $F(y) \ge F(z)$ for any and $y > z$ in E_x if such an interval exists for x. Note that we do not refer to a single point as an interval. It is straightforward to see that if $y \in E_x$ then $E_y = E_x$. Therefore, E_x is the maximum connected component in the sense that $F(y) \ge F(z)$ for any $y < z$ in E_x. Let U be the set of all these maximum connected components over $(0,a)$ and for an interval denote the left and the right end points of E by l_E and r_E respectively. It is shown in [24], utilizing the lower bound $\lambda_{\min}(C^{-1}A) = 1$, that the condition number of the preconditioned matrices for problems 1, 2 and 3 is bounded by

$$\mathrm{cond}(C^{-1}A) \le \gamma h^{-1} + o(h^{-1}),$$

$$\gamma = 2\left(\int_0^a \int_x^a \frac{F(x)}{F(y)} dy dx\right)^{\frac{1}{2}} + \sum_{E\in U} \int_{l_E}^a \frac{F(l_E+0) - F(\min(y, r_E - 0))}{F(y)} dy,$$

provided the sum $\sum_{E\in U} \int_{l_E}^a \frac{F(l_E+0)-F(\min(y,r_E-0))}{F(y)} dy$ exists. This result shows clearly how the coefficient $a_1(x,y)$ of the elliptic equation influences the upper bound of the condition number of the preconditioned matrix.

The condition (4.8) is indeed not strict. The assumption allows the coefficient $a_1(x,y)$ to have some jumps in both x and y direction. For example, assume that $a_1(x,y)$ is a piecewise differentiable function for x over Ω, i.e., there exist $0 = a_1 < a_2 < \ldots < a_{k+1} = a$ such that $a_1(x,y)$ is differentiable over (a_i, a_{i+1}) for $i = 1, 2, \ldots, k$. Let

$$g_i(x) = \max_{0 \le y \le h} \frac{\partial a_1(x,y)}{\partial x} / a_1(x,y),$$

for $x \in (a_i, a_{i+1})$, $i = 1, \ldots, k$. If $g_i(x)$ is integrable over (a_i, a_{i+1}) it is shown in [24] that there is a function $F(x)$ such that (4.6) holds.

Theorem 3.5 and 3.6 give more accurate upper bounds for the condition number of preconditioned matrices even for the cases where the previous results are applicable. For example, for the model problem, i.e. Problem 1 with constant coefficients, Theorem 3.5 yields an upper bound $0.3301m + 0.8301$.

For further discussion how to use Theorem 3.5 and Theorem 3.6 to obtain more accurate upper bounds for the condition number and every eigenvalue of the preconditioned matrix, see [24] for details.
A natural question to ask is if the order $O(h^{-1})$ of $\mathrm{cond}(C^{-1}A)$ is sharp or if it can be improved by some other choices of D_i. By deriving lower bounds of the condition number it can be shown that the order of the bound is sharp if D_i has a fixed sparsity pattern. This does not exclude that choices of D_i with other sparsity patterns and, in particular, a sparsity pattern which grows with i and h^{-1}, can reduce the order of the condition number. In the extreme case, if we let $D_i = A_{i,i} - A_{i,i-1}D_{i-1}^{-1}A_{i-1,i}$, then C becomes the exact factorization of A and $\mathrm{cond}(C^{-1}A) = 1$. However, here D_i, $i = 2, 3, \dots$ are full matrices.

The lower bounds of the condition number can be derived using Schur complements. For details, see [7].

As mentioned previously, bounds involving only m, the number of blocks are of particular interest when an elliptic second order difference equation is solved on an oblong rectangular domain with number of nodepoints $N_1 \times N_2$ where we assume that $N_1 \geq N_2$. If we number the points such that the order of the matrix blocks is N_1, i.e., there are $m = N_2$ blocks in the main diagonal, then applying Theorem 3.4 shows that

$$\mathrm{cond}(C^{-1}A) \leq N_2,$$

or $\sim \frac{1}{4}N_2$ for the model problem, both of which hence do not depend on N_1. It is therefore efficient to choose big blocks for such domains. In addition, this result is independent of the coefficients a, b in the differential equation. Such bounds are also of interest for domains decomposed in m strips. Using a generalized SSOR type preconditioner as above it shows that the upper bound of the condition number is m, or approximately $\frac{1}{4}m$ for the model type difference equation.

An application for difference matrices

As an application of the eigenvalue estimates in Theorem 2.4 we consider the finite difference approximation of the problem

$$-\delta u_{xx} - u_{yy} = f \text{ in } [0,1]^2$$

where $\delta > 0$, with Dirichlet boundary conditions, using a uniform mesh. Using a natural ordering, one finds

$$a_{i,i-m_1} = -1,\ a_{i,i-1} = -\delta,\ a_{i,i} = d,\ a_{i,i+1} = -\delta,\ a_{i,i+m_1} = -1,$$

where $d = 2(1+\delta)$, and the mesh width $h = 1/(m_1+1)$.

For simplicity we consider a pointwise factorization and let $X = \mathrm{diag}(x_1, x_2, \ldots)$ where we let x_i be defined by the recursion (method of relaxation, see [11])

$$x_i = d_i - \sum_{i<j} \ell_{i,j} x_j^{-1} \ell_{j,i} - \omega (Re)_i, \ i = 1, 2, \ldots$$

where $R = C - A$, or

$$x_i = 2(1+\delta) - \delta^2 x_{i-1}^{-1} - x_{i-m_1}^{-1} - \omega\delta(x_{i-m_1}^{-1} + x_{i-1}^{-1})$$

(apart from obvious corrections at points next to the boundary). Here $\omega < 1$ is a relaxation parameter and this will be chosen so that $2x - d > 0$, where $x = \min_i x_i$. We see readily that as $i \to \infty$ and $h \to 0$, x_i converges to a lower bound x, where

$$x = 2(1+\delta) - (1 + 2\omega\delta + \delta^2) x^{-1}$$

or,

$$x = 1 + \delta + \{2\delta(1-\omega)\}^{\frac{1}{2}}.$$

Similarly, it follows that the nonzero entries of R converge to

$$r_{i,i\pm(m_1-1)} = \delta x^{-1}, \ r_{i,i} = -2\omega\delta x^{-1}.$$

Hence

$$\lambda_{\max}(R) \leq 2\delta(1-\omega)x^{-1} \text{ and } \lambda_{\min}(R) \geq -2\delta(1+\omega)x^{-1}.$$

Note that

$$2x - d = 2\{2\delta(1-\omega)\}^{\frac{1}{2}} > 0.$$

Hence Theorem 2.4 shows that

$$\mu = \frac{x}{2x-d} = \frac{1}{2} + \frac{1+\delta}{2\{2\delta(1-\omega)\}^{\frac{1}{2}}}$$

is an upper bound of the eigenvalues of $C^{-1}A$. Furthermore, by Theorem 2.4

$$\lambda_i(C^{-1}A) \geq \underline{\lambda}_i = \frac{\lambda_i(A)}{\lambda_i(A) + \lambda_{\max}(C-A)}$$

holds (note that Theorem 2.4 with $\mu_1 = 1$ shows that it suffices that $\lambda_{\max}(C-A) = \lambda_{\max}(R) \geq 0$ for this to hold) and

$$\lambda_i(C^{-1}A) \leq \overline{\lambda}_i = \frac{4x\lambda_i(A)}{[2x - d + \lambda_i(A)]^2}, \tag{4.9}$$

where the latter holds for all $\lambda_i(A) \leq 2x - d$.

Hence, for the difference between these upper and lower bounds we find for all $\lambda_i(A) \leq 2x - d$,

$$\begin{aligned} \overline{\lambda}_i - \underline{\lambda}_i &\leq \left(\frac{4x}{[2x-d+\lambda_i(A)]^2} - \frac{1}{\lambda_i(A)+2\delta(1-\omega)x^{-1}}\right)\lambda_i(A) \\ &= \frac{[2d-\lambda_i(A)](\lambda_i(A))^2}{[\lambda_i(A)+2\{2\delta(1-\omega)\}^{\frac{1}{2}}]^2[\lambda_i(A)+2\delta(1-\omega)x^{-1}]} \end{aligned} \tag{4.10}$$

Hence, if for an eigenvalue $\lambda_i(A)$ it holds $\lambda_i(A) = O(h^\alpha),\ h \to 0$, for some $\alpha > 0$, then (4.9) and (4.10) show that

$$\lambda_i(C^{-1}A) = \frac{x}{2\delta(1-\omega)}\lambda_i(A) + O(\lambda_i(A)^2),\ h \to 0,$$

which shows that for any fixed value of $\omega,\ \omega < 1$ (independent of the problem parameter h), the eigenvalues λ_i of $C^{-1}A$ are very close to a factor, $x/[2\delta(1-\omega)]$ times the corresponding eigenvalues $\lambda_i(A)$, for all small eigenvalues, $\lambda_i = O(h^\alpha),\ h \to 0$.

This shows that the essentially unmodified incomplete factorization method changes the smallest eigenvalues mainly only by a constant factor. The consequence of this for the rate of convergence of the corresponding preconditioned conjugate gradient method has been discussed in [6], [7] and [8].

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References

[1] O. Axelsson, *A generalized SSOR method*, BIT 12 (1972), 443-467.

[2] O. Axelsson, *On preconditioning and convergence acceleration in sparse matrix problems*, CERN Technical Report 74-10, Data Handling Division, Geneva, 1974.

[3] O. Axelsson, *Solution of large systems of equations: iterative methods*, In Sparse Matrix Techniques, Lecture Notes in Mathematics # 572, pp. 1-51 (ed. V.A. Barker), Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[4] O. Axelsson, *A general incomplete block-matrix factorization method*, Linear Algebra Appl., 74 (1986), 179-190.

[5] O. Axelsson, *Preconditioning methods for block H-matrices*. In Computer Algorithms for Solving Linear Algebraic Equations (edited by E. Spedicato), NATO ASI Series, Vol. F77, Springer-Verlag, Berlin, Heidelberg 1991, pp. 169-184.

[6] O. Axelsson, *Bounds of eigenvalues of preconditioned matrices*, SIAM J. Matrix Anal. Appl. 13 (1992), pp. 847-862.

[7] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, New York, 1994.

[8] O. Axelsson, *Condition numbers for the study of the rate of convergence of the conjugate gradient method*, IMACS Conference of Iterative Solution Methods, Blagoevgrad, Bulgaria, June 1995, to appear.

[9] O. Axelsson and V.A. Barker, *Finite Element Solution of Boundary Value Problems*, Academic Press, Orlando, Florida, 1984.

[10] O. Axelsson, S. Brinkkemper and V.P. Il'in, *On some versions of incomplete block-matrix factorization iterative methods*, Linear Algebra and its Applications 58 (1984), 3-15.

[11] O. Axelsson, G. Lindskog, *On the eigenvalue distributions of a class preconditioning methods*, Numer. Math. 48 (1986), pp. 479-498.

[12] O. Axelsson and H. Lu, *On eigenvalue estimates for block incomplete factorization methods*, SIAM J. Matrix Analysis, 16 (1995), 1074-1085.

[13] O. Axelsson, H. Lu and B. Polman, *On the numerical radius of matrices and its application to iterative methods*, Linear and Multilinear Algebra 37 (1994), pp. 225-238.

[14] R. Beauwens, *Upper eigenvalue bounds for pencils of matrices*, Linear Algebra Appl.,62 (1984), pp. 87-104.

[15] R. Beauwens, *On Axelsson's perturbations*, Lin. Alg. Appl., 68 (1985), pp. 221-242.

[16] R. Beauwens, *Modified incomplete factorization strategies*, in Preconditioned Conjugate Gradient Methods, O. Axelsson and L. Kolotilina, eds., Lecture Notes in Mathematics No. 1457, Springer-Verlag, 1990, pp. 1-16.

[17] R. Beauwens, *Approximate factorizations with S/P consistenty ordered M-factors*, BIT, 29 (1989), pp. 658-681.

[18] R. Beauwens and M. Ben Bouzid, *Existence and conditioning properties of sparse approximate block factorizations*, SIAM J. Numer. Anal., 25 (1988), pp. 941-956.

[19] P. Concus, G. Golub and G. Meurant, *Block preconditioning for the conjugate gradient method*, SIAM J. Sci. Statist. Comput. 6 (1985), 220-252.

[20] T. Dupont, R. Kendall and H.H. Rachford, Jr., *An approximate factorization procedure for solving self-adjoint elliptic difference equations*, SIAM J. Numer. Anal. 5 (1968), 554-573.

[21] I. Gustafsson, *Stability and rate of convergence of modified incomplete Cholesky factorization methods*, Ph.d. thesis, Report 79.02R, Göteborg, Sweden, 1979.

[22] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[23] L. Kolotilina, *Eigenvalue Bounds for block preconditioned matrices*, December 1995, submitted.

[24] H. Lu and O. Axelsson, *Conditioning analysis of block incomplete factorizations and its application to elliptic equations* (submitted to SIAM J. Matrix Anal. Appl.).

[25] M.M. Magolu, *Modified block-approximate factorization strategies*, Numer. Math., 61 (1992), pp. 91-110.

[26] M.M. Magolu, *Sparse approximate block factorizations for solving symmetric positive (semi)definite linear systems*, PhD Thesis, Service de Métrologie Nucléaire, Université Libre de Bruxelles, Brussels, Belgium, 1992.

[27] M.M. Magolu, *Analytical bounds for block approximate factorization methods*, Lin. Alg. Appl. 179 (1993), pp. 33-57.

[28] M.M. Magolu, *Ordering strategies for modified block incomplete factorizations*, SIAM J. Sci. Comput., 16 (1995), pp. 378-399.

[29] M.M. Magolu and Y. Notay, *On the conditioning analysis of block approximate factorization methods*, Linear Algebra Appl., 154-156 (1991), pp. 583-599.

[30] M.M. Magolu and B. Polman, *Theoretical comparison of pointwise, linewise and planewise preconditioners for solving three-dimensional problems*, Scientific Report, Department of Mathematics, University of Nijmegen, The Netherlands, 1995. In press.

[31] J.A. Meijerink and H. van der Vorst, *An iterative method for linear systems of which the coefficient matrix is a symmetric M-matrix*, Math. Comp. 31 (1977), 148-162.

[32] Y. Notay, *Conditioning analysis of modified block incomplete factorizations*, Linear Algebra Appl., 154-156 (1991), pp. 711-722.

[33] Y. Notay, *Upper eigenvalue bounds and related modified incomplete factorization strategies*, in Iterative Methods in Linear Algebra, R. Beauwens and P. de Groen, eds., North-Holland, 1992, pp. 551-562.

[34] B.N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall, Englewood Cliffs, N.J. 1980.

[35] D.M. Young, *Iterative Solution of Large Systems*, Academic Press, New York, 1971.