Algebraic Geometry

Some background information for students

The aim of this text is to give you an idea of what the field of Algebraic Geometry is about, and how it connects to the master courses that are offered. We include some examples of nice things that have been done, or that are being studied in Algebraic Geometry, just to give a flavour of the field. In between we will try to give some indications of connections to other areas in mathematics.

Fermat's Last Theorem.

As you will probably know, in 1995 Andrew Wiles (with help of Richard Taylor) proved Fermat's Last Theorem, which had been an open problem for about 360 years:

> **Theorem.** *If n is an integer with $n \geq 3$, there are no positive integers x, y and z such that $x^n + y^n = z^n$.*

Wiles's proof is a stunning tour de force that contains many deep new ideas.

Sir Andrew Wiles (born 1953); Abel Prize 2016

The work of Wiles lies at the side of Algebraic Geometry that connects to Number Theory; this part is often referred to as *Arithmetic Algebraic Geometry.* The idea is that an equation such as $x^n + y^n = z^n$ defines a geometric object, which we call an algebraic variety, but that we can study solutions of this equation in any commutative ring, and this links it to number theory.

The proof of Wiles builds on the study of elliptic curves, Galois representations, modular forms, ... It is definitely not an end point; rather it is the beginning of a whole new era.

The name Algebraic Geometry comes from the fact that in this part of Mathematics one tries to study geometric objects (mainly) through algebraic techniques. This combination of algebra and geometry is extremely fruitful, and as a result the field of Algebraic Geometry has become big and very diverse. There are many connections to other areas/techniques in mathematics, such as Number Theory, Differential Geometry, Topology, Category Theory, Cryptography, Mathematical Physics, and so on. All in all, it's better to think of Algebraic Geometry as indicating a sub-area of mathematics as a whole, rather than a very precisely defined subfield.

Cohomology theories.

A sad fact of life is that we humans are unable to see objects in high-dimensional spaces. For a geometer this is a serious handicap. To compensate for this, mathematicians have invented very powerful techniques that still allow us to "see" a lot of important information about varieties of arbitrarily high dimension. Among these tools are what are called the *cohomology groups* of a variety. These are invariants that can be difficult to compute, but that give us a lot of important information about varieties that we would otherwise be unable to study.

The development of such advanced techniques has been greatly influenced by Alexander Grothendieck, who by many is regarded as one of the greatest mathematicians of the 20th century and who has completely changed the way we think about mathematics.

Left: Alexander Grothendieck (1928–2014), Fields Medal 1966
Right: Pierre Deligne (born 1944), Fields Medal 1978, Abel Prize 2013

There are many other mathematicians who have contributed to this but some of the deepest contributions were made by the Belgian mathematician Pierre Deligne who, among many other things, in 1974 proved the so-called *Weil Conjectures*. This result is of great theoretical importance (and beauty!) but also has important applications in Cryptography.

Nowadays, even more sophisticated tools are in constant development; much of this is directly connected to Algebraic Topology and to Category Theory.

There are also many things that we do not yet know. For instance, according to the famous *Hodge Conjecture*, cohomology groups are able to detect geometrical patterns that go far beyond what we humans can really see. If you solve that conjecture you will get 1 million dollars[a], plus eternal fame!

[a] see `https://en.wikipedia.org/wiki/Millennium_Prize_Problems`

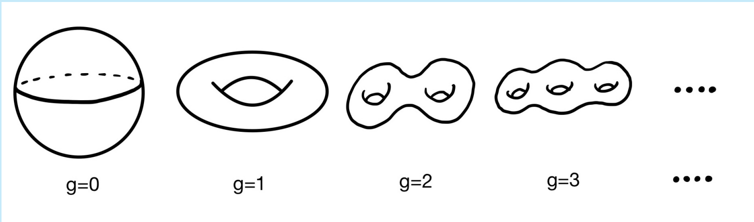
The basic idea behind Algebraic Geometry is that we study varieties that are defined using polynomial equations. This leads to connections to Commutative Algebra and Number Theory. On the other hand, there is a lot of nice geometry going on, which connects to Topology and Differential Geometry. It's the interaction between such techniques that makes the field so attractive—it is not uncommon to study problems in Number Theory using techniques from Differential Geometry, or you may be able to understand a theorem from abstract Algebra because you have a clear geometric intuition behind it. Such is the magic of Algebraic Geometry!

Here is a simple example. Consider the equation $y^2 = x^3 + 1$. Draw the solutions of this equation for $x, y \in \mathbb{R}$ and you will find a curve in the plane. Or you can study all solutions with x and y complex numbers—can you figure out what that solution set looks like as a topological space? (Answer: it is homeomorphic to a doughnut with one point deleted.) But you can also study all solutions with x and y in some finite field, such as the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Because $\mathbb{F}_p$ is finite, you can count how many solutions there are. Can you do it for $p = 5$? (Answer: 5 points.) It turns out that there is a close connection between the number of solutions over finite fields and the topology of the space of complex solutions, and in examples like this we can even determine the topological space of complex solutions by counting the numbers of points over finite fields. Isn't that cool?

Moduli Spaces.

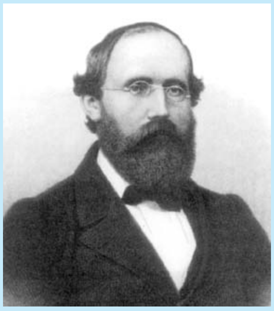
Moduli spaces are geometric solutions to classification problems. Suppose you want to understand all possible varieties of some specific type, for instance all elliptic curves. It is then often possible to construct a *moduli space*. Such a moduli space is itself a variety, and it has the property that each point of it corresponds to an isomorphism class of the objects that you are trying to understand. So the moduli space is some kind of *map* (as in an atlas), that completely describes how the objects that you are interested in all fit together.

Riemann introduced the notion of moduli when he began the classification of "Riemann surfaces" (surfaces with a complex structure). Topologically, these objects have the following shape:



The number of "holes" that you see is an important invariant, called the *genus* of the Riemann surface. Algebraically, Riemann surfaces can be viewed as complex algebraic curves (which are 1-dimensional algebraic varieties).

If you fix this genus g, there is a (connected) moduli space, called $\mathcal{M}_g$, that parametrizes all possible Riemann surfaces of genus g. Riemann was already able to calculate that $\mathcal{M}_g$ is a variety of dimension $3g - 3$ (if $g \geq 2$) but it was David Mumford who introduced many important ideas to construct and study these (and many other) moduli spaces. They belong to the most important spaces that we study, and apart from being very interesting in their own right, they have important connections to Mathematical Physics and Representation Theory, for instance.

Left: Bernhard Riemann (1826–1866)
Right: David Mumford (born 1937); Fields Medal 1974

Algebraic Geometry and the study programme at Radboud University

Which courses are most relevant for you?

The fact that Algebraic Geometry has so many connections to other fields also means that there is a lot to learn! For this reason, Algebraic Geometry is sometimes felt to be quite challenging, and even among the experts there are few people who know almost everything. To build up knowledge just takes time, but the good news is that already with a basic training there are many interesting things that you can work on, and that the interplay between different techniques makes things very lively and exciting! You will (finally?) see how many things that you've learned in the Bachelor programme are connected to each other.

How does the study programme in Mathematics prepare for the study of Algebraic Geometry? In the bachelor programme, there are a couple of courses that are essential if you want to continue in the direction of Algebraic Geometry. These include:

- Linear Algebra
- Algebra courses such as Group Theory and Rings and Fields
- Topology
- Manifolds

Such courses are really at the very basis of much of Pure Mathematics. If you have not taken some of these courses, you will miss some essential background knowledge in studying Algebraic Geometry.

There are also several elective courses that are highly relevant for Algebraic Geometry. Examples of these are:

- Introductory courses to Number Theory
- Introductory courses to Representation Theory and Lie Theory
- Complex Analysis

There are also several courses in the bachelor programme that are perhaps not strictly necessary for Algebraic Geometry but that still give very valuable extra technical background. Examples of these include Functional Analysis and Measure Theory.

In the master programme, there are several courses that are important for Algebraic Geometry. Some courses that are obvious choices, such as the Mastermath courses on Algebraic Geometry and the course on Riemann Surfaces. Other courses that directly connect to Algebraic Geometry are:

- Category Theory and Homological Algebra
- Commutative Algebra
- Galois Theory
- Courses on Number Theory (including topics such as Class Field Theory)
- Algebraic Topology
- Differential Geometry

At this stage, your personal taste will probably start playing a role. E.g., if your interests lie in an arithmetic direction (interactions with Number Theory), it's natural to take additional courses in Number Theory. Conversely, if you wish to focus more on the geometrical aspects, it seems natural to take some Differential Geometry and you may take fewer courses in Number Theory. Sometimes choosing is quite hard, so we encourage you to get in touch with staff members—it may help you a lot to have a chat with one of us!

Algebraic Geometers at Radboud University

Get in touch!

At Radboud University, several people work in Algebraic Geometry:

Lie Fu. I joined the Radboud University as an assistant professor in 2021. I work at the crossroad of Algebraic Geometry, Arithmetic Geometry and Complex Analytic Geometry. I study algebraic varieties from various angles, like the algebraic cycles (linear combination of subvarieties) they contain, the vector bundles they carry, deformations of the algebraic/complex structures, their topology etc. Those approaches lead to the theory of motives, derived categories, and Hodge theory. My favourite geometric objects are Calabi–Yau manifolds, hyper-Kähler manifolds, and moduli spaces of vector bundles. You are welcome to get in touch with me (lie.fu@math.ru.nl).

Victoria Hoskins. I joined Radboud University in 2020 as an assistant professor. I especially enjoy the interaction of algebraic geometry with different fields of mathematics. My background is in classical algebraic geometry over the complex numbers, which has many beautiful and surprising links with differential geometry and mathematical physics. Often these neighbouring fields are a powerful source of inspiration for algebraic geometers. My research focuses on moduli problems and the study of group actions in algebraic geometry, both of which have strong connections with representation theory. I have supervised students on topics related to elliptic curves and Riemann surfaces, as well as moduli spaces and the study of their cohomological invariants. If you'd like to have a chat about the different courses on offer or possible thesis topics, you are welcome to come by my office (HG03.717) or send me an email.

Ben Moonen. I have been full professor at Radboud University since 2013. My interests are broad; they include classical algebraic geometry, geometry over fields of characteristic p, arithmetic algebraic geometry, moduli spaces, cohomology theories, and the theory of motives. In the past I have supervised student projects on topics such as category theory, elliptic curves and cryptography, problems from the theory of algebraic surfaces, and so on. I'd be very happy to have a chat about which courses you could choose, or about possible projects. Feel free to drop in at HG03.720, or if I'm not around get in touch by email. (See below.)

In addition, several staff members work in areas that are related to Algebraic Geometry:

Wieb Bosma: Computer Algebra, Algebraic Number Theory

Magdalena Kędziorek: Algebraic Topology

Ioan Mărcuţ: Poisson Geometry

Steffen Sagave: Algebraic Topology

Maarten Solleveld: Representation Theory and Lie Theory

Wadim Zudilin: Number Theory, Special Functions

Contact anyone of us if you'd like to discuss your options or if we can help you with further information. If you don't know who to approach, please get in touch with Lie Fu (lie.fu@math.ru.nl), Victoria Hoskins (v.hoskins@math.ru.nl) or Ben Moonen (b.moonen@math.ru.nl). We will be happy to give you advice!