ALGEBRAIC GEOMETRY—Exercises for 4 March

Exercise 1. Do HAG, Chapter II, Exercise 1.4. Here is the text of this exercise:

(a) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injectve for each U. Show that the induced map $\varphi^+\colon \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

(b) Use part (a) to show that if $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\mathrm{Im}(\varphi)$ can be naturally identified with a subsheaf of $\mathscr{G}$.

Exercise 2. Given a topological space X, we denote by $\mathbb{Z}_X^{\mathrm{pre}}$ the constant presheaf given by

$$\begin{cases} \mathbb{Z}_X^{\mathrm{pre}}(U) = \mathbb{Z} & \text{for all open } U; \\ \rho_{U,V} = \mathrm{id}_{\mathbb{Z}} & \text{for all open } V \subset U. \end{cases}$$

By $\mathbb{Z}_X$ we denote the associated sheaf, which is the constant sheaf on X associated with the group $\mathbb{Z}$. Finally, $\theta_X\colon \mathbb{Z}_X^{\mathrm{pre}} \to \mathbb{Z}_X$ is the universal map as in the definition of an associated sheaf.

(i) Let X be a topological space and $Y \subset X$ a non-empty subset. We give Y the induced topology (subspace topology) and write $i\colon Y \hookrightarrow X$ for the inclusion map. Note that the presheaf $i_*(\mathbb{Z}_Y^{\mathrm{pre}})$ is the same as the presheaf $\mathbb{Z}_X^{\mathrm{pre}}$. Show that there is a unique homomorphism of sheaves $\varphi\colon \mathbb{Z}_X \to i_*(\mathbb{Z}_Y)$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}_X^{\mathrm{pre}} & \xrightarrow{\mathrm{id}} & i_*(\mathbb{Z}_Y^{\mathrm{pre}}) \\ {\scriptstyle \theta_X}\big\downarrow & & \big\downarrow{\scriptstyle i_*(\theta_Y)} \\ \mathbb{Z}_X & \xrightarrow{\varphi} & i_*(\mathbb{Z}_Y) \end{array}$$

is commutative.

In the rest of the exercise we consider the 4-point topological space $X = \{a, b, c, d\}$ in which the open sets are the following ones:

$$\emptyset\,, \quad \{a\}\,, \quad \{b\}\,, \quad Y = \{a, b\}\,, \quad V = \{a, b, c\}\,, \quad W = \{a, b, d\}\,, \quad X\,.$$

(You don't need to prove that this is indeed a topology.) We again write $i\colon Y = \{a, b\} \hookrightarrow X$ for the inclusion map. Note that the induced topology on Y is the discrete topology.

A presheaf $\mathscr{F}$ on X is given by a commutative diagram of groups

$$\begin{array}{ccccc} \mathscr{F}(X) & \xrightarrow{\rho_{X,V}} & \mathscr{F}(V) & & \\ {\scriptstyle \rho_{X,W}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{V,Y}} & & \\ \mathscr{F}(W) & \xrightarrow[\rho_{W,Y}]{} & \mathscr{F}(Y) & \xrightarrow{\rho_{Y,\{a\}}} & \mathscr{F}(\{a\}) \\ & & {\scriptstyle \rho_{Y,\{b\}}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{\{a\},\emptyset}} \\ & & \mathscr{F}(\{b\}) & \xrightarrow{\rho_{\{b\},\emptyset}} & \mathscr{F}(\emptyset) \end{array}$$

(Not all restriction maps are given in this diagram but the remaining maps are fully determined by the information in the diagram. For instance, $\rho_{X,Y} = \rho_{V,Y} \circ \rho_{X,V} = \rho_{W,Y} \circ \rho_{X,W}$.)

(ii) Give the sheaves $\mathbb{Z}_X$ and $i_*\mathbb{Z}_Y$ in the form of such a diagram.

(iii) With $\varphi\colon \mathbb{Z}_X \to i_*(\mathbb{Z}_Y)$ as in (i), applied to the situation considered here, give the diagram for the quotient sheaf $\mathrm{Coker}(\varphi)$, which is defined as the sheaf associated with the presheaf $U \mapsto i_*(\mathbb{Z}_Y)\big(U\big)/\mathrm{Im}\big(\varphi(U)\big)$. Is the natural homomorphism $i_*(\mathbb{Z}_Y) \to \mathrm{Coker}(\varphi)$ a surjective map of sheaves?