INTRODUCTION TO ALGEBRAIC GEOMETRY

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[Versie van 26 februari 2014]

PREFACE

These are course notes based on a Mastermath course *Algebraic Geometry* taught in the Spring of 2013.

As almost any author of an introductory text on Algebraic Geometry remarks, there is some tension between, on the one hand, the need to develop the technical machinery necessary for the study of more advanced results, and, on the other hand, the desire to show, in a concrete and down-to-earth manner, the beauty of the subject. I have tried to steer a middle course. The course is aimed at beginning Dutch master students, many of whom will hopefully take an advanced course in Algebraic Geometry at some later stage in their education. Thus, I do try to develop the theory with some rigour; but at the same time there is a clear focus on the geometry and on concrete examples and applications. I'm willing to sacrifice for this the proofs of some harder results, notably in commutative algebra.

For this course I assume a sound knowledge of basic Algebra, including linear algebra and the theory of groups, rings and fields. I also assume the reader knows, or is willing to study from another text, what is a module over a ring. Further we shall freely use the basic concepts from abstract (point-set) Topology. For some of the differential-geometric notions we discuss, it may be helpful if the reader has already seen their C^∞-analogues in a course on manifolds. Finally, I occasionally use some jargon from category theory, but as we won't need more than some very basic notions, I assume the reader will be able to catch up with this as we go. Where possible, I try to explain in elementary terms what is meant.

The chapters on Algebraic Geometry are interluded with sections on Commutative Algebra. These are somewhat different in style. Their sole purpose is to introduce the notions that are relevant for us and to state some important results. Hardly any proofs are given in these sections. For a deeper study of this material I can recommend the classic *Introduction to commutative algebra* by Atiyah and MacDonald.

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January 2013

CONTENTS

CHAPTER 1

Affine varieties

In Algebraic Geometry we study geometric objects—varieties—that are defined by polynomial equations. One fascinating aspect of this is that we can do geometry over fields of arbitrary characteristic. Though we can gain a lot of geometric intuition from looking at examples over familiar fields like $\mathbb{C}$, the general theory heavily relies on concepts and results from commutative algebra.

In this chapter we introduce affine varieties. These form the building blocks for the theory. We shall later define more general varieties by gluing affine pieces.

§1. *The Zariski topology on $\mathbb{A}^n$.*

1.1. In these course notes, k denotes an algebraically closed field. This field is the ground field over which we work. When we discuss polynomials with coefficients in k we shall usually call the variables x_1, x_2, ...; thus, $k[x_1, \ldots, x_n]$ denotes the k-algebra of polynomials in n variables with coefficients in k. If we have few variables, it is often more convenient to use letters x, y, z, ... without indices.

For $n \geqslant 0$ we define $\mathbb{A}^n = k^n$. We want to view this space as an algebraic variety, called *affine n-space*. The reason that we introduce a special notation for this variety is that we want to avoid confusion with k^n as a k-vector space. In particular, the origin $O = (0, \ldots, 0)$ does not play a special role.

A polynomial $f \in k[x_1, \ldots, x_n]$ defines a function $\hat{f}\colon \mathbb{A}^n \to k$, given by $(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$. The k-valued functions on $\mathbb{A}^n$ form a k-algebra via pointwise addition and multiplication. The map $f \mapsto \hat{f}$ is a k-algebra homomorphism

$$k[x_1, \ldots, x_n] \longrightarrow \bigl\{\text{functions } \mathbb{A}^n \to k\bigr\}\,.$$

Because k is an infinite field, this homomorphism is injective; see Exercise 1.1. This means that we can unambiguously identify a polynomial $f \in k[x_1, \ldots, x_n]$ with the function $\mathbb{A}^n \to k$ it defines. In what follows we shall therefore simply call this function again f instead of $\hat{f}$; this will not lead to confusion.

1.2. Definition. If $S \subset k[x_1, \ldots, x_n]$ is a subset, we define its *zero set* $\mathscr{Z}(S) \subset \mathbb{A}^n$ by

$$\mathscr{Z}(S) = \bigl\{P = (a_1, \ldots, a_n) \in \mathbb{A}^n \bigm| f(P) = 0 \quad \text{for all } f \in S\bigr\}\,.$$

1.3. Proposition.

(i) *If S is a subset of $k[x_1, \ldots, x_n]$ and $I \subset k[x_1, \ldots, x_n]$ is the ideal generated by S then $\mathscr{Z}(S) = \mathscr{Z}(I)$.*

(ii) *We have $\mathscr{Z}\bigl((0)\bigr) = \mathbb{A}^n$ and $\mathscr{Z}\bigl((1)\bigr) = \emptyset$.*

(iii) *If $\{S_\alpha\}_{\alpha \in A}$ is a collection of subsets of $k[x_1, \ldots, x_n]$ then $\mathscr{Z}\bigl(\cup_{\alpha \in A} S_\alpha\bigr) = \cap_{\alpha \in A} \mathscr{Z}(S_\alpha)$.*

(iv) *If I and J are ideals of $k[x_1, \ldots, x_n]$ then $\mathscr{Z}(IJ) = \mathscr{Z}(I) \cup \mathscr{Z}(J)$.*

The proof of this proposition presents no difficulties and is therefore left as an exercise.

By (i), if we start with an arbitrary subset $S \subset k[x_1, \ldots, x_n]$ and we want to study its zero locus, we may replace S by the ideal I it generates. On the other hand, the ring $k[x_1, \ldots, x_n]$ is noetherian, which means that any ideal of it is finitely generated. Hence we may find a finite number of polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ with $I = (f_1, \ldots, f_r)$. Again by (i) the zero set of S is then the set of points $P \in \mathbb{A}^n$ for which

$$f_1(P) = \cdots = f_r(P) = 0\,.$$

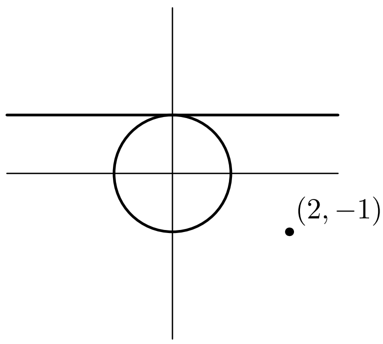
The conclusion, therefore, is that the subsets of the form $\mathscr{Z}(S) \subset \mathbb{A}^n$ are the sets given by a finite number of polynomial equations.

1.4. Definition. The *Zariski topology* on $\mathbb{A}^n$ is the topology for which the closed sets are the subsets of the form $\mathscr{Z}(S)$ for some $S \subset k[x_1, \ldots, x_n]$.

This definition is justified, as it follows from Proposition 1.3 that these subsets are indeed the closed sets of a topology.

In what follows, whenever we talk about open or closed subsets, this will refer to the Zariski topology, unless specified otherwise. In cases of possible confusion we will say "Zariski open" or "Zariski closed". Further, we give subsets of $\mathbb{A}^n$ the induced topology.

1.5. Examples. (i) The Zariski topology on $\mathbb{A}^0$ (a single point) is the only possible topology. The Zariski topology on the affine line $\mathbb{A}^1$ is the co-finite topology. An example of a closed subset in $\mathbb{A}^2$ is the union of the circle $\mathscr{Z}(x^2 + y^2 - 1)$ with the line $y = 1$ and the point $(2, -1)$.



This set is the zero locus of $S = \big\{(x^2 + y^2 - 1)(y - 1)(x - 2), (x^2 + y^2 - 1)(y - 1)(y + 1)\big\}$.

(ii) Every point $P = (a_1, \ldots, a_n)$ in $\mathbb{A}^n$ is closed. If we denote by $\mathfrak{m}_P$ the ideal $(x_1 - a_1, \ldots, x_n - a_n) \subset k[x_1, \ldots, x_n]$ then $\mathfrak{m}_P$ is a maximal ideal and $\{P\} = \mathscr{Z}(\mathfrak{m}_P)$. As we shall see in Corollary 1.11, the map $P \mapsto \mathfrak{m}_P$ gives a bijection between $\mathbb{A}^n$ and the set of maximal ideals of $k[x_1, \ldots, x_n]$.

For the purpose of Algebraic Geometry, the Zariski topology is very natural and useful. It is, however, a topology that is very different from metric topologies like the Euclidean topology on $\mathbb{C}^n$. E.g., $\mathbb{A}^n$ with its Zariski topology is not a Hausdorff space if $n > 0$.

1.6. Definition. If $Y \subset \mathbb{A}^n$ is a subset, we define its ideal $\mathscr{I}(Y) \subset k[x_1, \ldots, x_n]$ by

$$\mathscr{I}(Y) = \{f \in k[x_1, \ldots, x_n] \mid f(P) = 0 \quad \text{for all } P \in Y\}.$$

It is immediate that this is indeed an ideal of $k[x_1, \ldots, x_n]$.

With this definition we have a diagram

$$\text{ideals of } k[x_1, \ldots, x_n] \quad \underset{\mathscr{I}}{\overset{\mathscr{Z}}{\rightleftarrows}} \quad \text{subsets of } \mathbb{A}^n$$

and we may wonder if the maps "$\mathscr{Z}$" and "$\mathscr{I}$" are mutually inverse. It is clear, however, that this is too much to expect. For instance, the ideals (x) and (x^2) of $k[x]$ are distinct but they have the same zero locus; hence we cannot expect to be able to recover an ideal from its zero locus. Similarly, if we start with a subset $Y \subset \mathbb{A}^n$ that is not closed, we cannot hope to recover it from its ideal. As we shall see, we do get a bijective correspondence once we restrict to radical ideals and closed subsets; see Corollary 1.12 below.

1.7. Proposition.
(i) *The map $\mathscr{Z}$ is inclusion-reversing: if $I_1 \subset I_2 \subset k[x_1, \ldots, x_n]$ then $\mathscr{Z}(I_1) \supset \mathscr{Z}(I_2)$.*
(ii) *The map $\mathscr{I}$ is inclusion-reversing: if $Y_1 \subset Y_2$ are subsets of $\mathbb{A}^n$ then $\mathscr{I}(Y_1) \supset \mathscr{I}(Y_2)$.*
(iii) *Let $Y \subset \mathbb{A}^n$ be a subset. Then $\mathscr{Z}(\mathscr{I}(Y)) = \overline{Y}$.*

Proof. Parts (i) and (ii) are clear from the definitions. In (iii) it is clear that $Y \subseteq \mathscr{Z}(\mathscr{I}(Y))$ and since $\mathscr{Z}(\mathscr{I}(Y))$ is closed, this implies that $\overline{Y} \subseteq \mathscr{Z}(\mathscr{I}(Y))$. On the other hand, $\overline{Y} = \mathscr{Z}(J)$ for some ideal $J \subset k[x_1, \ldots, x_n]$. By (ii), $Y \subset \overline{Y}$ implies that $\mathscr{I}(Y) \supset \mathscr{I}(\overline{Y}) = \mathscr{I}(\mathscr{Z}(J))$. But it is clear that $J \subset \mathscr{I}(\mathscr{Z}(J))$; so we find that $J \subset \mathscr{I}(Y)$ and by (i) this gives $\overline{Y} \supset \mathscr{Z}(\mathscr{I}(Y))$. Hence, $\mathscr{Z}(\mathscr{I}(Y)) = \overline{Y}$. □

1.8. Definition. Let I be an ideal of a ring R. Then the *radical of I* is the ideal

$$\sqrt{I} := \{r \in R \mid r^m \in I \quad \text{for some } m > 0\}.$$

As Exercise 1.3 asks you to verify, this is indeed again an ideal of R.

An ideal $I \subset R$ is called a *radical ideal* if $I = \sqrt{I}$.

1.9. Example. For any $Y \subset \mathbb{A}^n$ the associated ideal $\mathscr{I}(Y) \subset k[x_1, \ldots, x_n]$ is a radical ideal.

1.10. Hilbert's Nullstellensatz. *If $I \subset k[x_1, \ldots, x_n]$ is an ideal, $\mathscr{I}(\mathscr{Z}(I)) = \sqrt{I}$.*

We shall not prove this theorem. There are many good proofs available in the literature. See for instance MRB, Chap. I, §§1–2. A very slick proof can be found in a short note by Daniel Allcock; see his webpage at `http://www.ma.utexas.edu/users/allcock/` .

Hilbert's Nullstellensatz (abbreviated to HNS) has several consequences that are important for us. The following Corollary is usually called the "weak Nullstellensatz". It should be noted that one usually proves the HNS by first proving the weak version.

1.11. Corollary. *The map that sends a point $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$ to the maximal ideal $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)$ of $k[x_1, \ldots, x_n]$ gives a bijection*

$$\mathbb{A}^n \xrightarrow{\sim} \bigl\{\text{maximal ideals of } k[x_1, \ldots, x_n]\bigr\} .$$

Note that $\{P\} = \mathscr{Z}(\mathfrak{m}_P)$, as discussed in Example (ii) of 1.5; this gives the inverse map.

Proof. If $\mathfrak{m}$ is a maximal ideal then by the HNS we have $\mathfrak{m} = \mathscr{I}(\mathscr{Z}(\mathfrak{m}))$. In particular, $\mathscr{Z}(\mathfrak{m})$ is non-empty. If $P = (a_1, \ldots, a_n) \in \mathscr{Z}(\mathfrak{m})$ then $\mathfrak{m} \subset \mathscr{I}\bigl(\{P\}\bigr)$. But $\{P\} = \mathscr{Z}(\mathfrak{m}_P)$; so the HNS gives $\mathscr{I}\bigl(\{P\}\bigr) = \mathfrak{m}_P$. The maximality of $\mathfrak{m}$ together with the inclusion $\mathfrak{m} \subset \mathfrak{m}_P$ then implies that $\mathfrak{m} = \mathfrak{m}_P$. □

1.12. Corollary. *The map $Z \mapsto \mathscr{I}(Z)$ defines a bijection*

$$\bigl\{\text{closed subsets of } \mathbb{A}^n\bigr\} \xrightarrow{\sim} \bigl\{\text{radical ideals of } k[x_1, \ldots, x_n]\bigr\}$$

with inverse given by $I \mapsto \mathscr{Z}(I)$.

§3. *Decomposition into components*

In the example of a closed subset of $\mathbb{A}^2$ given in 1.5(i), it is clear that this subset is the union of three parts, or, as we shall say, three irreducible components. We shall now make this precise.

1.13. Definition. A topological space X is *reducible* if $X = \emptyset$ or if X is the union of two proper closed subsets, i.e., if there are proper closed subsets Y, $Z \subsetneq X$ such that $X = Y \cup Z$. The space X is *irreducible* if it is not reducible.

A subset of a topological space is called reducible if it is reducible as a topological space with the induced topology. By convention, the empty space is reducible.

1.14. Example. The affine space $\mathbb{A}^n$ is irreducible. Indeed, suppose it were reducible; say $\mathbb{A}^n = Y \cup Z$. By Corollary 1.12 we have $Y = \mathscr{Z}(I)$ and $Z = \mathscr{Z}(J)$ for non-zero radical ideals I and J. Then $\mathbb{A}^n = Y \cup Z$ gives that $\mathscr{Z}(IJ) = \mathbb{A}^n$, so $\sqrt{IJ} = (0)$. But if $0 \neq f \in I$ and $0 \neq g \in J$ then $0 \neq fg \in IJ \subset \sqrt{IJ}$ because $k[x_1, \ldots, x_n]$ is a domain; this gives a contradiction.

By contrast, $\mathbb{C}^n$ with its Euclidean topology is reducible, as is demonstrated for instance by writing

$$\mathbb{C}^n = \bigl\{(z_1, \ldots, z_n) \in \mathbb{C}^n \bigm| \mathrm{Re}(z_1) \geqslant 0\bigr\} \cup \bigl\{(z_1, \ldots, z_n) \in \mathbb{C}^n \bigm| \mathrm{Re}(z_1) \leqslant 0\bigr\} .$$

We can recognize if a closed subset of $\mathbb{A}^n$ is irreducible in terms of the corresponding ideal:

1.15. Proposition. *A closed subset $Y \subset \mathbb{A}^n$ is irreducible if and only if the corresponding ideal $\mathscr{I}(Y) \subset k[x_1, \ldots, x_n]$ is a prime ideal.*

Proof. Suppose Y is reducible, say $Y = Y' \cup Y''$ with Y' and Y'' properly contained in Y. On ideals we then have $\mathscr{I}(Y') \cdot \mathscr{I}(Y'') \subset \mathscr{I}(Y)$, whereas $\mathscr{I}(Y)$ is properly contained in both $\mathscr{I}(Y')$ and $\mathscr{I}(Y'')$. But then $\mathscr{I}(Y)$ is not a prime ideal. Next suppose Y is irreducible and let $f, g \in k[x_1, \ldots, x_n]$ be elements with $fg \in \mathscr{I}(Y)$. Then $Y \subset \mathscr{Z}(f) \cup \mathscr{Z}(g)$, and because Y is irreducible this implies that either $Y \subset \mathscr{Z}(f)$ or $Y \subset \mathscr{Z}(g)$. Hence, either f or g is an element of $\mathscr{I}(Y)$, proving that $\mathscr{I}(Y)$ is prime. □

1.16. Corollary. *The bijection of Corollary* 1.12 *restricts to a bijection*

$$\big\{\text{closed irreducible subsets of } \mathbb{A}^n\big\} \xrightarrow{\sim} \big\{\text{prime ideals of } k[x_1, \ldots, x_n]\big\}.$$

1.17. Definition. A topological space X is *noetherian* if every descending chain of closed subsets

$$Z_1 \supset Z_2 \supset \cdots$$

is stationary, which means that there is an index s such that $Z_s = Z_{s+i}$ for all $i \geqslant 0$.

Note that we here consider *descending* chains, whereas in the corresponding property for rings (see A1.3) we consider *ascending* chains of ideals. This is natural in view of Proposition 1.7.

1.18. Example. The affine space $\mathbb{A}^n$ is noetherian. This follows from Corollary 1.12 together with the fact that the ring $k[x_1, \ldots, x_n]$ is noetherian.

1.19. Proposition. *Let Y be a non-empty closed subset of a noetherian topological space X. Then Y can be written as a finite union of closed irreducible subsets, say $Y = Z_1 \cup \cdots \cup Z_m$, such that $Z_i \not\subset Z_j$ whenever $i \neq j$. The collection of closed irreducible sets $\{Z_1, \ldots, Z_m\}$ is uniquely determined by Y.*

Proof. First we show that every non-empty closed $Y \subset X$ can be written as a finite union of closed irreducible subsets. Let $\mathscr{Y}$ be the collection of all Y for which this is *not* true. Our goal is to show that $\mathscr{Y} = \emptyset$. Suppose this is not so. Because X is noetherian, $\mathscr{Y}$ then contains a minimal element. (If not, we obtain a non-stationary descending chain.) Choose a minimal $Y \in \mathscr{Y}$. Because Y is in $\mathscr{Y}$, it is reducible, so we can write Y as a union of closed proper subsets, say $Y = Y' \cup Y''$. Because Y was minimal, Y' and Y'' are not in $\mathscr{Y}$, so they can be written as finite unions of closed irreducible subsets. But then it is clear that the same holds for Y, contradicting the assumption that $Y \in \mathscr{Y}$.

It is also clear that in the decomposition $Y = Z_1 \cup \cdots \cup Z_m$ we can assume that $Z_i \not\subset Z_j$ whenever $i \neq j$: just omit those terms Z_i that are contained in some Z_j with $i \neq j$.

Now suppose that we have $Y = Z_1 \cup \cdots \cup Z_m = Z'_1 \cup \cdots \cup Z'_n$ where all Z_i and Z'_j are closed and irreducible, and such that there are no inclusions among the Z_i or the Z'_j. For $i \in \{1, \ldots, m\}$ we have $Z_i = \cup_{j=1}^n (Z_i \cap Z'_j)$. As Z_i is irreducible this implies that there is an index $\nu(i) \in \{1, \ldots, n\}$ such that $Z_i \subset Z'_{\nu(i)}$. Similarly, for every $j \in \{1, \ldots, n\}$ there is an index $\mu(j) \in \{1, \ldots, m\}$ with $Z'_j \subset Z_{\mu(j)}$. Then $Z_i \subset Z_{\mu(\nu(i))}$, which by our assumptions implies that $\mu(\nu(i)) = i$. Similarly, $\nu(\mu(j)) = j$ for all j. It follows that $m = n$, that the maps μ and ν are mutually inverse permutations of $\{1, \ldots, m\}$, and that $Z_i = Z'_{\nu(i)}$ for all i. So up to a permutation of the indices, the two decompositions are the same. □

1.20. Definition. Let Y be a non-empty closed subset of a noetherian topological space X. Then the closed irreducible sets Z_i appearing in Proposition 1.19 are called the *irreducible components* of Y.

1.21. Because $\mathbb{A}^n$ is noetherian we can apply Proposition 1.19 to it. The conclusion, then, is that every non-empty closed set in $\mathbb{A}^n$ can be written as a union of its irreducible components.

1.22. Definition. An *affine variety* is a closed irreducible subset of $\mathbb{A}^n$ for some $n \geqslant 0$. A *quasi-affine variety* is a non-empty open part of an affine variety.

It is a matter of convention if one wants to call the empty set an affine variety; according to our definitions, affine varieties are non-empty (as the empty set is reducible). Further we note that we shall later give a more complete, and in some sense "better", definition of an affine variety; see Example 5.3.

1.23. Definition. Let $Y \subset \mathbb{A}^n$ be an affine variety. Then we call the k-algebra

$$A(Y) = k[x_1, \dots, x_n]/\mathscr{I}(Y)$$

the *coordinate ring* of Y.

Note that the coordinate ring is a domain by Proposition 1.15. It is a subalgebra of the algebra of k-valued functions on Y; see Exercise 1.7.

1.24. Proposition. *Let Y be an affine variety. The map $Z \mapsto \mathscr{I}(Z)A(Y)$ gives a bijection*

$$\bigl\{\text{closed irreducible subsets of } Y\bigr\} \xrightarrow{\sim} \bigl\{\text{prime ideals of } A(Y)\bigr\}.$$

This map restricts to a bijection

$$Y \xrightarrow{\sim} \bigl\{\text{maximal ideals of } A(Y)\bigr\},$$

sending a point $P = (a_1, \dots, a_n) \in Y$ to the maximal ideal $\mathfrak{m}_P A(Y)$ of $A(Y)$.

Proof. The prime ideals of $A(Y)$ are the ideals $\mathfrak{p}A(Y)$, where $\mathfrak{p} \subset k[x_1, \dots, x_n]$ is a prime ideal containing $\mathscr{I}(Y)$. Under the correspondence in Corollary 1.16, these prime ideals correspond to the affine varieties $Z \subset \mathbb{A}^n$ that are contained in Y. □

1.25. Let $Y \subset \mathbb{A}^n$ be an affine variety with coordinate ring $A(Y)$. Let $\varphi\colon k[x_1, \dots, x_n] \to A(Y)$ be the canonical homomorphism. If $I \subset A(Y)$ is an ideal, we define its zero locus (or zero set) $\mathscr{Z}(I) \subset Y$ to be the zero locus of $\varphi^{-1}(I)$. Note that $\mathscr{I}(Y) \subset \varphi^{-1}(I)$, so the zero locus of $\varphi^{-1}(I)$ is indeed contained in Y.

We can write $I = (\bar{f}_1, \dots, \bar{f}_r)$ for some polynomials $f_i \in k[x_1, \dots, x_n]$; here $\bar{f}_i$ denotes the class $(f_i \bmod \mathscr{I}(Y)) \in A(Y)$. Then $\varphi^{-1}(I) = \mathscr{I}(Y) + (f_1, \dots, f_r)$; note that the RHS is independent of the chosen representatives f_i. So, informally speaking, $\mathscr{Z}(I)$ is obtained by adjoining to the equations for Y the equations $f_1 = \cdots = f_r = 0$.

For $f \in A(Y)$ we denote by $D(f) \subset Y$ the complement of $\mathscr{Z}(f)$; so

$$D(f) = Y \setminus \mathscr{Z}(f).$$

Note that $D(1) = Y$ and $D(f) \cap D(g) = D(fg)$. Further, if $U = Y \setminus \mathscr{Z}(I)$ is open in Y and $P \in U$, there is an element $f \in I$ with $f(P) \neq 0$; we then have $P \in D(f) \subset U$. Hence the collection of open sets $\bigl\{D(f)\bigr\}_{f \in A(Y)}$ is a basis for the topology on Y.

1.26. As an application of the results we have discussed we shall give a proof of the Cayley-Hamilton theorem. As a preparation for this, consider a monic polynomial $f \in k[t]$ of degree m, say $f = t^m + c_{m-1}t^{m-1} + \cdots + c_1 t + c_0$. Via the map $f \mapsto (c_0, \ldots, c_{m-1})$ we may identify the space of all such polynomials f with the affine space $\mathbb{A}^m$.

Because k is algebraically closed, we can write f as a product of linear factors, say $f = (t - \alpha_1) \cdots (t - \alpha_m)$. The discriminant $\mathrm{disc}(f)$ is then defined to be the number

$$\mathrm{disc}(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2 = (-1)^{\frac{m(m-1)}{2}} \prod_{i \neq j} (\alpha_i - \alpha_j)\,.$$

A basic fact that we shall use is that the discriminant is a polynomial function of the coefficients c_ν. Hence the set of polynomials $f \in \mathbb{A}^m$ such that $\mathrm{disc}(f) = 0$ is Zariski closed.

1.27. Cayley-Hamilton Theorem. *Let $A \in M_m(k)$ be an $m \times m$ matrix with coefficients in a field k and let $P_A = \det(t \cdot \mathrm{id} - A) \in k[t]$ be its characteristic polynomial. Then $P_A(A) = 0$.*

Proof. Without loss of generality we may assume that k is algebraically closed. Let $n = m^2$. We identify $M_m(k)$ with the affine space $\mathbb{A}^n$ by sending a matrix A to its vector of coefficients (a_{ij}) sorted in lexicographical ordering.

The matrix coefficients of $P_A(A)$ are polynomials in the coefficients a_{ij}. Hence the subset $Y \subset M_m(k)$ of matrices A for which $P_A(A) = 0$ is Zariski closed. The assertion of the theorem is that $Y = M_m(k)$.

Consider a matrix A that is diagonalisable; this means there exists an invertible matrix Q such that $QAQ^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$. Then $P_A = \prod_{i=1}^m (t - \lambda_i)$ and if we write $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ then $P_A(A) = Q^{-1} P_A(D) Q = 0$. Hence Y contains all diagonalisable matrices.

The characteristic polynomial P_A of a matrix A is of the form $t^m + c_{m-1}t^{m-1} + \cdots + c_1 t + c_0$ for some $(c_0, c_1, \ldots, c_{m-1}) \in \mathbb{A}^m$. By Exercise 1.7 the map $\mathbb{A}^n \to \mathbb{A}^m$ given by $A \mapsto (c_0, c_1, \ldots, c_{m-1})$ is continuous. By what we discussed in 1.26, the set $Z \subset \mathbb{A}^n$ of matrices A for which $\mathrm{disc}(P_A) = 0$ is therefore closed.

If $\mathrm{disc}(P_A) \neq 0$ then A has m distinct eigenvalues and is therefore diagonalisable. This shows that $\mathbb{A}^n = Y \cup Z$. As $\mathbb{A}^n$ is irreducible, either $\mathbb{A}^n = Y$ or $\mathbb{A}^n = Z$. But $\mathbb{A}^n = Z$ is absurd: take m distinct elements $\lambda_1, \ldots, \lambda_m$ in k (which is possible because k is infinite); then $\mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ is not in Z. Hence $\mathbb{A}^n = Y$, which is what we had to prove. □

Exercises for Chapter 1.

Exercise 1.1. Prove that the homomorphism $k[x_1, \ldots, x_n] \to \{\text{functions } \mathbb{A}^n \to k\}$ introduced in 1.1 is indeed injective. [*Hint:* use induction on n.]

Exercise 1.2. Let I and J be ideals of $k[x_1, \ldots, x_n]$. Is it always true that $\mathscr{Z}(I \cap J) = \mathscr{Z}(I) \cup \mathscr{Z}(J)$? If yes, prove this; if no, give an explicit example to illustrate this.

Exercise 1.3. If I is an ideal of a ring R, show that $\sqrt{I}$ is again an ideal.

Exercise 1.4. (i) Let R be a commutative ring with $1 \neq 0$. Prove that $\sqrt{(0)}$ is the intersection of all prime ideals of R. [*Hint:* if $a \in R$ is not nilpotent, consider the set of ideals $J \subset R$ with $a \notin \sqrt{J}$. Use Zorn's lemma to prove that this set has a maximal element; then prove that this maximal element is a prime ideal.]

(ii) If $I \subset R$ is a proper ideal, prove that $\sqrt{I}$ is the intersection of all prime ideals containing I.

Exercise 1.5. (i) If X is an irreducible topological space and $U \subset X$ is a non-empty open subset, show that U is again irreducible and is dense in X. Conclude that X is a Hausdorff space only if it is a point.

(ii) If Y is an irreducible subset of a topological space X, show that its closure $\overline{Y}$ is again irreducible.

Exercise 1.6. Show that any affine variety is quasi-compact. (*Note:* We follow the terminology of Bourbaki. So, "quasi-compact" means that every open cover has a finite subcover, and "compact" means Hausdorff and quasi-compact.)

Exercise 1.7. (i) Let $(f_1, \dots, f_m)$ be an m-tuple of polynomials in $k[x_1, \dots, x_n]$ and let $f\colon \mathbb{A}^n \to \mathbb{A}^m$ be the map given by $P \mapsto \big(f_1(P), \dots, f_m(P)\big)$. Show that f is continuous, taking the Zariski topologies on $\mathbb{A}^m$ and $\mathbb{A}^n$. Also show, by means of a concrete example, that f need not be a closed map.

(ii) Let $Y \subset \mathbb{A}^n$ be an affine variety. Let $F(Y, k)$ be the k-algebra of k-valued functions on Y. Prove that the natural map $k[x_1, \dots, x_n] \to F(Y, k)$ given by $f \mapsto f|_Y$ induces an injective homomorphism $A(Y) \hookrightarrow F(Y, k)$.

(iii) If f, $g \in F(Y, k)$ are continuous functions (where we give $k = \mathbb{A}^1$ the Zariski topology) and $fg = 0$, show that $f = 0$ or $g = 0$.

Exercise 1.8. Let $X = \big\{(t^3, t^4, t^5) \in \mathbb{A}^3 \bigm| t \in k\big\}$. Prove that X is closed in $\mathbb{A}^3$ and give three polynomials f, g, $h \in k[x, y, z]$ that generate $\mathscr{I}(X)$. Determine the irreducible components of $\mathscr{Z}(f, g)$.

Exercise 1.9. Determine the irreducible components of $\mathscr{Z}(y^4 - x^6, y^3 - xy^2 - yx^3 + x^4)$ in $\mathbb{A}^2$.

Exercise 1.10. (i) Consider the lines L and M in $\mathbb{A}^4$ given parametrically by

$$L = \big\{(t, 0, t+1, 0) \bigm| t \in k\big\}, \qquad M = \big\{(0, u, u, 1) \bigm| u \in k\big\}.$$

Let X be the union of all lines $\overline{PQ}$ with $P \in L$ and $Q \in M$. Show that X is a closed subvariety of $\mathbb{A}^4$ and give a generator of its ideal. (Remark: X is called the *join* of L and M.)

(ii) What happens if we replace L and M by arbitrary (but distinct) lines? Is the variety X we get always a linear subvariety?

Exercise 1.11. (i) If $f \in k[x_1, \dots, x_n]$, explain how the decomposition of $\mathscr{Z}(f)$ into irreducible components is related to the factorization of f into irreducibles. (Recall that $k[x_1, \dots, x_n]$ is a UFD.)

(ii) If $f \in k[x, y]$ is an irreducible polynomial, prove that the Zariski topology on $\mathscr{Z}(f)$ is the co-finite topology. (Note: if we say that f is irreducible, this includes the assumption that it is non-constant.)

(iii) Give an example of closed subsets Y, $Z \subset \mathbb{A}^n$, for some n, that are both reducible but such that $Y \cap Z$ is irreducible.

Exercise 1.12. Generalizing Theorem 1.27, show that if A is an $m \times m$ matrix with coefficients in any commutative ring R and if $P_A \in R[t]$ is its characteristic polynomial, $P_A(A) = 0$. [*Hint:* if R is a domain, this follows from 1.27. In general, show that there is a homomorphism $R' \to R$ with R' a domain, and a matrix $A' \in M_m(R')$ whose image in $M_m(R)$ is A.]

COMMUTATIVE ALGEBRA 1

Finiteness conditions

A1.1. Definition. (i) Let R be a commutative ring. An *R-algebra* is a ring A, not necessarily commutative, together with a homomorphism $\varphi\colon R \to A$ such that $\varphi(R)$ is contained in the center of A.

(ii) If $(A_1, \varphi_1\colon R \to A_1)$ and $(A_2, \varphi_2\colon R \to A_2)$ are R-algebras then an R-algebra homomorphism $f\colon A_1 \to A_2$ is a ring homomorphism such that $f \circ \varphi_1 = \varphi_2$.

It is usually clear from the context which is the homomorphism $\varphi\colon R \to A$; in this case we drop φ from the notation and simply write ra (or $r \cdot a$) instead of $\varphi(r)a$.

To give some examples, any ring has a unique structure of a $\mathbb{Z}$-algebra. (In particular, this shows that the map $R \to A$ need not be injective.) If R is a ring and $I \subset R[x_1, \ldots, x_n]$ is an ideal then $R[x_1, \ldots, x_n]/I$ is an R-algebra; as we shall see, the R-algebras of this type are most relevant for this course. For a non-commutative example, the ring $M_n(R)$ of $n \times n$ matrices with coefficients in R is an R-algebra in the obvious way. (It is of course non-commutative only for $n \geqslant 2$.)

The R-algebra homomorphisms $A_1 \to A_2$ form a subset of the set of all ring homomorphisms, and in general this is a proper subset. For example, if we view $\mathbb{C}$ as an algebra over itself then complex conjugation $z \mapsto \bar{z}$ is not a homomorphism of $\mathbb{C}$-algebras.

The R-algebras form a category Alg_R, in which we take the R-algebra homomorphisms as morphisms. The commutative R-algebras form a full subcategory $\mathsf{ComAlg}_R \subset \mathsf{Alg}_R$.

A1.2. Definition. (i) If R is a ring, an R-module M is said to be *finitely generated* (or "of finite type", or even just "finite") if there is a finite collection $\{m_1, \ldots, m_n\}$ of elements of M that generate M as an R-module. This means that every element of M can be written as a linear combination $r_1 m_1 + \cdots + r_n m_n$ for some $r_1, \ldots, r_n \in R$.

(ii) If R is a ring, an R-algebra A is said to be *finitely generated* (or "of finite type") if there is a finite collection $\{a_1, \ldots, a_n\}$ of elements of A that generate A as an R-algebra. This means that every element of A can be written in the form $f(a_1, \ldots, a_n)$ for some polynomial $f \in R[x_1, \ldots, x_n]$. Equivalently, A is finitely generated as an R-algebra if and only if A is isomorphic, as R-algebra, to a quotient of $R[x_1, \ldots, x_n]$ for some n.

(iii) If $K \subset L$ is an extension of fields, L is said to be a *finitely generated field extension* of K if there exist $\alpha_1, \ldots, \alpha_n \in L$ such that $L = K(\alpha_1, \ldots, \alpha_n)$.

We sometimes abbreviate "finitely generated" to "f.g.".

A1.3. Definition. (i) Let R be a ring. Then an R-module M is called *noetherian* if every ascending chain of submodules

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

is stationary, which means that there is an index s such that $N_s = N_{s+i}$ for all $i \geqslant 0$.

(ii) A ring R is noetherian if R is noetherian as a module over itself; this is equivalent to the requirement that every ascending chain of ideals

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

is stationary.

A1.4. Proposition.
(i) *An R-module M is noetherian if and only if every R-submodule of M is finitely generated.*
(ii) *A ring R is noetherian if and only if every ideal of R is finitely generated.*

A1.5. Hilbert basis theorem. *If a ring R is noetherian then the polynomial ring $R[x]$ is noetherian, too.*

A1.6. Proposition.
(i) *If a ring R is noetherian then every quotient of R is noetherian, too.*
(ii) *If R is a noetherian ring and A is a f.g. (commutative) R-algebra then A is noetherian.*
(iii) *If R is a noetherian ring and M is a f.g. R-module then M is noetherian.*

A1.7. Examples. Clearly every field and every PID is noetherian. Applying the above results, it follows that any ring of the form $k[x_1, \dots, x_n]/I$ or $R[x_1, \dots, x_n]/I$ with R a PID is noetherian.

A1.8. Exercise. Let $R \subset k[x, y]$ be the subring consisting of all polynomials of the form $f(x) + x \cdot g(x, y)$. Show that R is not noetherian by exhibiting a non-stationary ascending chain of ideals. In particular, a subring of a noetherian ring need not be noetherian.

A1.9. Exercise. Let I be an ideal of a ring R such that $\sqrt{I}$ is finitely generated. Prove that there is a positive integer N such that $\sqrt{I}^N \subset I$. Conclude that in a noetherian ring, two ideals I and J have the same radical if and only if there exists an integer N with $I^N \subset J$ and $J^N \subset I$.

We conclude this section with a brief discussion of Nakayama's lemma. This is an innocuous-looking result that is easy to prove but is tremendously useful. It is based on the following "linear algebra" lemma, which is effectively just the Cayley-Hamilton theorem.

A1.10. Lemma. *Let M be a finitely generated R-module, $I \subset R$ an ideal, and $F\colon M \to M$ an endomorphism with $F(M) \subset IM$. Then there exist $a_0, \dots, a_{n-1} \in I$ such that $F^n + a_{n-1}F^{n-1} + \cdots + a_1 F + a_0 = 0$.*

Proof. Choose generators $m_1, \dots, m_n$ for M. By assumption there is a matrix $A = (\alpha_{ij})$ with coefficients in I such that $F(m_i) = \sum_{j=1}^n \alpha_{ij} m_j$. Let $P_A = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of A, and note that the coefficients a_i lie in I. By Cayley-Hamilton (use the version of Exercise 1.12), $P_A(A) = 0$; hence $P_A(F) = 0$ as an endomorphism of M. □

A1.11. Theorem. (Nakayama's Lemma) *Let M be a finitely generated R-module. If $I \subset R$ is an ideal such that $M = IM$ then there exists an element $r \in 1 + I$ such that $rM = 0$.*

Proof. Apply the lemma with $F = \mathrm{id}_M$. □

A1.12. Corollary.

(i) *Let R be a local ring with maximal ideal $\mathfrak{m}$. If M is a finitely generated R-module such that $M = \mathfrak{m}M$ then $M = 0$.*

(ii) *Let M be a finitely generated module over a ring R. If $N \subset M$ is a submodule and $I \subset R$ is an ideal such that $M = IM + N$, there exists an element $r \in 1 + I$ such that $rM \subset N$.*

For (i), note that any element $r \in 1 + \mathfrak{m}$ is a unit; hence $rM = 0$ implies $M = 0$. For (ii), apply Nakayama's Lemma to M/N.

CHAPTER 2

Regular functions and morphisms

In the previous chapter, the focus has been on the underlying topological spaces of affine varieties. To properly define affine varieties and morphisms between them we must also define what are the "good" functions on these spaces.

§1. *Regular functions.*

2.1. Definition. Let $Y \subset \mathbb{A}^n$ be a quasi-affine variety.

(i) Let $P \in Y$. Then a function $f\colon Y \to k$ is said to be *regular at the point* P if there is an open subset $U \subset Y$ containing P and polynomials g, $h \in k[x_1, \ldots, x_n]$ with $h \neq 0$ on U such that $f|_U = g/h$ as functions on U.

(ii) A function $f\colon Y \to k$ is called a *regular function* if f is regular at all points of Y.

(iii) If $U \subset Y$ is an open subset we write $\mathscr{O}_Y(U)$ for the k-algebra of regular functions on U. If there is no risk of confusion we write $\mathscr{O}(Y)$ instead of $\mathscr{O}_Y(Y)$.

Note that the regular functions indeed form a k-algebra; it is immediate from the definition that sums, products and scalar multiples of regular functions are again regular.

2.2. Lemma. *Let Y be a quasi-affine variety.*

(i) *If $f\colon Y \to k$ is a regular function, f is continuous for the Zariski topologies on Y and k.*

(ii) *If f and g are regular functions on Y that restrict to the same function on some non-empty open subset $U \subset Y$ then $f = g$.*

Proof. (i) As continuity is a local notion on the source, it suffices to consider the case that $f = g/h$ for some polynomials g and h with $h \neq 0$ on Y. The continuity of f then follows from the remark that, for $a \in k$, we have $f^{-1}(a) = Y \cap \mathscr{Z}(g - ah)$, which is closed in Y.

(ii) The set $Z = \{P \in Y \mid f(P) = g(P)\}$ is the inverse image of $0 \in k$ under the regular function $f - g$; so by (i) Z is closed. But U is dense in Y (Exercise 1.5); so if $f|_U = g|_U$ then $Z = Y$. □

Part (ii) of the Lemma tells us that for open sets $\emptyset \neq U \subset V$ in Y the restriction map $\mathscr{O}_Y(V) \to \mathscr{O}_Y(U)$ is injective.

2.3. Let Y be a quasi-affine variety. Consider pairs (U, f), where U is a non-empty open subset of Y and $f \in \mathscr{O}_Y(U)$ is a regular function on U. We call two such pairs (U_1, f_1) and (U_2, f_2) equivalent if f_1 and f_2 restrict to the same function on $U_{12} = U_1 \cap U_2$. Let $[U, f]$ denote the equivalence class of (U, f).

The equivalence classes form a field $k(Y)$, with addition and multiplication given by

$$[U_1, f_1] + [U_2, f_2] = [U_{12}, f_1 + f_2] \qquad [U_1, f_1] \cdot [U_2, f_2] = [U_{12}, f_1 f_2]\,.$$

(Here it is understood that we restrict f_1 and f_2 to functions on U_{12}.) Note that $U_{12} \neq \emptyset$; further one readily verifies that the addition and multiplication are well-defined on equivalence classes.

If $[U, f] \neq 0$ then the zero locus $f^{-1}(0)$ is a proper closed subset of U; hence $V = U \setminus f^{-1}(0)$ is a non-empty open subset of Y. It is clear from the definitions that $1/f$ is a regular function on V and that $[U, f] \cdot [V, 1/f] = [V, 1] = [Y, 1]$, which is the identity element of $k(Y)$; this proves our claim that $k(Y)$ is a field.

2.4. Definition. Let Y be a quasi-affine variety. Then the field $k(Y)$ just defined is called the *function field of* Y.

The construction of the function field heavily relies on the irreducibility of the space Y. It follows from (ii) of Lemma 2.2 that for any non-empty open $U \subset Y$ the natural map $\mathscr{O}_Y(U) \to k(Y)$ is injective; via this map we may (and shall) view $\mathscr{O}_Y(U)$ as a subring of $k(Y)$. By construction, $k(Y)$ is the union of all its subrings $\mathscr{O}_Y(U)$.

2.5. Definition. Let P be a point of a quasi-affine variety Y. Then the *local ring of* Y *at* P, notation $\mathscr{O}_{Y,P}$ is the subring of $k(Y)$ given by

$$\mathscr{O}_{Y,P} = \bigcup_{P \in U} \mathscr{O}_Y(U)$$

where the union is taken over the open subsets $U \subset Y$ that contain P.

Concretely, an element of $\mathscr{O}_{Y,P}$ is an equivalence class $[U, f]$ with U an open neighbourhood of P and f a regular function on U; as before $(U_1, f_1) \sim (U_2, f_2)$ if $f_1 = f_2$ on U_{12}.

2.6. Remark. Let U be a non-empty open subset of Y. Then U itself is again a quasi-affine variety, and U and Y have the same function field. Also, if $P \in U$ then $\mathscr{O}_{U,P} = \mathscr{O}_{Y,P}$.

2.7. Remark. The ring $\mathscr{O}_{Y,P}$ is an example of a *direct limit* (also called inductive limit) of rings; this is written as $\mathscr{O}_{Y,P} = \varinjlim \mathscr{O}_Y(U)$, where the limit is taken over the open neighbourhoods U of P. Similarly, the function field $k(Y)$ is the direct limit of the system of the $\mathscr{O}_Y(U)$ taken over all open U.

2.8. Lemma. *Let $\mathfrak{n}_P \subset \mathscr{O}_{Y,P}$ be the ideal of elements $[U, f]$ with $f(P) = 0$. Then $\mathfrak{n}_P$ is the unique maximal ideal of $\mathscr{O}_{Y,P}$.*

Proof. The map $\mathscr{O}_{Y,P} \to k$ given by $[U, f] \mapsto f(P)$ is a surjective homomorphism with kernel $\mathfrak{n}_P$; hence $\mathfrak{n}_P$ is a maximal ideal. If $f(P) \neq 0$ then $V = U \setminus f^{-1}(0)$ is an open neighbourhood of P and $[V, 1/f] \in \mathscr{O}_{Y,P}$ is an inverse of $[U, f]$. (See 2.3.) Hence all elements not in $\mathfrak{n}_P$ are units, which implies that $\mathfrak{n}_P$ is the only maximal ideal of $\mathscr{O}_{Y,P}$. □

2.9. Proposition. *Let $Y \subset \mathbb{A}^n$ be an affine variety.*

(i) *The natural map $i\colon A(Y) \to \mathscr{O}(Y)$ is an isomorphism.*

(ii) *For $P \in Y$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_P \subset A(Y)$ the natural map $i_P\colon A(Y)_\mathfrak{m} \to \mathscr{O}_{Y,P}$ is an isomorphism.*

(iii) *The homomorphism* $\mathrm{Frac}\big(A(Y)\big) \to k(Y)$ *induced by i is an isomorphism.*

Proof. Any polynomial in $k[x_1, \ldots, x_n]$ defines a regular function on Y; this gives us a homomorphism $k[x_1, \ldots, x_n] \to \mathscr{O}(Y)$. The kernel of this map is precisely $\mathscr{I}(Y)$; so the induced homomorphism $i\colon A(Y) \to \mathscr{O}(Y)$ is injective.

We now first prove (ii). By (ii) of Lemma 2.2 the natural homomorphism $\mathscr{O}(Y) \to \mathscr{O}_{Y,P}$ is injective. Hence we have $A(Y) \hookrightarrow \mathscr{O}_{Y,P}$. As all elements of $A(Y) \setminus \mathfrak{m}$ map to units in $\mathscr{O}_{Y,P}$, the map i induces an injective homomorphism $i_P\colon A(Y)_{\mathfrak{m}} \to \mathscr{O}_{Y,P}$. This map is surjective by definition of what it means for a function f to be regular at P. So indeed $i_P\colon A(Y)_{\mathfrak{m}} \xrightarrow{\sim} \mathscr{O}_{Y,P}$.

For (iii) we remark that, since $\mathscr{O}_{Y,P} \subset k(Y)$ for any $P \in Y$ we have

$$\mathrm{Frac}\big(A(Y)\big) = \mathrm{Frac}\big(A(Y)_{\mathfrak{m}_P}\big) = \mathrm{Frac}\big(\mathscr{O}_{Y,P}\big) \subset k(Y)\,.$$

But every element of $k(Y)$ is regular at at least one point; so it follows that $\mathrm{Frac}\big(A(Y)\big) = k(Y)$.

Now we complete the proof of (i). We use A2.11. This gives

$$A(Y) \subset \mathscr{O}(Y) \subseteq \bigcap_{P\in Y} \mathscr{O}_{Y,P} = \bigcap_{P\in Y} A(Y)_{\mathfrak{m}_P} = \bigcap_{\mathfrak{m}\subset A(Y)} A(Y)_{\mathfrak{m}} = A(Y)\,,$$

where in the fifth term the intersection is taken over all maximal ideals of $A(Y)$. Hence $i\colon A(Y) \xrightarrow{\sim} \mathscr{O}(Y)$. □

§2. *Morphisms between quasi-affine varieties.*

2.10. Definition. Let X and Y be quasi-affine varieties. Then a *morphism from X to Y* is a continuous map $\varphi\colon X \to Y$ such that for every open $U \subset Y$ and every regular function $f \in \mathscr{O}_Y(U)$, the function $f \circ \varphi\colon \varphi^{-1}(U) \to k$ is regular on $\varphi^{-1}(U) \subset X$.

It is immediate from the definition that the identity map on X is a morphism and that the composition of two morphisms is again a morphism. This means we obtain a category $\mathsf{QAffVar}_k$ of quasi-affine varieties over k. The full subcategory of affine varieties is denoted by AffVar_k. In particular, we now also have the notion of an isomorphism: a morphism $\varphi\colon X \to Y$ is an isomorphism if there exists a morphism $\psi\colon Y \to X$ with $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. Two quasi-affine varieties are called isomorphic if there exists an isomorphism between them.

2.11. Definition. If $\varphi\colon X \to Y$ is a morphism, we denote by

$$\varphi^*\colon \mathscr{O}(Y) \to \mathscr{O}(X)\,,$$

the homomorphism of k-algebras defined by $f \mapsto f \circ \varphi$. (Note that a morphism from X to Y induces a homomorphism from $\mathscr{O}(Y)$ to $\mathscr{O}(X)$.)

2.12. Example. Let $X \subset \mathbb{A}^m$ be a quasi-affine variety. Then the inclusion map $i\colon X \hookrightarrow \mathbb{A}^m$ is a morphism. This is immediate from the definitions.

As will intuitively be clear, morphisms $X \to Y$ are maps that are locally given by rational maps (fractions with polynomial numerator and denominator). It turns out that morphisms to affine varieties are even gobally given by polynomials, in a sense we shall make precise in Proposition 2.14. As a preparation we first prove a lemma.

2.13. Lemma. *Let $Y \subset \mathbb{A}^m$ be a quasi-affine variety, and let $x_1, \ldots, x_n$ be the coordinate functions on $\mathbb{A}^n$. If X is any quasi-affine variety and $\varphi\colon X \to Y$ is a map, then φ is a morphism if and only if the functions $\varphi_i = x_i \circ \varphi$, for $i = 1, \ldots, n$ are regular functions on X.*

Note that φ is the map given by $P \mapsto \big(\varphi_1(P), \ldots, \varphi_n(P)\big)$.

Proof. The necessity of the condition is clear: the x_i are regular functions on Y, so if φ is a morphism, the functions φ_i are regular functions on X. Conversely, suppose $\varphi_1, \ldots, \varphi_n$ are regular functions. If $Z \subset Y$ is closed, there are polynomials $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$ such that $Z = Y \cap \mathscr{Z}(g_1, \ldots, g_r)$. Then $\varphi^{-1}(Z) \subset X$ is the intersection of the sets $(g_j \circ \varphi)^{-1}(0)$. The assumption that $\varphi_1, \ldots, \varphi_n$ are regular functions implies that the functions $g_j \circ \varphi$ are regular, too. Hence it follows from Lemma 2.2 that $\varphi^{-1}(Z)$ is closed in X. This proves that φ is continuous.

If $P \in X$ maps to $Q \in Y$ and f is a regular function on an open set $V \subset Y$ containing Q then there exist an open $U \subset V$ containing Q and polynomials $g, h \in k[x_1, \ldots, x_n]$ with h nowhere zero on U, such that $f|_U = g/h$. Then $f \circ \varphi = (g \circ \varphi)/(h \circ \varphi)$ as functions on $\varphi^{-1}(U)$ and, again by the the assumption that $\varphi_1, \ldots, \varphi_n$ are regular, $g \circ \varphi$ and $h \circ \varphi$ are regular functions on $\varphi^{-1}(U)$, of course still with $h \circ \varphi$ nowhere zero. Hence $f \circ \varphi$ is a regular function on $\varphi^{-1}(U)$, and this proves that φ is a morphism. □

2.14. Proposition. *Let Y be an affine variety. If X is a quasi-affine variety, the map $\varphi \mapsto \varphi^*$ gives a bijection*

$$\big\{\text{morphisms } X \to Y\big\} \xrightarrow{\sim} \big\{k\text{-algebra homomorphisms } \mathscr{O}(Y) \to \mathscr{O}(X)\big\}\,.$$

Proof. Suppose $Y = \mathscr{Z}(\mathfrak{p}) \subset \mathbb{A}^n$ for some prime ideal $\mathfrak{p} \subset k[x_1, \ldots, x_n]$. If φ and ψ are morphisms from X to Y with $\varphi^* = \psi^*$ then $\varphi_i = x_i \circ \varphi = \varphi^*(x_i)$ equals $\psi_i = x_i \circ \psi = \psi^*(x_i)$ for all $i = 1, \ldots, n$. But as remarked before, $\varphi\colon X \to Y$ is the map given by $P \mapsto \big(\varphi_1(P), \ldots, \varphi_n(P)\big)$, and likewise for ψ. Hence $\varphi = \psi$.

It remains to be shown that if $F\colon \mathscr{O}(Y) = k[x_1, \ldots, x_n]/\mathfrak{p} \to \mathscr{O}(X)$ is a homomorphism of k-algebras, there is a morphism $\varphi\colon X \to Y$ with $\varphi^* = F$. Define $\varphi_i = F(\bar{x}_i)$, where the bar indicates the residue class modulo $\mathfrak{p}$. Let $\varphi\colon X \to Y$ be the map given by $\varphi(P) = \big(\varphi_1(P), \ldots, \varphi_n(P)\big)$. Note that this map indeed takes values in Y, because if $g \in \mathfrak{p}$ then $g\big(\varphi(P)\big) = F(\bar{g})\big(P\big) = 0$, as $\bar{g} = 0$. (Make sure you understand this in detail!) By Lemma 2.13, φ is a morphism, and $\varphi^* = F$ by construction. □

2.15. Definition. By an *affine k-algebra* we mean a finitely generated commutative k-algebra without zero divisors. (So the underlying ring is an integral domain.) Let $\mathsf{AffAlg}_k \subset \mathsf{ComAlg}_k$ be the full subcategory of affine k-algebras.

Of course, the motivating example is that the coordinate ring of an affine variety is an affine k-algebra.

2.16. Corollary. *The functor*

$$A\colon \mathsf{AffVar}_k^{\mathrm{opp}} \to \mathsf{AffAlg}_k$$

that sends an affine variety X to the k-algebra $\mathscr{O}(X)$ and that sends a morphism $\varphi\colon X \to Y$ to the induced homomorphism $\varphi^*\colon \mathscr{O}(Y) \to \mathscr{O}(X)$ is an equivalence of categories.

Proof. Let us first explain what this means, without any category language. First of all, that A is a functor means that for each affine variety X we have the affine k-algebra $\mathscr{O}(X)$, and that every morphism $\varphi\colon X \to Y$ induces a k-algebra homomorphism $\varphi^*\colon \mathscr{O}(Y) \to \mathscr{O}(X)$. Saying that A is an equivalence of categories then means:

(a) every k-algebra in AffAlg_k is isomorphic to some $\mathscr{O}(X)$;

(b) for affine varieties X and Y the map $\varphi \mapsto \varphi^*$ is a bijection between the set of morphisms $X \to Y$ and the set of k-algebra homomorphisms $\mathscr{O}(Y) \to \mathscr{O}(X)$.

Now (b) is a special case of Proposition 2.14, and (a) follows from the definitions, together with (i) of Proposition 2.9. □

2.17. Corollary. *Two affine varieties X and Y are isomorphic if and only if $\mathscr{O}(X)$ and $\mathscr{O}(Y)$ are isomorphic as k-algebras.*

2.18. Example. The quasi-affine variety $X = \mathbb{A}^1 \setminus \{0\}$ is isomorphic to the affine variety $Y = \mathscr{Z}(xy-1) \subset \mathbb{A}^2$. The map $X \to Y$ given by $a \mapsto (a, a^{-1})$ is an isomorphism, with inverse given by $(t,u) \mapsto t$. So, although X is not given to us as an affine variety, it is isomorphic to an affine variety.

2.19. Remark. Extending the terminology introduced in Definition 1.22, we call *affine variety* any quasi-affine variety that is isomorphic to an affine variety in the previous sense. The above results are valid with this more general meaning of affineness.

Our definitions imply that the notion of a regular function on a quasi-affine variety $X \subset \mathbb{A}^m$ is independent of the chosen embedding into an affine space. More precisely, if $Y \subset \mathbb{A}^n$ is a quasi-affine variety that is isomorphic to X then an isomorphism $\varphi\colon X \xrightarrow{\sim} Y$ induces an isomorphism $\varphi^*\colon \mathscr{O}(Y) \xrightarrow{\sim} \mathscr{O}(X)$. Note, however, that for quasi-affine varieties (as opposed to only affine ones), $\mathscr{O}(X)$ in general no longer determines X.

The same idea as in Example 2.18 works more generally, yielding the following result.

2.20. Proposition. *Let Y be an affine variety. For $0 \neq f \in A(Y)$ the open subset $D(f) \subset Y$ is again an affine variety, with coordinate ring $A(Y)_f = A(Y)\big[1/f\big]$. Consequently, $\mathscr{O}_Y\big(D(f)\big) = A(Y)_f$.*

Proof. If $Y \subset \mathbb{A}^n$ be defined by the ideal I. Consider the affine variety $Z \subset \mathbb{A}^{n+1}$ defined by $I + (f \cdot x_{n+1} - 1)$. The projection map $\mathbb{A}^{n+1} \to \mathbb{A}^n$ given by $(a_1, \ldots, a_n, a_{n+1}) \mapsto (a_1, \ldots, a_n)$ induces an isomorphism $Z \xrightarrow{\sim} D(f)$, with inverse given by $P \mapsto \big(P, 1/f(P)\big)$. Noting that $A(Z) = A(Y)\big[x_{n+1}\big]/\big(f \cdot x_{n+1} - 1\big) \cong A(Y)\big[1/f\big]$, this proves the proposition. □

2.21. Example. The quasi-affine variety $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not affine. To see this, consider the inclusion map $i\colon X \hookrightarrow \mathbb{A}^2$. We are going to show that any regular function on X extends (necessarily uniquely) to a regular function on $\mathbb{A}^2$, so that $i^*\colon k[x,y] \hookrightarrow \mathscr{O}(X)$ is an isomorphism. If X were affine, this would imply that i is an isomorphism, which is visibly not the case.

To see that i^* is an isomorphism, note that $\mathscr{O}(X)$ is a subring of the function field $k(x,y)$ that contains $k[x,y]$. For $P = (a,b) \in X$ we have $\mathscr{O}_{X,P} = \mathscr{O}_{\mathbb{A}^2,P} = k[x,y]_{\mathfrak{m}_P}$, where $\mathfrak{m}_P = (x-a, y-b)$ is the maximal ideal of $k[x,y]$ corresponding with P. Hence,

$$\mathscr{O}(X) \subset \cap_{P \neq (0,0)}\, k[x,y]_{\mathfrak{m}_P}$$

where the intersection is taken inside $k(x, y)$. Now consider an element $f = g/h \in k(x, y)$ with g, $h \in k[x, y]$ coprime. Then $f \in k[x, y]_{\mathfrak{m}_P}$ only if $h(P) \neq 0$. (Use that $k[x, y]$ is a UFD.) If h is not a unit, there are infinitely many points $P \in \mathbb{A}^2$ for which $h(P) = 0$; hence there are also such points in X, and therefore $f \notin \mathscr{O}(X)$. So $f \in \mathscr{O}(X)$ only if h is a unit, which means that $f \in k[x, y]$. This proves that $k[x, y] = \mathscr{O}(X)$; hence X is not affine.

2.22. Definition. Let X be a quasi-affine variety. A *closed subvariety* of X is an irreducible closed subset $Z \subset X$ with its induced structure of quasi-affine variety. An *open subvariety* of X is a non-empty open subset $U \subset X$ with its induced structure of quasi-affine variety. A *subvariety* of X is an open subvariety of a closed subvariety of X, or, what is the same, a non-empty locally closed irreducible subset of X with its induced structure of quasi-affine variety.

A morphism $f: V \to X$ is called a *closed immersion* (resp. *open immersion*, resp. *immersion*) if f factors as $V \xrightarrow{\sim} W \hookrightarrow X$, where the morphism $W \hookrightarrow X$ is the inclusion morphism of a closed subvariety (resp. an open subvariety, resp. a subvariety).

2.23. Example. Assume $\mathrm{char}(k) = p > 0$. Then the map $F: \mathbb{A}^n \to \mathbb{A}^n$ given by $(a_1, \ldots, a_n) \mapsto (a_1^p, \ldots, a_n^p)$ is a morphism, called the Frobenius endomorphism of $\mathbb{A}^n$. It corresponds to the homomorphism of k-algebras $F^*: k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ given by $x_i \mapsto x_i^p$. (Caution: this is *not* the Frobenius endomorphism of the ring $k[x_1, \ldots, x_n]$, as that would not be a homomorphism of k-algebras. The map F^* is k-linear.)

The morphism F is bijective (why?) but if $n > 0$ it is *not* an immersion; if it were, it had to be an isomorphism but by looking at F^* we see this is not the case.

More generally, let $X \subset \mathbb{A}^n$ be an affine variety, and suppose its ideal $\mathscr{I}(X) \subset k[x_1, \ldots, x_n]$ is generated by elements $f_1, \ldots, f_r$ that lie in the subring $\mathbb{F}_q[x_1, \ldots, x_n] \subset k[x_1, \ldots, x_n]$ for some $q = p^m$. In this case, the morphism $F^m: \mathbb{A}^n \to \mathbb{A}^n$ restricts to a morphism $\varphi_X: X \to X$, called the q-Frobenius endomorphism of X. Again, φ_X is bijective but is not an immersion, unless X is a single point.

2.24. Example. Consider the morphism $\Phi: \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t^2, t^3)$. The corresponding homomorphism of k-algebras is the map $k[x, y] \to k[t]$ given by $x \mapsto t^2$ and $y \mapsto t^3$. The image of Φ is the closed subvariety $C \subset \mathbb{A}^2$ (a curve, but we have not yet defined that notion) given by $y^2 - x^3 = 0$.

The map $\varphi: \mathbb{A}^1 \to C$ induced by Φ is a bijection; the inverse map sends $(a, b) \in C$ to b/a if $a \neq 0$ and sends $(0, 0)$ to 0. As $\mathbb{A}^1$ and C both have the co-finite topology (see (ii) of Exercise 1.11), it follows that φ is even a homeomorphism. But it is clear from looking at the map on rings that φ is not an isomorphism. Hence Φ is not an immersion. Though we won't be able to make this aspect more precise until later, the reason that Φ is not an immersion is that its tangent map for $t = 0$ fails to be injective.

§3. *Products of quasi-affine varieties.*

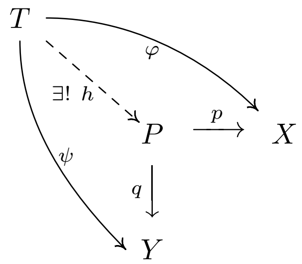
2.25. Definition. Let X and Y be objects of a category C. Then a *product* of X and Y (in C)

is a diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle q}\big\downarrow & & \\ Y & & \end{array}$$

in C that has the following *universal property:* Whenever we have an object T and morphisms $\varphi\colon T \to X$ and $\psi\colon T \to Y$, there is a unique morphism $h\colon T \to P$ such that $\varphi = p \circ h$ and $\psi = q \circ h$.

The universal property is symbolically expressed through the diagram



It is a basic fact that if a product exists, it is unique up to unique isomorphism. That is, if (P, φ, ψ) and (P', φ', ψ') are both products of X and Y, there is a unique morphism $i\colon P \to P'$ such that $\varphi = \varphi' \circ i$ and $\psi = \psi' \circ i$ (apply the universal property), and i is an isomorphism. This allows us to talk about *the* product. It is usually denoted by $X \times Y$, and the morphisms $X \times Y \to X$ and $X \times Y \to Y$ are often called the projection maps.

2.26. Proposition. *Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be quasi-affine varieties. Identifying $\mathbb{A}^{m+n}$ with $\mathbb{A}^m \times \mathbb{A}^n$ (as sets), the product set $X \times Y \subset \mathbb{A}^{m+n}$ is again a quasi-affine variety, and the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ make $X \times Y$ into a product of X and Y in the category $\mathsf{QAffVar}_k$ of quasi-affine k-varieties. If X and Y are affine, so is $X \times Y$, and $A(X \times Y) \cong A(X) \otimes_k A(Y)$.*

We give the proof in three steps.

2.27. Let us first look at the special case $X = \mathbb{A}^m$ and $Y = \mathbb{A}^n$. The assertion is that the affine space $\mathbb{A}^{m+n}$ is the product of $\mathbb{A}^m$ and $\mathbb{A}^n$, with projection maps $p\colon \mathbb{A}^{m+n} \to \mathbb{A}^m$ and $q\colon \mathbb{A}^{m+n} \to \mathbb{A}^n$ given by $p(a_1, \ldots, a_{m+n}) = (a_1, \ldots, a_m)$ and $q(a_1, \ldots, a_{m+n}) = (a_{m+1}, \ldots, a_{m+n})$. This easily follows from Proposition 2.14. Indeed, let T be any quasi-affine variety, and suppose we are given morphisms $\varphi\colon T \to \mathbb{A}^m$ and $\psi\colon T \to \mathbb{A}^n$. On k-algebras these give us homomorphisms $\varphi^*\colon k[x_1, \ldots, x_m] \to \mathscr{O}(T)$ and $\psi^*\colon k[x_{m+1}, \ldots, x_{m+n}] \to \mathscr{O}(T)$. Further, p^* and q^* are the inclusion maps $k[x_1, \ldots, x_m] \hookrightarrow k[x_1, \ldots, x_{m+n}]$ and $k[x_{m+1}, \ldots, x_{m+n}] \hookrightarrow k[x_1, \ldots, x_{m+n}]$, respectively. Clearly, there is a unique homomorphism $H\colon k[x_1, \ldots, x_{m+n}] \to \mathscr{O}(T)$ with $H \circ p^* = \varphi^*$ and $H \circ q^* = \psi^*$. By Proposition 2.14 this means that there is a unique morphism $h\colon T \to \mathbb{A}^{m+n}$ such that $p \circ h = \varphi$ and $q \circ h = \psi$, which proves the assertion.

Note that the topology on the product $\mathbb{A}^{m+n}$ is *not* the product topology (unless $m = 0$ or $n = 0$). For a simple concrete example, in $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ the set $\mathscr{Z}(xy - 1)$ is closed; but it is certainly not closed in the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

2.28. Next we prove Proposition 2.26 in case X and Y are affine. Note that $X \times Y \subset \mathbb{A}^{m+n}$ is a closed subset, for if $X = \mathscr{Z}(f_1, \ldots, f_r)$ for some polynomials $f_i \in k[x_1, \ldots, x_m]$ and $Y = \mathscr{Z}(g_1, \ldots, g_s)$ for some $g_j \in k[x_{m+1}, \ldots, x_{m+n}]$ (note the choice of variables) then clearly $X \times Y$ is the zero set of the ideal $(f_1, \ldots, f_r, g_1, \ldots, g_s) \subset k[x_1, \ldots, x_{m+n}]$.

Next we show that $X \times Y$ is irreducible, for the topology induced by the Zariski topology on $\mathbb{A}^{m+n}$. Suppose this were false; then there exist closed subsets $Z_1, Z_2 \subsetneq X \times Y$ whose union is $X \times Y$. Let $X_1 = \{P \in X \mid \{P\} \times Y \subset Z_1\}$ and $X_2 = \{P \in X \mid \{P\} \times Y \subset Z_2\}$. If $P \in X \setminus (X_1 \cup X_2)$ then

$$\{P\} \times Y = \Big[\big(\{P\} \times Y\big) \cap Z_1\Big] \cup \Big[\big(\{P\} \times Y\big) \cap Z_2\Big]$$

realizes $\{P\} \times Y$ as a union of proper closed subsets, which contradicts the irreducibility of Y. Hence $X = X_1 \cup X_2$.

Next we observe that X_1 and X_2 are closed in X. Indeed, if $P \notin X_1$ then this means there is a point $Q \in Y$ with $(P, Q) \notin Z_1$. The set $\{R \in X \mid (R, Q) \notin Z_1\}$ is then an open set in X, disjoint from X_1, and containing P; hence $X \setminus X_1$ is open. In the same way we see that X_2 is closed in X. Because X is irreducible, it follows that $X = X_1$ or $X = X_2$, which means that either Z_1 or Z_2 is the whole $X \times Y$. This proves that $X \times Y$ is irreducible, making it an affine variety. It is clear that the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are morphisms.

Finally, let T be any quasi-affine variety, and suppose we have morphisms $\varphi\colon T \to X$ and $\psi\colon T \to Y$. Clearly there is a unique map of sets $h\colon T \to X \times Y$ such that $p \circ h = \varphi$ and $q \circ h = \psi$, namely the map given by $A \mapsto \big(\varphi(A), \psi(A)\big)$. By Lemma 2.13 this map is a morphism. This shows that $X \times Y$ is indeed a product of X and Y. Also we note that, with notation as above, the coordinate ring $A(X \times Y) = k[x_1, \ldots, x_{m+n}]/(f_1, \ldots, f_r, g_1, \ldots, g_s)$ is isomorphic to

$$A(X) \otimes_k A(Y) = k[x_1, \ldots, x_m]/(f_1, \ldots, f_r) \otimes_k k[x_{m+1}, \ldots, x_{m+n}]/(g_1, \ldots, g_s)\,.$$

2.29. Finally, we prove the general case of Proposition 2.26. By definition of quasi-affine varieties, X is open in its closure $\overline{X} \subset \mathbb{A}^m$, which is an affine variety. Likewise, Y is open in $\overline{Y} \subset \mathbb{A}^n$, which is also an affine variety. Then $X \times Y$ is open in $\overline{X} \times \overline{Y}$ (the topology on the product is finer than the product topology) and is therefore again a quasi-affine variety. By the same argument as in the affine case (see the end of the previous step), $X \times Y$ is a product. $\square$

2.30. Corollary. *If A and B are affine algebras over the algebraically closed field k, the tensor product $A \otimes_k B$ is again an affine k-algebra.*

Note that if k is not algebraically closed a tensor product of f.g. k-algebras without zero divisors is in general no longer a domain. For instance, if $k \subsetneq K$ is a finite field extension, $K \otimes_k K$ is not a domain.

2.31. Example. The *diagonal morphism* $\Delta = \Delta_X\colon X \to X \times X$, given by $P \mapsto (P, P)$, is the unique morphism with $\mathrm{pr}_1 \circ \Delta = \mathrm{pr}_2 \circ \Delta = \mathrm{id}_X$. (Here we write $\mathrm{pr}_i\colon X \times X \to X$ for the projections.) If X is affine, the corresponding homomorphism $A(X) \otimes_k A(X) \to A(X)$ is the product map $f \otimes g \mapsto fg$.

§4. *Linear algebraic groups.*

The techniques introduced thus far already suffice to start discussing an important class of objects, namely linear algebraic groups.

2.32. Definition. A *linear algebraic group* is an affine variety G that is equipped with the structure of a group, such that the group multiplication $m\colon G\times G \to G$ and the inverse $i\colon G \to G$ are morphisms of algebraic varieties.

If the context allows it we shall sometimes drop the adjective "linear" and simply refer to algebraic groups.

2.33. Let G be a linear algebraic group and $A(G) = \mathscr{O}(G)$ its coordinate algebra. As we shall explain now, the structure of an algebraic group can be given entirely in terms of algebra.

The structure of a group involves the existence of a unit element $e \in G$; we shall use the same letter for the inclusion map $e\colon \{e\} \to G$. The group law m, the inverse i and the unit e correspond to homomorphisms of k-algebras

$$\mu = m^*\colon A(G) \to A(G) \otimes_k A(G)\,, \qquad \iota = i^*\colon A(G) \to A(G)\,, \qquad \varepsilon = e^*\colon A(G) \to k\,,$$

called the *co-multiplication*, the *co-inverse* (or antipode), and the *co-unit* (or augmentation).

The group axioms can be translated into identities between these homomorphisms. The associativity axiom says that the two compositions

$$G \times G \times G \xrightarrow{m\times \mathrm{id}_G} G \times G \xrightarrow{m} G \qquad \text{and} \qquad G \times G \times G \xrightarrow{\mathrm{id}_G\times m} G \times G \xrightarrow{m} G$$

are equal; on k-algebras this means that

$$(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu \tag{1}$$

as homomorphisms $A(G) \to A(G) \otimes_k A(G) \otimes_k A(G)$. Similarly, the defining property of the identity element says that the morphisms $m \circ (\mathrm{id}_G, e)\colon G \to G \times G \to G$ and $m \circ (e, \mathrm{id}_G)\colon G \to G \times G \to G$ are both equal to the identity on G; on k-algebras:

$$(\mathrm{id} \cdot \varepsilon) \circ \mu = \mathrm{id} = (\varepsilon \cdot \mathrm{id}) \circ \mu\,. \tag{2}$$

(By $\mathrm{id}\cdot\varepsilon\colon A(G) \otimes_k A(G) \to A(G)$ we mean the map $\alpha \otimes \beta \mapsto \alpha \cdot \varepsilon(\beta)$; likewise for $\varepsilon \cdot \mathrm{id}$.) Finally,

$$G \xrightarrow{\Delta} G \times G \xrightarrow{\mathrm{id}_G\times i} G \times G \xrightarrow{m} G \qquad \text{and} \qquad G \xrightarrow{\Delta} G \times G \xrightarrow{i\times \mathrm{id}_G} G \times G \xrightarrow{m} G$$

are both the constant map with value e, which translates into the identities

$$(\mathrm{id} \cdot \iota) \circ \mu = a \circ \varepsilon = (\iota \cdot \mathrm{id}) \circ \mu\,, \tag{3}$$

where $a\colon k \to A(G)$ is the structural homomorphism. (So $a \circ \varepsilon\colon A(G) \to A(G)$ is the map $f \mapsto f(e)$, viewing $f(e) \in k$ as a constant function on G.)

2.34. Definition. A *Hopf algebra* over a field K is a K-vector space A together with K-linear maps

$$\begin{array}{ccc} a\colon K \to A & & \varepsilon\colon A \to K \\ & \iota\colon A \to A & \\ \rho\colon A \otimes_k A \to A & & \mu\colon A \to A \otimes_k A \end{array} \tag{4}$$

such that:

(a) taking ρ as ring multiplication and a as structural morphism, A is an associative K-algebra;

(b) the maps ε, ι and μ are homomorphisms of K-algebras;

(c) the identities (1), (2) and (3) hold.

The Hopf algebra is called *commutative* if $\rho \circ \mathrm{sw} = \rho$, where $\mathrm{sw}\colon A \otimes_K A \to A \otimes_K A$ is the map $\alpha \otimes \beta \mapsto \beta \otimes \alpha$; this just means that A is commutative as a K-algebra. The Hopf algebra is called *co-commutative* if $\mathrm{sw} \circ \mu = \mu$.

2.35. Remark. Conditions (a), (b) and (c) can all be expressed as the commutativity of certain diagrams. It turns out that in the collection of diagrams thus obtained there is a great deal of symmetry, as already suggested by the way we have displayed the maps in (4). (In brief, for every notion involved, there is also a "co"-version and the symmetry involves exchanging these.) This has as an interesting consequence that if A is finite dimensional over K, the dual vector space $A^\vee$ again has the structure of a Hopf algebra. This plays an important role in the theory of finite commutative group schemes.

The conclusion of the previous discussion is that $A(G)$, for G a linear algebraic group, is a commutative Hopf algebra over k. This Hopf algebra is co-commutative if and only if G is abelian.

2.36. Example. The *additive group* $\mathbb{G}_\mathrm{a}$ is just the affine line, with group law given by addition. The coordinate algebra is the polynomial ring $k[t]$, with co-unit $\varepsilon\colon k[t] \to k$ given by evaluation at 0 (so: $t \mapsto 0$), co-multiplication $\mu\colon k[t] \to k[t_1,t_2]$ given by $t \mapsto t_1 + t_2$ and co-inverse $\iota\colon k[t] \to k[t]$ given by $t \mapsto -t$. (Note: we here identify $k[t] \otimes_k k[t]$ with $k[t_1,t_2]$.)

2.37. Example. The group GL_n of invertible $n \times n$ matrices (with coefficients in k) is another very basic example of a linear algebraic group. To describe it in detail, we use coordinates x_{ij} with $1 \leqslant i,j \leqslant n$ (corresponding to the matrix coefficients). The determinant of the matrix (x_{ij}) is then a polynomial $\det \in k[x_{ij}]$ of degree n, and GL_n can be identified with the basic open subset $D(\det)$ in affine n^2-space $\mathbb{A}^{n^2}$. The coordinate algebra is

$$A(\mathrm{GL}_n) = k\big[x_{ij},\xi\big]/\big(\xi \cdot \det - 1\big)\,.$$

(We formally adjoin the inverse of det as a new variable ξ; cf. the proof of Proposition 2.20.)

The co-unit $\varepsilon\colon A(\mathrm{GL}_n) \to k$ is the map sending x_{ij} to 0 if $i \neq j$, sending x_{ii} to 1 and ξ to 1. (Here we are just giving the coefficients and the inverse determinant of the identity matrix!) To express the co-multiplication, we identify

$$A(\mathrm{GL}_n) \otimes A(\mathrm{GL}_n) = k\big[y_{ij},z_{ij},\eta,\zeta\big]/\big(\eta \cdot \det_Y - 1, \zeta \cdot \det_Z - 1\big)\,,$$

where $\det_Y$ and $\det_Z$ are the determinants of the matrices $Y = (y_{ij})$ and $Z = (z_{ij})$. The co-multiplication sends x_{ij} to the coefficient in position (i, j) of the matrix product $Y \cdot Z$ and sends the inverse determinant ξ to $\det^{-1}(Y) \cdot \det^{-1}(Z)$; so:

$$\mu(x_{ij}) = \sum_{\nu=1}^{n} y_{i\nu} z_{\nu j}\,, \qquad \mu(\xi) = \eta\zeta\,.$$

The co-inverse, finally, expresses the coefficients of the inverse matrix $(x_{ij})^{-1}$ as functions of the original matrix:

$$\iota(x_{ij}) = \xi \cdot (-1)^{i+j} \mathrm{minor}_{i,j}\,, \qquad \iota(\xi) = \det\,,$$

where $\mathrm{minor}_{i,j} \in A(\mathrm{GL}_n)$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and the jth column.

For $n = 1$ we get the commutative algebraic group GL_1, which is usually denoted by $\mathbb{G}_\mathrm{m}$ and which is called the *multiplicative group*. This group is just $k^* = \mathbb{A}^1 \setminus \{0\}$ with multiplication as group structure. For $n > 1$ the group GL_n is of course not commutative.

2.38. If G is a linear algebraic group and $g \in G$ then we have morphisms $\lambda_g\colon G \to G$ and $\rho_g\colon G \to G$, defined by $\lambda_g(x) = gx$ and $\rho_g(x) = xg$. These morphisms, called left and right multiplication by g, are automorphisms of G as a variety, for the obvious relations $\lambda_e = \mathrm{id}_G = \rho_e$ and $\lambda_h \circ \lambda_g = \lambda_{hg}$ and $\rho_h \circ \rho_g = \rho_{gh}$ give $\lambda_{g^{-1}} = (\lambda_g)^{-1}$ and $\rho_{g^{-1}} = (\rho_g)^{-1}$. Note further that the λ_g and ρ_h mutualy commute and that they are not group automorphisms, unless $g = e$ or $h = e$. The inner automorphisms $\mathrm{Inn}(g) = \lambda_g \circ \rho_{g^{-1}}$ are automorphisms of G as a linear algebraic group.

2.39. Examples. We get many (and in fact all, see Remark 2.42) interesting examples of linear algebraic groups by considering closed subgroups of GL_n for some n. Such examples include:

— the subgroup $\mathrm{SL}_n \subset \mathrm{GL}_n$ of matrices with determinant 1,

— the group $B \subset \mathrm{GL}_n$ of upper triangular matrices,

— the group $N \subset B$ of upper triangular matrices with all diagonal coefficients equal to 1,

— the subgroup $\mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$ consisting of the transformations $A\colon k^{2n} \to k^{2n}$ that preserve a given symplectic form. A standard choice of a symplectic form is the form $J\colon k^{2n} \times k^{2n} \to k$ given with respect to the basis $\{e_1, \ldots, e_{2n}\}$ by

$$J(e_i, e_j) = \begin{cases} 0 & \text{if } i + j \neq 2n+1; \\ 1 & \text{for } i \leqslant n \text{ and } j = 2n+1-i; \\ -1 & \text{for } i > n \text{ and } j = 2n+1-i. \end{cases}$$

(At first glance, other choices of a symplectic form may seem more natural, but this choice turns out to yield the most convenient realization of the group Sp_{2n}.)

Similarly we may consider a non-degenerate symmetric bilinear form $\varphi\colon k^n \times k^n \to k$ (best choice: the one given by the anti-diagonal matrix with coefficients 1 on the anti-diagonal) and consider the corresponding orthogonal group $\mathrm{O}_n \subset \mathrm{GL}_n$ of transformations that preserve φ. In characteristic 2 this will not give a subgroup variety, and in fact a more refined treatment of orthogonal groups is required. But even in characteristic $\neq 2$, the group O_n is not a linear algebraic group according to our definition, as it is not connected. (An element of O_n has determinant ± 1 and this allows us to distinguish two connected components.) Here we see that in some settings it is more natural to allow algebraic varieties that are not necessarily irreducible.

The examples given here all have nice explicit realizations as matrix groups, but it should be realized that this is by no means a natural thing to expect. For instance, if we start with the subgroup $B \subset \mathrm{GL}_2$ and conjugate it by some $g \in \mathrm{GL}_2$ we get another subgroup variety gBg^{-1} whose matrix realization may be less convenient. See for instance Exercise 2.3.

2.40. Proposition. *Let G be a linear algebraic group and $H \subset G$ a subgroup variety, meaning that H is a subvariety of G that at the same time is a subgroup. Then H is closed in G.*

Proof. We first show that the closure $\overline{H} \subset G$ is again a subgroup variety. It is clear that $\overline{H}$ is a subvariety. If $h \in H$ then λ_h maps H into itself, and therefore $\lambda_h\bigl(\overline{H}\bigr) \subset \overline{H}$. Hence $H \cdot \overline{H} \subset \overline{H}$. If $y \in \overline{H}$ we therefore find that $\rho_y(H) \subset \overline{H}$; hence also $\rho_y\bigl(\overline{H}\bigr) \subset \overline{H}$. This shows that $\overline{H} \subset G$ is closed under multiplication. But also the inverse $i\colon G \to G$ sends H into itself, which implies that $i\bigl(\overline{H}\bigr) \subset \overline{H}$, so $\overline{H}$ is also closed under inverses. This proves that $\overline{H}$ is again a subgroup variety.

To prove that $H = \overline{H}$ we may now replace G by $\overline{H}$ and assume H is open in G. Let $g \in G$ and consider the coset $gH = \lambda_g(H)$. Because G is irreducible, $H \cap gH \neq \emptyset$, which means there exist elements h_1, $h_2 \in H$ with $h_1 = gh_2$. Then $g = h_1 h_2^{-1} \in H$, which shows that $H = G$. $\square$

2.41. Remark. Let V be an n-dimensional k-vector space. The choice of a basis for k gives an identification $\alpha\colon \mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n$. Via this identification we can view $\mathrm{GL}(V)$ as an algebraic group. This structure of an algebraic group is in fact independent of the choice of a basis. Indeed, a different choice of basis changes α by an inner automorphism of GL_n. In particular, the Zariski topology on $\mathrm{GL}(V)$ is well-defined, and for $U \subset \mathrm{GL}(V)$ open it is well-defined what it means for a function $f\colon U \to k$ to be regular.

2.42. Remark. It can be shown that every linear algebraic group is isomorphic to a closed subgroup of GL_n for some n. See for instance Springer's book [7].

2.43. Example. Perhaps a less obvious example of an algebraic group is the group $\mathrm{PGL}_n = \mathrm{GL}_n/k^* \cdot \mathrm{id}$ of invertible $n \times n$ matrices modulo scalars. To realize this as a closed subgroup of GL_N for some N, let GL_n act on the space $M_n(k)$ of all $n \times n$ matrices by conjugation. Taking $N = n^2$ and identifying $M_n(k)$ with k^N, this gives a homomorphism $\mathrm{GL}_n \to \mathrm{GL}_N$ (called the adjoint representation) whose kernel is precisely the subgroup $k^* \cdot \mathrm{id} \subset \mathrm{GL}_n$ of invertible scalar multiplications. The induced injective homomorphism $\mathrm{PGL}_n \to \mathrm{GL}_N$ realizes PGL_n as a closed subgroup of GL_N. Exercise 2.5 asks you to make this explicit in the simplest non-trivial case $n = 2$. In fact, one equation is clear: as inner automorphisms of $M_n(K)$ have determinant 1, we get that $\mathrm{PGL}_n \hookrightarrow \mathrm{SL}_N$.

Exercises for Chapter 2.

Exercise 2.1. Give an example of a morphism $X \to \mathbb{A}^2$, for some affine variety X, whose image is $\bigl\{(a,b) \in \mathbb{A}^2 \bigm| a \neq 0\bigr\} \cup \bigl\{(0,0)\bigr\}$. (This shows that images of morphisms are in general not even locally closed. A theorem of Chevalley says that images are, however, constructible. Cf. HAG, Chap II, Exercises 3.18–19.)

Exercise 2.2. Let Y be a quasi-affine variety and let $Z \subset Y$ be a subvariety. Define the local ring of Y along Z, notation $\mathscr{O}_{Y,Z}$, to be the k-subalgebra of the function field $k(Y)$ consisting of all equivalence classes $[U, f]$ such that $U \cap Z \neq \emptyset$.

(i) Prove that $\mathscr{O}_{Y,Z}$ is indeed a k-subalgebra of $k(Y)$ and that $\mathscr{O}_{Y,Z}$ is a local ring. What is the maximal ideal?

(ii) If $\overline{Z} \subset Y$ is the closure of Z, show that $\mathscr{O}_{Y,Z} = \mathscr{O}_{Y,\overline{Z}}$.

(iii) Assume Y is affine and Z is closed in Y. Let $\mathfrak{p} \subset A(Y)$ be the prime ideal corresponding to Z. Prove that $A(Y)_{\mathfrak{p}} \xrightarrow{\sim} \mathscr{O}_{Y,Z}$.

Exercise 2.3. Let $B \subset \mathrm{GL}_2$ be the subgroup of upper triangular matrices. Give generators of the ideal in $k\big[x_{11}, x_{12}, x_{21}, x_{22}, 1/\det\big]$ that defines the subgroup $gBg^{-1} \subset \mathrm{GL}_2$, where $g = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$.

Exercise 2.4. (i) Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be local rings. If $f\colon R_1 \to R_2$ is a homomorphism, show that $f^{-1}(\mathfrak{m}_2) \subseteq \mathfrak{m}_1$.

(ii) A homomorphism f as in (i) is called a *local homomorphism* if $\mathfrak{m}_1 = f^{-1}(\mathfrak{m}_2)$. Given an example of a homomorphism of local rings that is not a local homomorphism.

(iii) Let $\varphi\colon X \to Y$ be a morphism of quasi-affine varieties. If $P \in X$ and $Q = \varphi(P)$, we have an induced homomorphism $\varphi^*\colon \mathscr{O}_{Y,Q} \to \mathscr{O}_{X,P}$. Prove that this map is a local homomorphism.

Exercise 2.5. (i) In Example 2.43 take $n = 2$ and consider the injective homomorphism $\mathrm{PGL}_2 \hookrightarrow \mathrm{SL}_4$. Prove that this realizes PGL_2 as a closed subgroup of SL_4, and give generators for its defining ideal. (Linear and quadratic equations will suffice.)

(ii) Assume $n > 1$. By a slight modification of the construction in Example 2.43, we get a closed embedding $\mathrm{PGL}_n \hookrightarrow \mathrm{SL}_M$ with $M = n^2 - 1$. (So for $n = 2$ we get $\mathrm{PGL}_2 \hookrightarrow \mathrm{SL}_3$.) Can you see how this is done? [*Hint:* For any $g \in \mathrm{GL}_n$ and $A \in M_n(k)$ we have $\mathrm{tr}(gAg^{-1}) = \mathrm{tr}(A)$.]

COMMUTATIVE ALGEBRA 2

Localization.

A2.1. Let R be a ring and $S \subset R$ a multiplicatively closed subset; by this we mean that $1 \in S$ and $s, t \in S \Rightarrow st \in S$. We introduce a relation $\sim$ on $R \times S$ by

$$(r_1, s_1) \sim (r_2, s_2) \quad \Leftrightarrow \quad \text{there exists an element } t \in S \text{ with } t(r_1s_2 - r_2s_1) = 0.$$

It is immediate that this relation is reflexive and symmetric. If $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$ then we have elements t, $u \in S$ with $t(r_1s_2 - r_2s_1) = 0$ and $u(r_2s_3 - r_3s_2) = 0$; but then

$$s_2tu(r_1s_3 - r_3s_1) = s_3u \cdot t(r_1s_2 - r_2s_1) + s_1t \cdot u(r_2s_3 - r_3s_2) = 0\,,$$

and since $s_1tu \in S$ this shows that $\sim$ is an equivalence relation.

Let $S^{-1}R$ be the set of equivalence classes in $R \times S$. We denote the equivalence class of (r, s) by r/s; as this notation suggests, it should be thought of as a fraction with numerator r and denominator s. We put a ring structure on $S^{-1}R$ by letting

$$r_1/s_1 + r_2/s_2 = (r_1s_2 + r_2s_1)/(s_1s_2) \qquad \text{and} \qquad r_1/s_1 \cdot r_2/s_2 = (r_1r_2)/(s_1s_2)\,.$$

It is easy to verify that these give a well-defined addition and multiplication, making $S^{-1}R$ into a ring with identity element $1/1$.

There is a canonical ring homomorphism φ: $R \to S^{-1}R$, by $r \mapsto r/1$.

A2.2. Definition. The ring $S^{-1}R$ obtained in this way is called the *localization* of R with respect to S.

A2.3. Exercise. If R is a domain, $S = R \setminus \{0\}$ is a multiplicatively closed subset of R. Show that $S^{-1}R$ is in this case the fraction field $\mathrm{Frac}(R)$ and that φ is simply the inclusion of R in its fraction field.

A2.4. Exercise. Let R be a ring and $S \subset R$ a multiplicatively closed subset. Prove that the canonical homomorphism φ: $R \to S^{-1}R$ is injective if and only if S contains no zero divisors of R.

A2.5. Example. The definition of $S^{-1}R$ simplifies if R is a domain. In this case, if $0 \in S$ then $S^{-1}R$ is the zero ring. If $0 \notin S$ then $S^{-1}R$ is the subring of $\mathrm{Frac}(R)$ consisting of all fractions x/y with $x \in R$ and $y \in S$. It is readily verified that this indeed forms a subring. The homomorphism φ: $R \to S^{-1}R$ is in this case just the inclusion map.

A2.6. Examples. (i) If $\mathfrak{p} \subset R$ is a prime ideal, $S = R \setminus \mathfrak{p}$ is a multiplicatively closed subset. The localization $S^{-1}R$ is in this case denoted by $R_\mathfrak{p}$; it is called the localization of R at $\mathfrak{p}$. It is a local ring, with maximal ideal $\mathfrak{p}R_\mathfrak{p}$, the ideal of $R_\mathfrak{p}$ generated by the image of $\mathfrak{p}$.

(ii) Let $f \in R$. Then $S = \{1, f, f^2, \ldots\}$ is a multiplicatively closed subset of R. The localization $S^{-1}R$ is in this case denoted by R_f.

The notation just introduced may occasionally lead to confusion. For instance, $\mathbb{Z}_p$ usually refers to the ring of p-adic integers, which is a certain completion of $\mathbb{Z}$. The localization of $\mathbb{Z}$ with respect to $\{1, p, p^2, \ldots\}$ should therefore be denoted by $\mathbb{Z}[1/p]$, instead of $\mathbb{Z}_p$. (To make things worse, group theorists sometimes use $\mathbb{Z}_p$ for $\mathbb{Z}/p\mathbb{Z}$.) The localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ is written as $\mathbb{Z}_{(p)}$; this is the subring of $\mathbb{Q}$ consisting of all rational numbers m/n such that the denominator n is not divisible by p.

If R is a domain with fraction field $K = \mathrm{Frac}(R)$ and $f \neq 0$, the localization R_f is just the subring $R[1/f] \subset K$ obtained by adjoining f^{-1} to R.

A2.7. Exercise. Let R be an affine k-algebra for some field k, i.e., a f.g. k-algebra without zero divisors.

(i) If $f \neq 0$, show that R_f is again an affine k-algebra.

(ii) Let $R = k[x]$. Prove that if $\mathfrak{p} \subset R$ is a prime ideal, $R_\mathfrak{p}$ is not of finite type as a k-algebra.

A2.8. Proposition. *Let S be a multiplicatively closed subset of a ring R and let $\varphi\colon R \to S^{-1}R$ be the canonical homomorphism.*

(i) *For all $s \in S$ the element $\varphi(s)$ is a unit in $S^{-1}R$.*

(ii) *Let $g\colon R \to A$ be a homomorphism of rings such that $g(s) \in A^*$ for all $s \in S$. Then there is a unique homomorphism $h\colon S^{-1}R \to A$ such that $g = h \circ \varphi$.*

A2.9. Proposition. *Notation as in* A2.8. *Every ideal of $S^{-1}R$ is of the form $S^{-1}I = \{a/s \mid a \in I, s \in S\}$ for some ideal $I \subset R$. More precisely, for an ideal $J \subset S^{-1}R$ we have $J = S^{-1}\varphi^{-1}(J)$. The map $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$ gives a bijection*

$$\{\text{prime ideals of } S^{-1}R\} \xrightarrow{\sim} \{\text{prime ideals } \mathfrak{p} \subset R \text{ with } \mathfrak{p} \cap S = \emptyset\},$$

whose inverse sends $\mathfrak{p}$ to $S^{-1}\mathfrak{p}$.

A2.10. Corollary. *If R is a noetherian ring, any localization $S^{-1}R$ is again noetherian.*

A2.11. Proposition. *Let R be a domain. Then*

$$R = \bigcap_{\mathfrak{m}} R_\mathfrak{m}$$

where the intersection is taken inside the fraction field $\mathrm{Frac}(R)$ *and runs over all maximal ideals $\mathfrak{m}$ of R.*

A2.12. Exercise. Prove this proposition. The inclusion "$\subset$" is obvious. For the opposite inclusion, let $x \in \mathrm{Frac}(R)$ and consider $I = \{r \in R \mid rx \in R\}$, which is an ideal of R. If $x \in \cap_\mathfrak{m} R_\mathfrak{m}$, show that $I = R$, and conclude that $x \in R$.

A2.13. Just as we can localize rings, we can also define localization for modules. Given a ring R, a multiplicatively closed subset $S \subset R$ and an R-module M, define $S^{-1}M = (M \times S)/\sim$, where "$\sim$" is the equivalence relation on the set $M \times S$ given by

$$(m_1, s_1) \sim (m_2, s_2) \quad \Leftrightarrow \quad \text{there exists an element } t \in S \text{ with } t(s_2 m_1 - s_1 m_2) = 0.$$

The verification that this is indeed an equivalence relation works the same as before. We usually write m/s for the class of (m, s). Now $S^{-1}M$ has the structure of a module over $S^{-1}R$ via the rules

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}\,, \qquad \frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}\,.$$

Similar to the notation for rings, we write $M_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1}M$ if $\mathfrak{p} \subset R$ is a prime ideal and $M_f = \{1, f, f^2, \ldots\}^{-1}M$, for $f \in R$.

A2.14. Proposition. *Let $F\colon M \to N$ be a homomorphism of R-modules. Let P be one of the properties "injective", "surjective" or "bijective". Then the following are equivalent:*

(a) *$F\colon M \to N$ has property P;*

(b) *for all prime ideals $\mathfrak{p} \subset R$ the induced homomorphism $F_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ has property P;*

(c) *for all maximal ideals $\mathfrak{m} \subset R$ the induced homomorphism $F_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ has property P.*

CHAPTER 3

Projective varieties.

In this chapter we introduce projective space and projective varieties. To a large extent, the correspondence between algebra and geometry that we have seen in the previous chapters, carries over to the projective setting, provided we work with homogeneous polynomials.

As always, k denotes an algebraically closed field. If $R = \oplus_{d=0}^{\infty} R_d$ is a graded ring, let $R_{>n} = \oplus_{d>n} R_d$ and $R_{<n} = \oplus_{d<n} R_d$. (Similarly for $R_{\geqslant n}$ and $R_{\leqslant n}$.)

§1. *The Zariski topology on $\mathbb{P}^n$.*

3.1. Definition. Let $n \geqslant 0$. Then projective n-space $\mathbb{P}^n$ (over k) is the set of lines through the origin in k^{n+1}.

3.2. Let $O = (0, \ldots, 0)$ be the origin in k^{n+1}. If $A = (a_0, a_1, \ldots, a_n) \in k^{n+1} \setminus \{O\}$, we denote by $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n$ the line through O and A. We call $a_0, \ldots, a_n$ the homogeneous coordinates of this point. Note that we let the indices run from 0 to n.

Note that two points $A = (a_0, \ldots, a_n)$ and $A' = (a'_0, \ldots, a'_n)$ in $k^{n+1} \setminus \{O\}$ determine the same line through O if and only if there exists a constant $c \in k^*$ such that $a'_i = c \cdot a_i$ for all $i = 0, 1, \ldots, n$. So $\mathbb{P}^n$ is the space of points $(a_0 : \cdots : a_n)$ with $a_0, \ldots, a_n \in k$ not all equal to zero, and $(a_0 : \cdots : a_n) = (a'_0 : \cdots : a'_n)$ if there is a $c \in k^*$ such that $a'_i = c \cdot a_i$ for all i.

To formulate this differently, we may let the multiplicative group $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{A}^{n+1}$ by $c \cdot (a_0, \ldots, a_n) = (ca_0, \ldots, ca_n)$. The open subvariety $\mathbb{A}^{n+1} \setminus \{O\}$ is stable under this action, and the quotient map is a surjective map $q\colon \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$.

3.3. Let $k[X_0, X_1, \ldots, X_n]$ be the polynomial ring over k in the variables $X_0, \ldots, X_n$. We view this as a graded k-algebra, with each variable X_i of degree 1. So the degree we consider is the total degree; a monomial $X_0^{m_0} X_1^{m_1} \cdots X_n^{m_n}$ has degree $m_0 + \cdots + m_n$.

An ideal $I \subset k[X_0, \ldots, X_n]$ is called a *homogeneous ideal* if I is generated by homogeneous elements. This is equivalent to the requirement that for any $f \in I$, if we write f as a sum of homogeneous polynomials, say $f = f_0 + f_1 + \cdots + f_N$, the homogeneous parts f_j are again in I.

It is essential to remark that, even though we describe points of $\mathbb{P}^n$ through a system of (homogeneous) coordinates, a homogeneous polynomial $F \in k[X_0, \ldots, X_n]$ does *not* define a k-valued function on $\mathbb{P}^n$ (unless F is a constant). For instance, it does not make sense to ask for the value of the ith coordinate of a point $P \in \mathbb{P}^n$. However, because of the homogeneity of F, the condition that $F(P) = 0$ is unambiguous.

3.4. Definition. If S is a set of homogeneous polynomials in $k[X_0, \ldots, X_n]$, we define its *zero set* $\mathscr{Z}(S) \subset \mathbb{P}^n$ by

$$\mathscr{Z}(S) = \left\{ P = (a_0 : a_1 : \ldots : a_n) \in \mathbb{P}^n \;\middle|\; F(P) = 0 \quad \text{for all } F \in S \right\}.$$

If I is a homogeneous ideal of $k[X_0, \ldots, X_n]$, we define its *zero set* $\mathscr{Z}(I) \subset \mathbb{P}^n$ to be the zero set of the set of all homogeneous polynomials in I.

3.5. Proposition.

(i) *If S is a set of homogeneous elements of $k[X_0,\ldots,X_n]$ and $I \subset k[X_0,\ldots,X_n]$ is the ideal generated by S then $\mathscr{Z}(S) = \mathscr{Z}(I)$.*

(ii) *We have $\mathscr{Z}\big((0)\big) = \mathbb{P}^n$ and $\mathscr{Z}\big((1)\big) = \emptyset$.*

(iii) *If $\{S_\alpha\}_{\alpha\in A}$ is a collection of sets of homogeneous elements in $k[X_0,\ldots,X_n]$ then we have $\mathscr{Z}\big(\cup_{\alpha\in A} S_\alpha\big) = \cap_{\alpha\in A}\mathscr{Z}(S_\alpha)$.*

(iv) *If I and J are homogeneous ideals of $k[X_0,\ldots,X_n]$ then $\mathscr{Z}(IJ) = \mathscr{Z}(I) \cup \mathscr{Z}(J)$.*

The proof presents no difficulties and is left to the reader.

3.6. Definition. The *Zariski topology* on $\mathbb{P}^n$ is the topology for which the closed sets are the subsets of the form $\mathscr{Z}(I)$ for some homogeneous ideal $I \subset k[X_0,\ldots,X_n]$.

As in the affine case, because $k[X_0,\ldots,X_n]$ is noetherian, the closed sets in $\mathbb{P}^n$ are defined by a finite number of homogeneous equations $F_1(P) = \cdots = F_r(P) = 0$.

If $F \in k[X_0,\ldots,X_n]$ is a homogeneous polynomial, define

$$D_+(F) = \mathbb{P}^n \setminus \mathscr{Z}(F) = \big\{P \in \mathbb{P}^n \bigm| F(P) \neq 0\big\}\,.$$

The same argument as in 1.25 shows that the open sets $D_+(F)$ form a basis for the topology on $\mathbb{P}^n$.

§2. *The standard affine open covering.*

3.7. Define open subsets $U_i \subset \mathbb{P}^n$, for $i \in \{0,1,\ldots,n\}$, by

$$U_i = D_+(X_i) = \big\{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n \bigm| a_i \neq 0\big\}\,.$$

(Once again, the value of a_i is not well-determined but the condition that $a_i \neq 0$ makes sense.) Clearly these $n+1$ open sets cover $\mathbb{P}^n$. Further, if $P \in U_i$ then P can be written as

$$P = (a_0 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n)\,.$$

for uniquely determined $a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n \in k$. In other words: if the ith homogeneous coordinate is non-zero, we can scale it to become 1 and after that no further rescaling is possible. This means that we have bijections

$$\psi_i \colon \mathbb{A}^n \xrightarrow{\sim} U_i \subset \mathbb{P}^n \qquad \text{given by} \quad (b_1,\ldots,b_n) \mapsto (b_1 : \cdots : b_i : 1 : b_{i+1} : \cdots : b_n)$$

with inverse given by

$$(a_0 : \cdots : a_n) \mapsto \Big(\frac{a_0}{a_i},\ldots,\frac{a_{i-1}}{a_i},\frac{a_{i+1}}{a_i},\ldots,\frac{a_n}{a_i}\Big)\,,$$

which indeed is a well-defined map on U_i.

3.8. As we shall show next, the maps ψ_i are in fact homeomorphisms. As a preparation for the proof, we need to explain a simple construction with polynomials.

Consider the polynomial rings $R = k[y_1, \dots, y_n]$ and $S = k[X_0, X_1, \dots, X_n]$ with their natural gradings by total degree. If $F \in S_d$ then $f = F(y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n)$ is a polynomial in $R_{\leqslant d}$. (The degree may drop; for instance, $F = X_i^d$ gives $f = 1$.) This gives us a map $\mathrm{dehom}_i\colon S_d \to R_{\leqslant d}$ that we may call dehomogenization with respect to X_i. In the opposite direction, if $f \in R_d$ then there is a unique $F \in S_d$ with $\mathrm{dehom}_i(F) = f$; formally we may give it by $F = X_i^d \cdot f(X_0/X_i, \dots, X_{i-1}/X_i, X_{i+1}/X_i, \dots, X_n/X_i)$. This gives us a map $\mathrm{hom}_i\colon R_d \to S_d$ called homogenization. By construction, $\mathrm{dehom}_i \circ \mathrm{hom}_i = \mathrm{id}_R$. If $F \in S_d$, write $F = X_i^{d-e} \cdot G$ for some $G \in S_e$ not divisible by X_i; then $\mathrm{hom}_i \circ \mathrm{dehom}_i(F) = G$.

3.9. Proposition.

(i) *If* $Z = \mathscr{Z}(F_1, \dots, F_r) \subset \mathbb{P}^n$ *for some homogeneous polynomials* $F_1, \dots, F_r$ *in* $k[X_0, \dots, X_n]$ *then* $\psi_i^{-1}(Z \cap U_i) \subset \mathbb{A}^n$ *is the closed subset defined by the ideal* $(f_1, \dots, f_r)$, *where* $f_j = \mathrm{dehom}_i(F_j)$.

(ii) *If* $Y = \mathscr{Z}(I) \subset \mathbb{A}^n$ *for some ideal* $I \subset k[y_1, \dots, y_n]$ *then* $\overline{\psi_i(Y)} \subset \mathbb{P}^n$ *is the closed subset defined by the homogeneous ideal* $\mathrm{hom}_i(I) = \{\mathrm{hom}_i(f) \mid f \in I\}$.

(iii) *The bijections* $\psi_i\colon \mathbb{A}^n \xrightarrow{\sim} U_i$ *are homeomorphisms.*

Proof. Part (i) is clear. For (ii), note that $Z = \mathscr{Z}\big(\mathrm{hom}_i(I)\big)$ is closed, and by (i) we have $Z \cap U_i = \psi_i(Y)$; hence $\overline{\psi_i(Y)} \subset Z$. On the other hand, $\overline{\psi_i(Y)} = \mathscr{Z}(J)$ for some homogeneous ideal J. As explained above, if $F \in J$ we have $F = X_i^{d-e} \cdot \mathrm{hom}_i\big(\mathrm{dehom}_i(F)\big)$ for some e. As $\mathrm{dehom}_i(F) \in I$, it follows that $F \in \mathrm{hom}_i(I)$. Hence $J \subset \mathrm{hom}_i(I)$, which gives $\overline{\psi_i(Y)} \supset Z$. As we already know that the maps ψ_i are bijections, (iii) follows from (i) and (ii). □

3.10. Caution. In (ii), if $I \subset k[y_1, \dots, y_n]$ is generated by $f_1, \dots, f_r$ then it is not always true that $\mathrm{hom}_i(I)$ is generated by $\mathrm{hom}_i(f_1), \dots, \mathrm{hom}_i(f_r)$. See Exercise 3.5. Whereas the maps dehom_i give a k-algebra homomorphism $S \to R$, the homogenization maps hom_i are not even additive.

3.11. Example. We have $\mathbb{P}^1 = U_0 \cup U_1$, where U_1 is the open subset of points of the form $(a : 1)$ and U_0 is the open set of points $(1 : b)$. Write φ_i for the inverse of ψ_i. We have $\varphi_1\colon U_1 \xrightarrow{\sim} \mathbb{A}^1$ and use x as a coordinate on U_1; similarly, $\varphi_0\colon U_0 \xrightarrow{\sim} \mathbb{A}^1$ and we use y as a coordinate on U_0. Then the intersection $U_{01} = U_0 \cap U_1$ is the subset $D(y) = \mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$ when viewed as a subset of U_0 and is $D(x) = \mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$ when viewed as a subset of U_1. The "gluing map"

$$\varphi_1 \circ \varphi_0^{-1}\colon \mathbb{A}^1 \setminus \{0\} \xrightarrow{\sim} \mathbb{A}^1 \setminus \{0\}$$

is the map given by $b \mapsto b^{-1}$. This just expresses the fact that a point $(1 : b)$ in U_0 with $b \neq 0$ is the same as the point $(b^{-1} : 1)$ in U_1.

3.12. Example. Use $X : Y : Z$ as homogeneous coordinates on $\mathbb{P}^2$ and identify $\mathbb{A}^2$ (with coordinates x, y) with the open subset $U_2 \subset \mathbb{P}^2$ given by $Z \neq 0$. If $C \subset \mathbb{A}^2$ is the curve given by $y^2 = x^3 + x + 1$ then its closure in $\mathbb{P}^2$ is the projective curve given by $Y^2Z = X^3 + XZ^2 + Z^3$. If on the open set U_1 we use (t, u) as coordinates, the intersection $\overline{C} \cap U_1$ is given by the equation $u = t^3 + tu^2 + u^3$. (Set $Y = 1$ and substitute $X = t$ and $Z = u$.)

3.13. Definition. A *projective variety* is a closed irreducible subset $Y \subset \mathbb{P}^n$ for some n. A *quasi-projective variety* is a non-empty open subset of a projective variety.

3.14. Lemma. *Let $Y \subset \mathbb{P}^n$ be a quasi-projective variety. Let $P \in Y$ and let $i \in \{0, 1, \dots, n\}$ be an index such that P lies in the standard open subset $U_i \subset \mathbb{P}^n$. We identify U_i with $\mathbb{A}^n$ via the homeomorphism φ_i. If $f\colon Y \to k$ is a function, then the following two conditions are equivalent:*

(a) *the function $f\colon Y \cap U_i \to k$ is regular at P, in the sense of* Definition 2.1,

(b) *there exist an open $V \subset Y$ about P and homogeneous polynomials G, $H \in k[X_0, \dots, X_n]$ of the same degree, with $H \neq 0$ on V, such that $f = G/H$ as functions on V.*

In (b), note that G/H makes sense as a function on V because G and H are homogeneous of the same degree.

Proof. Assume (b) holds. We may assume $V \subset U_i$. Then $f = \mathrm{dehom}_i(G)/\mathrm{dehom}_i(H)$ as functions on V, so (a) holds. Conversely, if (a) holds, there exist an open $V \subset (Y \cap U_i)$ containing P and polynomials g, $h \in k[y_1, \dots, y_n]$ with $h \neq 0$ on V such that $f = g/h$ as functions on V. Let $d = \deg(g)$ and $e = \deg(h)$. If $e \geqslant d$ then $f = X_i^{e-d}\mathrm{hom}_i(g)/\mathrm{hom}_i(h)$ as functions; if $e \leqslant d$ then $f = \mathrm{hom}_i(g)/X_i^{d-e}\mathrm{hom}_i(h)$. □

3.15. Definition. Let $Y \subset \mathbb{P}^n$ be a quasi-projective variety.

(i) Let $P \in Y$. Then a function $f\colon Y \to k$ is said to be *regular at the point* P if it satisfies the equivalent conditions of Lemma 3.14.

(ii) A function $f\colon Y \to k$ is called a *regular function* if f is regular at all points of P.

(iii) If $U \subset Y$ is an open subset we write $\mathscr{O}_Y(U)$ for the k-algebra of regular functions on U. If there is no risk of confusion we write $\mathscr{O}(Y)$ instead of $\mathscr{O}_Y(Y)$.

As in the affine case, the regular functions indeed form a k-algebra, as it is immediate from the definition that sums, products and scalar multiples of regular functions are again regular.

The definition of a morphism is now a repetition of Definition 2.10.

3.16. Definition. Let X and Y be any quasi-affine or quasi-projective varieties. Then a morphism $\varphi\colon X \to Y$ is a continuous map $\varphi\colon X \to Y$ such that for every open $U \subset Y$ and every regular function $f \in \mathscr{O}_Y(U)$, the function $f \circ \varphi\colon \varphi^{-1}(U) \to k$ is regular on $\varphi^{-1}(U) \subset X$.

3.17. Remark. With this definition it is immediate that the homeomorphisms $\psi_i\colon \mathbb{A}^n \xrightarrow{\sim} U_i$ are isomorphisms. If $Y \subset \mathbb{A}^n$ is a quasi-affine variety, its image $\psi_i(Y) \subset \mathbb{P}^n$ (for any choice of i) is a quasi-projective variety and ψ_i induces an isomorphism $Y \xrightarrow{\sim} \psi_i(Y)$. In this way we see that any quasi-affine variety "is" also a quasi-projective variety, which means that—at least for now—the quasi-projective varieties are the most general among the varieties we are considering. The four classes are related as follows

$$\begin{array}{ccccc} & & & & \{\text{projective varieties}\} \\ & & & & \cap \\ \{\text{affine varieties}\} & \subset & \{\text{quasi-affine varieties}\} & \subset & \{\text{quasi-projective varieties}\} \end{array}$$

In Chapter 5 we shall give a completely general definition of an algebraic variety. We then call (quasi-)affine or (quasi-)projective any variety that is isomorphic with a (quasi-)affine, resp. (quasi-)projective variety as defined here.

3.18. We start with the observation that $\mathrm{GL}_{n+1}(k)$ naturally acts on $\mathbb{P}^n$ (from the left). Indeed, if $A \in \mathrm{GL}_{n+1}(k)$ and $\ell \subset k^{n+1}$ is a line through the origin, the image $A{\cdot}\ell$ is another such line. As the scalar matrices in $\mathrm{GL}_{n+1}(k)$ act trivially, we in fact have an induced action of $\mathrm{PGL}_{n+1}(k)$.

3.19. Lemma. *The group* $\mathrm{PGL}_{n+1}(k)$ *acts on* $\mathbb{P}^n$ *through automorphisms of* $\mathbb{P}^n$ *as a projective variety, so that we have a homomorphism* $\alpha\colon \mathrm{PGL}_{n+1}(k) \to \mathrm{Aut}(\mathbb{P}^n)$.

Proof. Let $g = \bigl(g_{ij}\bigr)_{0\leqslant i,j\leqslant n} \in \mathrm{GL}_{n+1}(k)$. To g we can associate an automorphism g^* of $k[X_0,\dots,X_n]$, given by linear substitutions in the variables:

$$g^*(F) = F(g_{00}X_0 + g_{01}X_1 + \cdots + g_{0n}X_n, \dots, g_{n0}X_0 + \cdots + g_{nn}X_n)\,.$$

Then $g \mapsto g^*$ gives a homomorphism $\mathrm{GL}_{n+1}(k) \to \mathrm{Aut}_k\bigl(k[X_0,\dots,X_N]\bigr)$ and by construction, g^* is an automorphism of $k[X_0,\dots,X_n]$ as a graded k-algebra.

Consider the bijection $\alpha(g)\colon \mathbb{P}^n \to \mathbb{P}^n$ defined by g. If $I \subset k[X_0,\dots,X_n]$ is a homogeneous ideal, $\alpha(g)^{-1}\bigl(\mathscr{Z}(I)\bigr) = \mathscr{Z}\bigl(g^*(I)\bigr)$. Hence $\alpha(g)$ is a homeomorphism. Further, if G and H are homogeneous polynomials of the same degree and $f = G/H$ on some open set V then the function $f \circ \alpha(g)\colon \alpha(g)^{-1}(V) \to k$ is given by g^*G/g^*H. Hence $\alpha(g)$ is an automorphism of $\mathbb{P}^n$. □

3.20. Caution. The automorphism of $\mathbb{P}^n$ that is given by g only depends on its class $[g] \in \mathrm{PGL}_{n+1}(k)$ but the automorphism g^* that we used depends on the choice of a representative g. It is not true that an automorphism of $\mathbb{P}^n$ induces an automorphism of $k[X_0,\dots,X_n]$. In fact, unlike what we have seen in the affine case, the polynomial ring $k[X_0,\dots,X_n]$ is not intrinsically associated with the variety $\mathbb{P}^n$. See also Exercise 3.8.

3.21. Remark. The homomorphism $\mathrm{PGL}_{n+1}(k) \to \mathrm{Aut}(\mathbb{P}^n)$ is in fact an isomorphism. Remarkably, the structure of the automorphism group of affine n-space $\mathbb{A}^n$ is much more complicated; it is the subject of a long-standing open problem, the Jacobian conjecture. In any case it is easy to write down automorphisms of $\mathbb{A}^n$ that are not linear; see Exercise 3.7.

3.22. The action of $\mathrm{PGL}_{n+1}(k)$ through automorphisms of $\mathbb{P}^n$ is extremely useful, and many results from classical projective geometry become very easy to prove if we first apply a suitable coordinate transformation. See below for examples. To prepare for these examples, we note that the automorphism of $\mathbb{P}^n$ given by an element $g \in \mathrm{PGL}_{n+1}(k)$ transforms linear subvarieties into linear subvarieties. (By a linear subvariety we mean a subvariety defined by linear equations; see Exercise 3.6 for further discussion.) Thus, points are mapped to points, lines to lines, etc. The automorphism obviously also preserves incidence relations.

The next point we want to make is that $\mathrm{PGL}_{n+1}(k)$ acts transitively on the set of $(n+2)$-tuples of points in general position in $\mathbb{P}^n$. In order to explain what this means, we start with the example $n = 1$; in this case the assertion is that $\mathrm{PGL}_2(k)$ acts transitively on the set of 3-tuples of distinct points in $\mathbb{P}^1$. Thus, if P_1, P_2 and P_3 are three distinct points in $\mathbb{P}^1$ and Q_1, Q_2 and Q_3 is another set of three distinct points, there is a $g \in \mathrm{PGL}_2(k)$—in fact, a unique such g—such that $g(P_i) = Q_i$.

Next take $n = 2$. If we look at 3-tuples of distinct points in $\mathbb{P}^2$, we cannot hope that PGL_3 acts transitively. Indeed, 3 points may lie on a line, or they may not, and the PGL_3-action preserves this distinction. This is where the "in general position" comes in. In $\mathbb{P}^2$ we say that four distinct points are in general position if no three of them lie on a line. Then the assertion is that PGL_3 acts transitively on the set of 4-tuples of points in general position.

3.23. Definition. Given $n+2$ points $P_1, \ldots, P_{n+2}$ in $\mathbb{P}^n$, these points are said to be *in general position* if no $n+1$ of them are contained in a hyperplane.

3.24. Proposition. *Let $P_1, \ldots, P_{n+2}$ and $Q_1, \ldots, Q_{n+2}$ be two collections of $n+2$ points in $\mathbb{P}^n$ in general position. Then there is a unique $g \in \mathrm{PGL}_{n+1}(k)$ such that $g(P_i) = Q_i$ for all i.*

Proof. Choose vectors e_i and f_i in k^{n+1} $(i = 1, \ldots, n+2)$ such that $P_i = k \cdot e_i$ and $Q_i = k \cdot f_i$ as lines in k^{n+1}. The assuption that the points P_i are in general position means that any $n+1$ of the vectors e_i are linearly independent; likewise for the f_i. In particular, $\{e_1, \ldots, e_{n+1}\}$ is a k-basis for k^{n+1} and so is $\{f_1, \ldots, f_{n+1}\}$. Hence we can find a matrix $A \in \mathrm{GL}_{n+1}(k)$ such that $A(e_i) = f_i$ for $i = 1, \ldots, n+1$. Next write

$$e_{n+2} = a_1 e_1 + \cdots + a_{n+1} e_{n+1}\,, \qquad f_{n+2} = b_1 f_1 + \cdots + b_{n+1} f_{n+1}\,,$$

and note that the coefficients a_i and b_i are all non-zero, because the P_i and the Q_i are in general position. In general, for the A we have chosen we cannot expect that $A(e_{n+2})$ is proportional to f_{n+2}. To remedy this, consider, for given non-zero constants $\lambda_1, \ldots, \lambda_{n+1} \in k^*$, the matrix $B = B(\lambda_1, \ldots, \lambda_{n+1}) \in \mathrm{GL}_{n+1}(k)$ such that $B(e_i) = \lambda_i \cdot e_i$ for all $i = 1, \ldots, n+1$. (So B is conjugate to the diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$.) Then $C = A \circ B$ has the property that $C(e_i)$ is proportional to f_i for $i = 1, \ldots, n+1$, and any matrix C with this property is of the form $C = A \circ B(\lambda_1, \ldots, \lambda_{n+1})$ for some $\lambda_1, \ldots, \lambda_{n+1} \in k^*$. We now want to choose the λ_i such that also $C(e_{n+2})$ is proportional to f_{n+2}. So we want that there is a constant $\mu \in k^*$ such that $a_i \lambda_i = \mu \cdot b_i$ for all i. As the a_i and b_i are all non-zero, this has a solution for the λ_i, and up to a simultaneous rescaling of the λ_i (changing μ) the solution is unique. Hence there is a unique $g = \big[A \circ B(\lambda_1, \ldots, \lambda_{n+1})\big] \in \mathrm{PGL}_{n+1}(k)$ with $g(P_i) = Q_i$ for $i = 1, \ldots, n+2$. □

3.25. Theorem of Desargues. *Let A, A', B, B', C, C' be six distinct points in $\mathbb{P}^2$ such that the lines $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$ are distinct and concurrent. Then the points*

$$P = \overline{AB} \cap \overline{A'B'}\,, \quad Q = \overline{AC} \cap \overline{A'C'} \quad \textit{and} \quad R = \overline{BC} \cap \overline{B'C'}$$

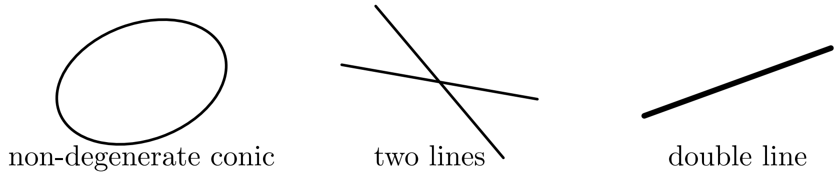
are collinear.

Proof. We may choose projective coordinates such that the three lines intersect in $O = (1:1:1)$ and that $A = (1:0:0)$, $B = (0:1:0)$ and $C = (0:0:1)$. Then we have $A' = (a:1:1)$, $B' = (1:b:1)$ and $C' = (1:1:c)$ for some a, b and c not equal to 1. This gives

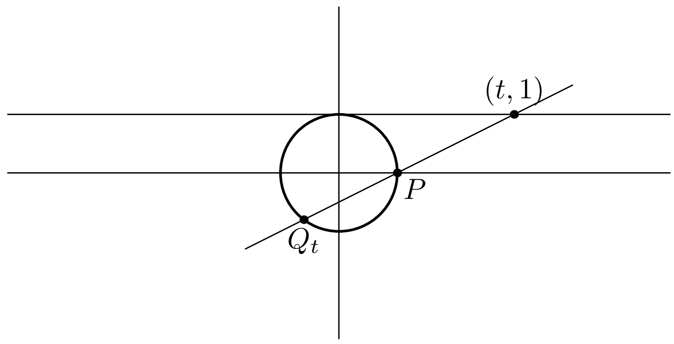
$$P = (a-1 : 1-b : 0)\,, \quad Q = (a-1 : 0 : 1-c)\,, \quad R = (0 : b-1 : 1-c)$$

which are on the line $\big((b-1)(c-1)\big)X + \big((a-1)(c-1)\big)Y + \big((a-1)(b-1)\big)Z = 0$. □

3.26. As our next example, we look at conics in $\mathbb{P}^2$. By definition, a *conic* is a curve $C \subset \mathbb{P}^2$ defined by a quadratic equation, i.e., $C = \mathscr{Z}(F)$ with $F \in k[X,Y,Z]$ homogeneous of degree 2. Though we shall usually assume that F is irreducible, our considerations will also lead us to consider degenerate cases, where F is a product of two linear forms, or even the square of a linear form.



It turns out that any non-degenerate conic is isomorphic to $\mathbb{P}^1$. We shall give more details on this later, when we have more techniques at our disposal. For now, to explain the idea, assume $\mathrm{char}(k) \neq 2$ and consider the circle $C_0 \subset \mathbb{A}^2$ given by $x^2 + y^2 = 1$. Pick a point on it, say $P = (1,0)$. For $t \in k$, the line L_t through P and the point $(t,1)$ intersects the circle in one other point, call it Q_t.



Direct calculation gives

$$Q_t = \left(\frac{1-(t-1)^2}{1+(t-1)^2}, \frac{-2(t-1)}{1+(t-1)^2} \right),$$

as long as $1 + (t-1)^2 \neq 0$. Now, projectively $C \subset \mathbb{P}^2$ is the circle with equation $X^2 + Y^2 = Z^2$ and C_0 is just the part with $Z \neq 0$. The map $t \mapsto Q_t$ is then "part" of a morphism

$$\varphi\colon \mathbb{P}^1 \to C \quad \text{given by} \quad (t:u) \mapsto \big(u^2 - (t-u)^2 : -2u(t-u) : u^2 + (t-u)^2\big).$$

(Note that the RHS is never $(0:0:0)$.)

We claim that φ is an isomorphism. Going through the geometric construction in the inverse direction (start with Q_t and find back t) gives as a candidate for the inverse the morphism $\psi\colon C \setminus \{P\} \to \mathbb{P}^1$ given by $(a:b:c) \mapsto (a+b-c:b)$. (Check this!) The given recipe does not work for $(a:b:c) = (1:0:1)$ because $(0:0)$ is not allowed as answer. Set-theoretically it is of course clear that φ^{-1} should send $(1:0:1)$ to $(1:1)$.

Now the whole point is that we can use the equation for C to obtain an extension of ψ to a morphism defined on all of C. Namely, the assumption that $(a:b:c) \in C$ gives us the relation $a^2 + b^2 = c^2$; hence $(a+b-c:b) = (2a:a+b+c)$. (Just note that $(a+b+c)(a+b-c) = 2ab$.)

Now $\chi\colon C \setminus \{(0:1:-1)\} \to \mathbb{P}^1$ given by $(a:b:c) \mapsto (2a : a+b+c)$ is a well-defined morphism (with $(1:0:1) \mapsto (1:1)$, as expected), and ψ and χ are equal on $C \setminus \{P, (0:1:-1)\}$. So together they indeed define a morphism $\psi\colon C \to \mathbb{P}^1$, which is the inverse of φ.

Let us also note that ψ is well-defined as a morphism $\mathbb{P}^2 \setminus \{P\} \to \mathbb{P}^1$; this morphism is the projection from the point P onto the line $L \subset \mathbb{P}^2$ given by $Y = Z$. (In affine coordinates L is the line $y = 1$.) This projection map does *not* extend to a morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$. (In fact, it can be shown that there are no non-constant morphisms from $\mathbb{P}^2$ to $\mathbb{P}^1$.) So we really need to use the equation of C to extend ψ over the point P.

Exercises for Chapter 3.

Exercise 3.1. Consider the quotient map $q\colon \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ as in 3.2. Prove that the Zariski topology on $\mathbb{P}^n$ is the finest topology for which q is continuous. Also prove that q is a morphism.

Exercise 3.2. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{A}^{n+1}$ by rescaling of the coordinates:

$$c \cdot (a_0, \ldots, a_n) = (ca_0, \ldots, ca_n)\,.$$

(Note our choice of coordinates on $\mathbb{A}^{n+1}$.)

(i) In the correspondence between closed irreducible subsets of $\mathbb{A}^{n+1}$ and prime ideals of $k[X_0, \ldots, X_n]$, show that $Y \subset \mathbb{A}^{n+1}$ is stable under the $\mathbb{G}_{\mathrm{m}}$-action if and only if the corresponding prime ideal $\mathfrak{p} \subset k[X_0, \ldots, X_n]$ is homogeneous.

(ii) Let $q\colon \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the projection map. Prove that a closed subset $Z \subset \mathbb{P}^n$ is irreducible if and only if its pre-image $q^{-1}(Z) \subset \mathbb{A}^{n+1} \setminus \{O\}$ is irreducible.

(iii) Write $S = k[X_0, \ldots, X_n]$ and note that $S_{>0} = \oplus_{d>0} S_d$ is a maximal ideal. Prove that there is an inclusion-reversing bijection

$$\big\{\text{closed irreducible subsets of } \mathbb{P}^n\big\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{homogeneous prime ideals} \\ \mathfrak{p} \subset k[X_0, \ldots, X_n] \text{ with } \mathfrak{p} \neq S_{>0} \end{matrix} \right\}.$$

Exercise 3.3. Let $A = (a_0 : a_1 : a_2)$ and $B = (b_0 : b_1 : b_2)$ be distinct points in $\mathbb{P}^2$. Give an equation for the line AB passing through A and B.

Exercise 3.4. (Pascal's theorem) Let C be a non-degenerate conic in $\mathbb{P}^2$, and let P, Q, R, P', Q', R' be six distinct points on C. Prove that the points $\alpha = PQ' \cap P'Q$, $\beta = PR' \cap P'R$ and $\gamma = QR' \cap Q'R$ are collinear.

Exercise 3.5. Consider the homogenization map $\mathrm{hom}_0\colon R \to S$ (with respect to the variable X_0) as in 3.8.

(i) Give examples showing that in general $\mathrm{hom}_0(f+g)$ is not the same as $\mathrm{hom}_0(f) + \mathrm{hom}_0(g)$.

(ii) Use this to construct an example of an ideal $I = (f_1, \ldots, f_r) \subset k[y_1, \ldots, y_n]$ such that $\mathrm{hom}_0(I) = \big\{\mathrm{hom}_0(f) \bigm| f \in I\big\}$ is not generated by the $\mathrm{hom}_0(f_i)$.

Exercise 3.6. If $X \subset \mathbb{P}^n$ is a subset, define its ideal $\mathscr{I}(X) \subset k[X_0, \ldots, X_n]$ to be the homogeneous ideal generated by all homogeneous F such that $F(P) = 0$ for all $P \in X$.

(i) A closed subvariety $X \subset \mathbb{P}^n$ is called a *linear subvariety* if its ideal $\mathscr{I}(X) \subset k[X_0, \ldots, X_n]$ is generated by linear forms (homogeneous elements of degree 1). If $V \subset k^{n+1}$ is a linear subspace with $V \neq \{O\}$, define $\mathbb{P}(V) \subset \mathbb{P}^n$ to be the set of lines through the origin that lie in V. Show that $\mathbb{P}(V)$ is a linear subvariety and that the map $V \mapsto \mathbb{P}(V)$ gives a bijection between the set of non-zero linear subspaces $V \subset k^{n+1}$ and the set of linear subvarieties of $\mathbb{P}^n$.

(ii) Let $L = \mathbb{P}(V)$ and $M = \mathbb{P}(W)$ be linear subvarieties of $\mathbb{P}^n$. Define their join to be the union of all (projective) lines $\overline{PQ}$ with $P \in L$ and $Q \in M$. Prove that this join equals $\mathbb{P}(U)$, where $U \subset k^{n+1}$ is the linear span of V and W.

(iii) Redo Exercise 1.10.

Exercise 3.7. (i) Given $v = (v_1, \ldots, v_n) \in k^n$, let $t_v\colon \mathbb{A}^n \to \mathbb{A}^n$ be the translation over v. Identifying $\mathbb{A}^n$ with the open subset $U_0 \subset \mathbb{P}^n$, show that t_v extends to an automorphism of $\mathbb{P}^n$ and give this extended automorphism as an element of $\mathrm{PGL}_{n+1}(k)$.

(ii) An automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ is called an *affine transformation* if φ can be written as a linear automorphism of $\mathbb{A}^n$ (given by an element of $\mathrm{GL}_n(k)$) followed by a translation. Show that these transformation form a subgroup $\mathrm{Aff}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ and that $\mathrm{Aff}(\mathbb{A}^n)$ is a semi-direct product $k^n \rtimes \mathrm{GL}_n(k)$.

(iii) Show that any affine transformation extends to an automorphism of $\mathbb{P}^n$. Can you see how this realizes $k^n \rtimes \mathrm{GL}_n(k)$ as a subgroup of $\mathrm{PGL}_{n+1}(k)$?

(iv) If $f \in k[x]$, show that $(a, b) \mapsto \big(a + f(b), b\big)$ is an automorphism of $\mathbb{A}^2$. Under what conditions on f is this an affine transformation?

Exercise 3.8. It follows from Proposition 2.9(i) that for an affine variety Y the coordinate ring $A(Y)$ is an intrinsic object. There is no analogous result for projective varieties. Indeed, if $Y \subset \mathbb{P}^m$ is isomorphic to $Z \subset \mathbb{P}^n$, it is not true, in general, that $k[X_0, \ldots, X_m]/\mathscr{I}(Y)$ is isomorphic to $k[Y_0, \ldots, Y_n]/\mathscr{I}(Z)$. Give an explicit example that demonstrates this.

Exercise 3.9. Formulate and prove the Desargues theorem in $\mathbb{P}^3$ (or even $\mathbb{P}^n$ for $n \geqslant 3$).

Exercise 3.10. Let $P_1, \ldots, P_4$ be four points in general position in $\mathbb{P}^2$, over a field of characteristic $\neq 2$.

(i) If S_2 is the 6-dimensional vector space of homogeneous polynomials in $k[X, Y, Z]$ of degree 2, show that there is a 2-dimensional subspace $V \subset S_2$ of homogeneous forms F such that $\mathscr{Z}(F)$ contains the four points P_i. As $\mathscr{Z}(F)$, for $F \neq 0$, only depends on F up to scalars, this means the conics through the P_i are parametrized by the projective line $\mathbb{P}(V) \cong \mathbb{P}^1$. We say that we have a *pencil of conics* $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ through the given points.

(ii) Prove that there are exactly 3 parameter values $\lambda \in \mathbb{P}^1 = \mathbb{P}(V)$ such that the corresponding conic C_λ is degenerate. Explain this geometrically, without calculation!

(iii) If P_5 is a fifth point, distinct from $P_1, \ldots, P_4$, prove that there is a unique conic through all five points.

CHAPTER 4

Sheaves.

En 53-54, nouveau grand Séminaire sur les fonctions de plusieurs variables complexes. Et d'abord, un travail de "fondation" : Cartan a l'idée de définir la structure d'espace analytique (éventuellement à singularités) par un faisceau, le faisceau des fonctions holomorphes. Cette idée a eu un tel succès, elle a été transposée à tant de situations, qu'elle nous paraît maintenant naturelle, presque banale (bientôt, l'Enseignement Secondaire (*) fera réciter "Qu'est-ce qu'une fonction ? C'est une section du faisceau des germes de fonctions... ").

(*) Primaire ! Primaire ! (interruption de J. Dieudonné)

(From: J-P. Serre, Les Séminaires Cartan; in Serre's Œuvres, Vol. III)

One of the drawbacks of the approach thus far, is that varieties always need to be given as closed subsets of some affine or projective space. This is unnatural, as the enveloping space has no intrinsic meaning. To arrive at a satisfactory general definition of a variety, we need the notion of a sheaf.

In brief, the two key ingredients in the definition of a variety are the underlying topological space, and the notion of what are the "good" (in our context: regular) functions. In the context of affine or projective varieties we have seen that for every open set $U \subset X$ we have an algebra $\mathscr{O}_X(U)$ of regular functions, and whenever we have open sets $V \subset U$ we have a restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$. The notion of a sheaf is just an abstract version of this.

§1. *Sheaves of abelian groups.*

4.1. Definition. Let X be a topological space. A *presheaf $\mathscr{F}$ of abelian groups on X* consists of:

1. for each open $U \subset X$ an abelian group $\mathscr{F}(U)$;
2. for each inclusion of open sets $V \subset U$ a homomorphism $\rho_{U,V} \colon \mathscr{F}(U) \to \mathscr{F}(V)$;

such that $\rho_{U,U}$ is the identity map on $\mathscr{F}(U)$ and, for open sets $W \subset V \subset U$, we have $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The elements of $\mathscr{F}(U)$ are called the *sections* of $\mathscr{F}$ over the open set U. The maps $\rho_{U,V}$ are usually referred to as the *restriction maps*. If there is no risk of confusion, we often write $s|_V$ instead of $\rho_{U,V}(s)$, for $s \in \mathscr{F}(U)$.

4.2. Definition. Let X be a topological space. Then a presheaf $\mathscr{F}$ of abelian groups on X is called a *sheaf* if the following *sheaf axiom* is satisfied: Whenever we have an open cover $\{U_\alpha\}_{\alpha \in I}$ of an open set U and sections $s_\alpha \in \mathscr{F}(U_\alpha)$ such that

$$s_\alpha|_{(U_\alpha \cap U_\beta)} = s_\beta|_{(U_\alpha \cap U_\beta)}$$

for all α, $\beta \in I$, there exists a unique section $s \in \mathscr{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all α.

4.3. Remark. The sheaf axiom involves the *existence* and *uniqueness* of a section. We may separate these aspects; this gives that the sheaf axiom is equivalent to the conjunction of the following two conditions:

(a) If $\{U_\alpha\}_{\alpha\in I}$ is an open cover of an open set U and if we have sections $s_\alpha \in \mathscr{F}(U_\alpha)$ such that

$$s_\alpha|_{(U_\alpha\cap U_\beta)} = s_\beta|_{(U_\alpha\cap U_\beta)}$$

for all α, $\beta \in I$, there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all α.

(b) If $\{U_\alpha\}_{\alpha\in I}$ is an open cover of an open set U and s, $t \in \mathscr{F}(U)$ are sections such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$.

Moreover, as we are considering presheaves of abelian groups, (b) is equivalent to

(b') If $\{U_\alpha\}_{\alpha\in I}$ is an open cover of an open set U and $s \in \mathscr{F}(U)$ is a section such that $s|_{U_\alpha} = 0$ for all α, then $s = 0$.

4.4. Remark. If $\mathscr{F}$ is a sheaf on X then (b) implies that $\mathscr{F}(\emptyset)$ is the trivial group $\{0\}$. (Take $I = \emptyset$, or, to put it differently, consider the empty covering of the empty set.)

4.5. Examples. (i) Let X be a topological space. For an open $U \subset X$, let $C_X(U)$ be the group of continuous $\mathbb{C}$-valued functions on U. If $V \subset U$ are open sets, let $\rho_{U,V}\colon C_X(U) \to C_X(V)$ be the map given by restriction of functions. Then C_X is clearly a presheaf, and one checks without any trouble that it satisfies the sheaf condition. Hence this gives us a sheaf C_X.

(ii) More generally, we may fix some abelian topological group A and consider the sheaf $C_{X,A}$ of A-valued functions. By definition, $C_{X,A}(U)$ is the abelian group of continuous functions $U \to A$ (with pointwise addition as the group law), and $\rho_{U,V}$ is again given by restriction of functions. This is again a sheaf; it is called the sheaf of continuous A-valued functions on X.

Note that if we start with an abelian group A and give it the indiscrete topology, any map $U \to A$ is continuous; so for this choice of a topology on A we get the sheaf of arbitrary A-valued functions. (So: arbitrary functions are special examples of continuous functions!)

(iii) If X is an affine variety, we have defined $\mathscr{O}_X(U)$ to be the algebra of regular functions on U. Let us temporarily forget the algebra structure and only remember the structure on $\mathscr{O}_X(U)$ of an abelian group, by addition of functions. (See §3 for the full story.) Then the $\mathscr{O}_X(U)$ form a sheaf, once again taking $\rho_{U,V}$ the obvious restriction map.

(iv) Fix some abelian group A. For $\emptyset \neq U \subset X$ define $\mathscr{F}(U) = A$, and for $V \subset U$ let $\rho_{U,V}$ be the identity on A. Together with the rule that $\mathscr{F}(\emptyset) = \{0\}$ this defines a presheaf $\mathscr{F}$, called the *constant presheaf* associated with A. This presheaf is usually not a sheaf. The explanation is the following: suppose that X contains two disjoint open sets V and W. Further suppose $\#A > 1$, so that we may choose elements $a \neq b$ in A. Now take $U = V \cup W$. Then $\{V, W\}$ is an open cover of U and the sheaf condition requires that there is a section $s \in \mathscr{F}(U) = A$ that restricts to a on V and to b on W, because these are the same on the intersection $V \cap W = \emptyset$. As there is no such s, we see that $\mathscr{F}$ is not a sheaf.

(v) Fix an abelian group and a point $Q \in X$. Define a presheaf $i_{Q,*}(A)$ on X by the rule that $i_{Q,*}(A)\big(U\big) = A$ if $Q \in U$ and $i_{Q,*}(A)\big(U\big) = \{0\}$ otherwise, with the obvious restruction maps. (The notation will be explained later; see Example 4.15(i).) One readily verifies that $i_{Q,*}(A)$ is indeed a sheaf; it is called the *skyscraper sheaf* at Q with value A.

4.6. Definition. If $\mathscr{F}$ and $\mathscr{G}$ are presheaves on a topological space X, a *homomorphism of presheaves* $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a collection of homomorphisms $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$, for $U \subset X$ open, such that for all open $V \subset U$ the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ {\scriptstyle \rho_{\mathscr{F},U,V}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathscr{G},U,V}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

is commutative.

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, a *homomorphism of sheaves* is the same as a homomorphism of presheaves.

We denote by Ab_X (resp. PreAb_X) the category of sheaves (resp. presheaves) of abelian groups on X. Note that if X is a point, Ab_X is just the category Ab of abelian groups.

4.7. Definition. Let $\mathscr{F}$ be a presheaf or a sheaf on X. If $P \in X$, the *stalk of* $\mathscr{F}$ *at* P is the direct limit of the groups $\mathscr{F}(U)$, for $U \subset X$ open containing P, with respect to the restriction maps $\mathscr{F}(U) \to \mathscr{F}(V)$, for $P \in V \subset U$.

Concretely, this means that an element of $\mathscr{F}_P$ is represented by a pair (U, s) with U an open neighbourhood of P and $s \in \mathscr{F}(U)$, and that two such pairs (U_1, s_1) and (U_2, s_2) are equivalent if and only if there exists an open $V \subset U_1 \cap U_2$ containing P such that $s_1|_V = s_2|_V$.

Note that a homomorphism of (pre)sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ induces homomorphisms on stalks $\varphi_P\colon \mathscr{F}_P \to \mathscr{G}_P$: map the class of (U, s) to the class of $\bigl(U, \varphi(s)\bigr)$.

If $s \in \mathscr{F}(U)$ for some open U and $P \in U$, we denote by $s_P \in \mathscr{F}_P$ the element in the stalk defined by s. This is sometimes called the *germ* of s at P.

4.8. Examples. (i) If X is a quasi-affine variety, the stalk of $\mathscr{O}_X$ at P is precisely the local ring $\mathscr{O}_{X,P}$ introduced in Chapter 2.

(ii) If $\mathscr{F}$ is the constant presheaf associated with a group A, as in Example 4.5(iv), $\mathscr{F}_P = A$ for all $P \in X$.

(iii) If $\mathscr{F} = i_{Q,*}(A)$ is the skyscraper sheaf at Q with value A then $\mathscr{F}_P = A$ if $P \in \overline{\{Q\}}$ and $\mathscr{F}_P = \{0\}$ otherwise. In particular, if Q is a closed point and $A \neq \{0\}$ then Q is the only point at which the stalk is non-trivial; this explains the name "skyscraper sheaf".

The sheaf axiom guarantees that the collection of stalks contain all essential information about a sheaf. The following proposition is a first concrete instance of this. The analogous statement does not hold for presheaves; see Remark 4.12.

4.9. Proposition. *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves on X. Then φ is an isomorphism if and only if all homomorphisms $\varphi_P\colon \mathscr{F}_P \to \mathscr{G}_P$ are isomorphisms.*

For the proof we refer to HAG, Chap II, Proposition 1.1.

4.10. Proposition. *Let $\mathscr{F}$ be a presheaf on a topological space X. Then there exists a sheaf $\mathscr{F}^+$ and a homomorphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ that is universal for homomorphisms from $\mathscr{F}$ to a sheaf; by this we mean that for any sheaf $\mathscr{G}$ on X and any homomorphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, there is a unique homomorphism $\psi\colon \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$.*

Proof. Let F be the disjoint union of all stalks $\mathscr{F}_P$; so: $F = \coprod_{P\in X} \mathscr{F}_P$. Let $\pi\colon F \to X$ be the map that sends an element of $\mathscr{F}_P$ to P. By definition, a section of the map π is a map $s\colon X \to F$ such that $\pi \circ s = \mathrm{id}_X$; in our case this just means that $s(P) \in \mathscr{F}_P \subset F$ for all P. More generally, if $U \subset X$ then by a section of π on U we mean a map $s\colon U \to F$ such that $\pi \circ s = \mathrm{id}_U$.

Let us call a section $s\colon U \to F$ a "good section" if for every $P \in U$ there is an open $V \subset U$ containing P and a section $t \in \mathscr{F}(V)$ such that $s(Q) = t_Q$ for all $Q \in V$. (Recall that $t_Q \in \mathscr{F}_Q \subset F$ is the germ of t at Q.) If $U' \subset U$ and $s\colon U \to F$ is a good section, the restriction of s to U' is again a good section; hence we obtain a presheaf $\mathscr{F}^+$ by

$$\mathscr{F}^+(U) = \bigl\{\text{good sections } s\colon U \to F\bigr\}\,.$$

(Note that $\mathscr{F}^+(U)$ is an abelian group via pointwise addition of sections.)

As the good sections are defined by a condition that is of a local nature, one verifies without any trouble that $\mathscr{F}^+$ is a sheaf. We have a morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$, sending a section $t \in \mathscr{F}(U)$ to the good section $P \mapsto t_P$. Now let $\mathscr{G}$ be a sheaf and $\varphi\colon \mathscr{F} \to \mathscr{G}$ a homomorphism. If $s \in \mathscr{F}^+(U)$, there exist an open covering $U = \cup V_\alpha$ plus sections $t_\alpha \in \mathscr{F}(V_\alpha)$ such that $s|_{V_\alpha} = \theta(t_\alpha)$ for all α. Writing $V_{\alpha\beta} = V_\alpha \cap V_\beta$, the images $\varphi(t_\alpha)$ satisfy $\varphi(t_\alpha)|_{V_{\alpha\beta}} = \varphi(t_\beta)|_{V_{\alpha\beta}}$ for all α and β, because they agree on stalks; see Exercise 4.1. Hence the sheaf property of $\mathscr{G}$ gives us a unique section $\sigma \in \mathscr{G}(U)$ such that $\sigma|_{V_\alpha} = \varphi(t_\alpha)$ for all α. Moreover, this σ is independent of any choices becasue it is characterized by the propery that $\sigma_P = \varphi_P(s(P))$ for all P. (Again use Exercise 4.1.) It is immediate that the map $\psi\colon \mathscr{F}^+ \to \mathscr{G}$ given by $s \mapsto \sigma$ is a homomorphism, that $\psi \circ \theta = \varphi$, and that ψ is the only possible homomorphism with this property. □

4.11. Definition. The sheaf $\mathscr{F}^+$ associated to the presheaf $\mathscr{F}$ is called the *sheafification* of $\mathscr{F}$, or the sheaf associated to $\mathscr{F}$ by sheafification.

The universal property of the pair $(\mathscr{F}^+, \theta)$ implies its uniqueness; hence we can unambiguously call $\mathscr{F}^+$ *the* sheaf associated to $\mathscr{F}$. Associating $\mathscr{F}^+$ to $\mathscr{F}$ gives a functor $\mathsf{PreAb}_X \to \mathsf{Ab}_X$. (For those familiar with category theory: this functor is left adjoint to the inclusion functor $\mathsf{Ab}_X \to \mathsf{PreAb}_X$; this is no more than a reformulation of the universal property of the associated sheaf.)

4.12. Remark. Taking a closer look at proof of the proposition, we find that the induced homomorphisms on stalks $\theta_P\colon \mathscr{F}_P \to \mathscr{F}^+_P$ are isomorphisms. In particular, we see that Proposition 4.9 has no analogue for presheaves.

4.13. Example. Let A be an abelian group and consider the constant presheaf $\mathscr{F}$ on X associated to A, as in Example 4.5(iv). The associated sheaf is called the *constant sheaf* associated to A; we denote it by A_X. It is given by

$$A_X(U) = \bigl\{\text{locally constant maps } U \to A\bigr\}\,.$$

Here we recall that a map $s\colon U \to A$ is said to be locally constant if every $P \in U$ has an open neighbourhood on which s is constant. This is the same as saying that s is continuous when we equip A with the discrete topology.

§2. *Operations on sheaves.*

4.14. Definition. Let $f\colon X \to Y$ be a continuous map of topological spaces.

(i) If $\mathscr{F}$ is a sheaf on X, we define the *push-forward sheaf* $f_*\mathscr{F}$ on Y by the rule $f_*\mathscr{F}(V) = \mathscr{F}\big(f^{-1}(V)\big)$, with the obvious restriction maps.

(ii) If $\mathscr{G}$ is a sheaf on Y, we define the *inverse image sheaf* $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf given by the rule that $f^{-1}\mathscr{G}(U)$ is the direct limit of the groups $\mathscr{G}(V)$, where V runs over the open sets of Y that contain $f(U)$. Concretely, an element of $f^{-1}\mathscr{G}(U)$ is given by a pair (V,t) with $f(U) \subset V$ and $t \in \mathscr{G}(V)$, and two such pairs (V_1,t_1) and (V_2,t_2) are equivalent if and only if there exists an open $W \subset V_1 \cap V_2$ containing $f(U)$ such that $t_1|_W = t_2|_W$. The restriction maps for $f^{-1}\mathscr{G}$ are again the obvious ones.

Implicit in the definition is the assertion that $f_*\mathscr{F}$ is again a sheaf (rather than just a presheaf); this is easy to verify and we omit the details. For $f^{-1}\mathscr{G}$ the sheafification is needed, in general. (For instance, what happens if Y is a single point?) One case that is of particular interest is when $i\colon U \hookrightarrow X$ is the inclusion map of an open subset. In this case, given a sheaf $\mathscr{F}$ on X we write $\mathscr{F}|_U = i^{-1}\mathscr{F}$; it is the sheaf given by $\mathscr{F}|_U(V) = \mathscr{F}(V)$, for V open in U (and hence open in X).

4.15. Examples. (i) Given a point $Q \in X$, write $i_Q\colon \{Q\} \hookrightarrow X$ for the inclusion map. If A is an abelian group, the push-forward under i_Q of the constant sheaf A on the set $\{Q\}$ is the skyscraper sheaf $i_{Q,*}(A)$ introduced in Example 4.5(v). On the other hand, given a sheaf $\mathscr{F}$ on X, the inverse image sheaf $i_Q^{-1}\mathscr{F}$ is simply the stalk $\mathscr{F}_Q$, now considered as a sheaf on the 1-point space $\{Q\}$.

(ii) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, there are the obvious relations $(gf)_* = g_* \circ f_*$ and $(gf)^{-1} = f^{-1} \circ g^{-1}$. Combining this with the previous example, it follows that the stalk of $f^{-1}\mathscr{G}$ at a point $P \in X$ is isomorphic to the stalk $\mathscr{G}_{f(P)}$ at the image of P. There is not a similarly simple rule for the stalks of the push-forward of a sheaf. To give an idea of what such stalks mean, consider the case where f is a morphism of quasi-affine varieties. Let $Q \in Y$ and assume, for simplicity, that the pre-image $Z = f^{-1}\{Q\}$ is irreducible, so that Z is a subvariety of X. Then $(f_*\mathscr{O}_X)_Q$ is the local ring $\mathscr{O}_{X,Z}$ of X along Z that was introduced in Exercise 2.2.

4.16. Let $f\colon X \to Y$ be a continuous map. Let $\mathscr{F}$ be a sheaf on X and $\mathscr{G}$ a sheaf on Y. Then $f^{-1}f_*\mathscr{F}$ is the sheaf associated to the presheaf whose group of sections on an open set U is the direct limit of the groups $\mathscr{F}\big(f^{-1}(V)\big)$, where V runs over the open sets of Y that contain $f(U)$. In particular, for each such V we have $U \subset f^{-1}(V)$; hence the restriction of sections defines a natural homomorphism $\alpha\colon f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

Similarly, $f_*f^{-1}\mathscr{G}$ is the sheaf associated to the presheaf whose group of sections on an open $V \subset Y$ is the direct limit of the groups $\mathscr{G}(W)$, where W runs over the open sets of Y that contain $f\big(f^{-1}(V)\big) = V \cap \mathrm{Im}(f)$. In this case we obtain a natural homomorphism $\beta\colon \mathscr{G} \to f_*f^{-1}\mathscr{G}$.

4.17. Proposition. *Let $f\colon X \to Y$ be a continuous map. If $\mathscr{F}$ is a sheaf on X and $\mathscr{G}$ is a sheaf on Y, the map $\mathrm{Hom}_{\mathsf{Ab}_X}\big(f^{-1}\mathscr{G},\mathscr{F}\big) \to \mathrm{Hom}_{\mathsf{Ab}_Y}\big(\mathscr{G},f_*\mathscr{F}\big)$ given by $\varphi \mapsto f_*(\varphi)\circ\beta$ is an isomorphism, with inverse given by $\psi \mapsto \alpha \circ f^{-1}(\psi)$.*

The details of the proof are left to the reader.

4.18. Definition. If $\mathscr{F}$ is a sheaf on X, a *subsheaf* of $\mathscr{F}$ is a sheaf $\mathscr{K}$ such that $\mathscr{K}(U)$ is a subgroup of $\mathscr{F}(U)$ for all open U, and such that the restriction map $\rho_{\mathscr{K},U,V}\colon \mathscr{K}(U) \to \mathscr{K}(V)$ is just the restriction of $\rho_{\mathscr{F},U,V}$ to the subgroup $\mathscr{K}(U) \subset \mathscr{F}(U)$.

If $\mathscr{K} \subset \mathscr{F}$ is a subsheaf, we define the *quotient sheaf* $\mathscr{F}/\mathscr{K}$ to be the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{K}(U)$.

4.19. Definition. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves of abelian groups.

(i) The *kernel* of φ is the subsheaf $\mathrm{Ker}(\varphi) \subset \mathscr{F}$ given by

$$\mathrm{Ker}(\varphi)\bigl(U\bigr) = \mathrm{Ker}\bigl(\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)\bigr)\,.$$

(ii) The *image* of φ is the sheaf $\mathrm{Im}(\varphi)$ associated to the presheaf given by

$$U \mapsto \mathrm{Im}\bigl(\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)\bigr)\,.$$

One readily verifies that $\mathrm{Ker}(\varphi)$ is indeed again a sheaf. For the image, the sheafification is in general really necessary. The image sheaf is a subsheaf of $\mathscr{G}$.

4.20. Proposition. *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves of abelian groups.*

(i) *The following conditions are equivalent:*
  (a) $\mathrm{Ker}(\varphi)$ *is the zero sheaf;*
  (b) φ_P *is injective for all* $P \in X$*;*
  (c) $\varphi(U)$ *is injective for all open* $U \subset X$*.*

(ii) *The following conditions are equivalent:*
  (a) $\mathrm{Im}(\varphi) = \mathscr{G}$*;*
  (b) φ_P *is surjective for all* $P \in X$*.*

If the equivalent conditions in (i) are satisfied, φ is called an injective homomorphism. If the conditions in (ii) are satisfied, φ is said to be surjective. Note that surjectivity does *not* imply that the maps $\varphi(U)$ are always surjective; see Example 4.22.

4.21. Remark. The category Ab_X of sheaves of abelian groups is an abelian category. Without going into full detail, the most important aspects of this are:

1. All sets $\mathrm{Hom}_{\mathsf{Ab}_X}(\mathscr{F},\mathscr{G})$ are abelian groups.
2. In Ab_X we can form sums and products.
3. The notions of a kernel, image and quotient exist.
4. For any homomorphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ we have an isomorphism $\mathscr{F}/\mathrm{Ker}(\varphi) \xrightarrow{\sim} \mathrm{Im}(\varphi)$.

The notions of a kernel and image can in fact be defined in a purely categorical way. Instead, we have given direct definitions. It can be shown that the kernels and images we have defined are indeed kernels and images in the categorical sense. Further, the notions of injectivity and surjectivity of a homomorphism, as we have defined them, agree with their categorical counterparts (monomorphism and epimorphism).

4.22. Example. Let $X = \mathbb{C} \setminus \{0\}$ with the Euclidean topology. Let $\mathscr{O}_X$ be the sheaf of holomorphic functions on X. Let $\mathscr{O}_X^*$ denote the sheaf of nowhere vanishing holomorphic functions;

the group structure for this sheaf is the multiplication of functions. Then the exponential map defines a homomorphism of sheaves exp: $\mathscr{O}_X \to \mathscr{O}_X^*$. The kernel of this homomorphism is the constant sheaf $(2\pi i\mathbb{Z})_X \subset \mathscr{O}_X$ of locally constant functions with values in $2\pi i\mathbb{Z}$.

The homomorphism exp is surjective; indeed, if g is a nowhere vanishing holomorphic function on a non-empty open set U then for every $P \in U$ we can find a small open ball $B(P,\varepsilon) \subset U$ on which a logarithm $\log(g)$ is defined as a holomorphic function. But on global sections, exp is certainly not surjective; indeed, the "tautological" function z (sending $z \in X$ to itself as a complex number) is nowhere vanishing (because we have removed 0) but there is no globally defined function $\log(z)$.

The short exact sequence

$$0 \longrightarrow 2\pi i\mathbb{Z}_X \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^* \longrightarrow 0$$

is called the exponential sequence on X and turns out to be extremely useful. Note that we can only define it in an analytic setting, working with holomorphic functions; there is no immediate analogue in a purely algebraic setting.

4.23. Glueing sheaves. Suppose $X = \cup_{i=1}^N U_i$ is a finite open cover of a topological space X. (The finiteness is included only for simplicity of exposition and is not essential.) Suppose further that on each U_i we are given a sheaf $\mathscr{F}_i$ and that we are given isomorphisms

$$\varphi_{ij}\colon \mathscr{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_j|_{U_{ij}}$$

such that φ_{ii} is the identity on $\mathscr{F}_i$ and for all indices i, j and k we have

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}\colon \mathscr{F}_i|_{U_{ijk}} \xrightarrow{\sim} \mathscr{F}_k|_{U_{ijk}}\,.$$

(As usual, $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Further note that in the glueing condition, structly speaking we should write $\varphi_{ik}|_{U_{ijk}}$ etc.) We may then glue the sheaves $\mathscr{F}_i$ into a single sheaf $\mathscr{F}$ with the property that $\mathscr{F}|_{U_i} = \mathscr{F}_i$ for all i.

A direct definition of $\mathscr{F}$ is that, for $V \subset X$ open,

$$\mathscr{F}(V) = \Big\{(s_1,\ldots,s_N) \in \prod_{i=1}^N \mathscr{F}(V\cap U_i) \;\Big|\; \varphi_{ij}(s_i|_{V\cap U_{ij}}) = s_j|_{V\cap U_{ij}}) \quad \text{for all } i,\, j\Big\}\,.$$

§3. *Other types of sheaves.*

So far we have only considered sheaves of abelian groups, as they form a natural starting point for the theory of sheaves. We may, however, also look at other types of sheaves, where the sections $\mathscr{F}(U)$ carry a different structure.

4.24. Let us begin with an example where we have less structure: (pre)sheaves of sets. In the definition we now only require that each $\mathscr{F}(U)$ is a set, and the restriction maps $\mathscr{F}(U) \to \mathscr{F}(V)$ only need to be maps of sets, without further constraints. Note that in the formulation of the sheaf axiom, the alternative version (b') of Remark 4.3 is no longer strong enough. Further, if $\mathscr{F}$ is a sheaf we have $\mathscr{F}(\emptyset) = \{*\}$, the 1-point set.

A typical example of a sheaf of sets is the following. Suppose $\pi\colon F \to X$ is a continuous map of topological spaces. Recall that, for $U \subset X$, a map $s\colon U \to F$ is called a section of π (over U) if $\pi \circ s = \mathrm{id}_U$. Then one readily verifies that

$$\mathscr{F}(U) = \bigl\{\text{continuous sections } s\colon U \to F\bigr\}$$

defines a sheaf of sets on X. A remarkable fact is that every sheaf of sets on X is isomorphic to a sheaf $\mathscr{F}$ obtained in this way. (This example maybe helps to explain why we talk about the sections of a sheaf.) See for instance HAG, Chap. II, Exercise 1.13 or (better), Godement [4], Chap. II, §1.

The category of sheaves of sets is not abelian. We can not add homomorphisms, and there are no notions like kernels. The reason, of course, is that the category of sets (which is the category of sheaves of sets on a point) itself is not abelian.

4.25. We may equally well work with sheaves that carry more structure, such as sheaves of R-modules, for some ring R, or sheaves of k-algebras, for some ring k. What we have discussed goes through in this setting without significant changes.

4.26. A next generalization does require some further comments. The situation we consider is that we have a topological space X equipped with a sheaf $\mathscr{O}_X$ of k-algebras, for some field k. The example we have in mind is the sheaf of regular functions on an algebraic variety. In this setting, a *sheaf of $\mathscr{O}_X$-modules*, often just called an $\mathscr{O}_X$-module, is a sheaf of abelian groups $\mathscr{M}$, such that each group $\mathscr{M}(U)$ has the structure of a module over $\mathscr{O}_X(U)$, such that the restriction maps of $\mathscr{M}$ are compatible with the module structures. By this last condition we mean that for all open sets $V \subset U$ and sections $f \in \mathscr{O}_X(U)$ and $m \in \mathscr{M}(U)$, we have $(f \cdot m)|_V = f|_V \cdot m|_V$.

A homomorphism of $\mathscr{O}_X$-modules $\varphi\colon \mathscr{M} \to \mathscr{N}$ is a homomorphism of sheaves of abelian groups with the additional property that $\varphi(f \cdot m) = f \cdot \varphi(m)$ for all local sections $f \in \mathscr{O}_X(U)$ and $m \in \mathscr{M}(U)$.

The category $\mathrm{Mod}_{\mathscr{O}_X}$ of sheaves of $\mathscr{O}_X$-modules is again an abelian category. The kernel and image of a $\varphi\colon \mathscr{M} \to \mathscr{N}$ are the same as before; they are naturally sub-$\mathscr{O}_X$-modules of $\mathscr{M}$ and $\mathscr{N}$, respectively.

Exercises for Chapter 4.

Exercise 4.1. Let $\mathscr{F}$ be a sheaf on a topological space X. If $s, t \in \mathscr{F}(X)$ and $s_P = t_P$ for all $P \in X$, prove that $s = t$.

Exercise 4.2. Prove that the constant sheaf A_X described in Example 4.13 is indeed the sheaf associated to the constant presheaf of Example 4.5(iv).

Exercise 4.3. Let $\Gamma\colon \mathsf{Ab}_X \to \mathsf{Ab}$ be the "global sections functor", given by $\Gamma(\mathscr{F}) = \mathscr{F}(X)$. Prove that this is a left exact functor; this means that for any short exact sequence $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$, application of the functor Γ gives an exact sequence

$$0 \longrightarrow \Gamma(\mathscr{F}') \longrightarrow \Gamma(\mathscr{F}) \longrightarrow \Gamma(\mathscr{F}'')\,.$$

(As discussed in Example 4.22, Γ is not right exact, in general.)

Exercise 4.4. Let Y be a closed irreducible subset of an affine variety X. Write $i\colon Y \hookrightarrow X$ for the inclusion map. Show that the natural homomorphism of sheaves $i^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ is surjective.

CHAPTER 5

Algebraic varieties.

In this chapter we define abstract algebraic varieties in an intrinsic way, using the language of sheaves.

§1. *Pre-varieties.*

5.1. Given a topological space X, let $\mathscr{F}_X$ denote the sheaf of (arbitrary) k-valued functions on X. Note that if $\varphi\colon X \to Y$ is a continuous map, φ induces a homomorphism of sheaves $\varphi^*\colon \mathscr{F}_Y \to \varphi_*\mathscr{F}_X$ on Y.

5.2. Definition. By a *k-space* we mean a pair $(X, \mathscr{O}_X)$ consisting of

1. a topological space X;
2. a sheaf of k-algebras $\mathscr{O}_X$ that is a subsheaf of $\mathscr{F}_X$.

If $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are k-spaces, a morphism $\varphi\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $\varphi\colon X \to Y$ such that the induced homomorphism $\mathscr{F}_Y \to \varphi_*\mathscr{F}_X$ maps the subsheaf $\mathscr{O}_Y \subset \mathscr{F}_Y$ into the subsheaf $\varphi_*\mathscr{O}_X \subset \varphi_*\mathscr{F}_X$. Concretely, this last condition means that for every open $V \subset Y$ and every $f \in \mathscr{O}_Y(V)$, the function $f \circ \varphi\colon \varphi^{-1}(V) \to k$ is a section of $\mathscr{O}_X\bigl(\varphi^{-1}(V)\bigr)$.

In practice we often drop the sheaf from the notation; if there is no risk of confusion we denote a k-space by a single letter, and we write morphisms simply as maps $\varphi\colon X \to Y$.

5.3. Example. Any (quasi-)affine or (quasi-)projective variety defines a k-space. Further, if X and Y are such varieties then the morphisms from X to Y are precisely the morphisms as k-spaces. We call a k-space a (quasi-)affine or (quasi-)projective variety if it is isomorphic, as a k-space, to such a variety.

5.4. Definition. (i) A *pre-variety* over k is an irreducible k-space $(X, \mathscr{O}_X)$ such that there exists a finite covering $X = \cup_{i=1}^m U_i$ with the property that each $(U_i, \mathscr{O}_X|_{U_i})$ is an affine variety. If $(X, \mathscr{O}_X)$ is a pre-variety, the sections of $\mathscr{O}_X$ are called the regular functions and $\mathscr{O}_X$ is called the sheaf of regular functions on X, or also the *structure sheaf*.

(ii) By a morphism of pre-varieties $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ we mean a morphism of k-spaces.

(iii) The category of pre-varieties over k is denoted by PreVar_k.

5.5. Remark. On an affine variety the affine open subsets form a basis for the topology. This implies that on any pre-variety the affine open sets form a basis. There are many results that are of a local nature, for which we can reduce the proof to the case of an affine variety. Here are some examples.

(i) For U a non-empty open subset of a pre-variety X, the pair $(U, \mathscr{O}_X|_U)$ is again a pre-variety.

(ii) Regular functions are continuous.

(iii) Let X be a pre-variety and let g, $h \in \mathscr{O}(X)$ be regular functions on X with $h(P) \neq 0$ for all $P \in X$. Then also the function g/h is regular on X.

5.6. Constructing pre-varieties. Suppose we have a set X and a (set-theoretic) cover $X = \cup_{i=1}^N U_i$. Suppose further that each U_i has the structure of an affine variety. Under some "glueing conditions" we can then glue the affine varieties U_i and obtain the structure of a pre-variety on X.

For each pair of indices i and j we have $U_i \supset U_{ij} \subset U_j$. The first assumption we make is that each U_{ij} is non-empty and open both in U_i and in U_j. This gives two structures of a quasi-affine variety on U_{ij}; the one inherited from U_i, the other inherited from U_j. Our second assumption is that these structures are the same. This means that the topology on U_{ij} induced by the topology on U_i is the same as the topology induced by that on U_j, and that for $V \subset U_{ij}$ open, $\mathscr{O}_{U_i}(V) = \mathscr{O}_{U_j}(V)$ as sub-algebras of the algebra of all k-valued functions on V.

Under these assumptions we obtain a topology on X, determined by the rule that a subset $V \subset X$ is open if and only if $V \cap U_i$ is open in U_i for all i. Further we obtain a subsheaf $\mathscr{O}_X \subset \mathscr{F}_X$ determined by the rule that, for $V \subset X$ open, a function $f\colon V \to k$ is a section of $\mathscr{O}_X(V)$ if and only if $f|_{V\cap U_i} \in \mathscr{O}_{U_i}(V \cap U_i)$ for all i. By construction, the induced topology on $U_i \subset X$ is the topology we started out with, and $\mathscr{O}_X|_{U_i}$ is the structure sheaf of U_i; also, X is irreducible. (See Exercise 5.2.) Hence $(X, \mathscr{O}_X)$ is a pre-variety. (Note that the assumptions we have made imply that $\mathscr{O}_{U_i}|_{U_{ij}} = \mathscr{O}_{U_j}|_{U_{ij}}$ as subsheaves of $\mathscr{F}_{U_{ij}}$; this means we can glue the sheaves $\mathscr{O}_{U_i}$ following the procedure of 4.23, and the sheaf we get is $\mathscr{O}_X$.)

If Y is any other pre-variety, to give a morphism $\varphi\colon X \to Y$ is the same as giving morphisms $\varphi_i\colon U_i \to Y$ such that $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$ for all i and j. A map $\psi\colon Y \to X$ is a morphism if and only if each of the induced maps $\psi^{-1}(U_i) \to U_i$ is a morphism.

5.7. Proposition. *In the category* PreVar_k *there exist products. If X and Y are quasi-affine varieties, their product in the category* PreVar_k *is the same as their product as quasi-affine varieties.*

We first show that if X and Y are affine, the product variety $X \times Y$ with its projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ as in Chapter 2 is a product in the larger category of pre-varieties. For this, let T be any pre-variety and suppose given morphisms $\varphi\colon T \to X$ and $\psi\colon T \to Y$. Then $h\colon T \to X \times Y$ given by $P \mapsto \big(\varphi(P), \psi(P)\big)$ is the unique set-theoretic map for which $p \circ h = \varphi$ and $q \circ h = \psi$. What we have to show is that h is a morphism. So we must show that for any open $V \subset X \times Y$ and any $f \in \mathscr{O}_{X\times Y}(V)$, the pre-image $h^{-1}(V) \subset T$ is open and $f \circ h$ is a regular function on $h^{-1}(V)$. But T can be written as a union of finitely many affine open subsets, say $T = U_1 \cup \cdots \cup U_m$. Because we already know that $X \times Y$ is a product in the category of affine varieties, $h^{-1}(V) \cap U_i$ is open in U_i for all i and $f \circ h|_{U_i}$ is a regular function on U_i. It follows that $h^{-1}(V)$ is open and $f \circ h$ is a regular.

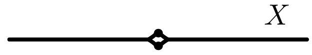
Next consider two arbitrary pre-varieties X and Y. Choose finite open coverings by affine varieties $X = U_1 \cup \cdots \cup U_m$ and $Y = V_1 \cup \cdots \cup V_n$. We can now apply the construction procedure of 5.6: the product set $X \times Y$ is the union of the sets $U_i \times V_j$, these all have the structure of affine varieties, and the intersections $(U_i \times V_j) \cap (U_p \times V_q) = (U_i \cap U_p) \times (V_j \cap V_q)$ have their natural structure of quasi-affine variety. Hence we obtain a pre-variety $(X \times Y, \mathscr{O}_{X\times Y})$ such that the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are morphisms.

To prove that $X \times Y$ is a product of X and Y, consider a pre-variety T and morphisms $\varphi\colon T \to X$ and $\psi\colon T \to Y$. Then again $h\colon T \to X \times Y$ given by $P \mapsto \big(\varphi(P), \psi(P)\big)$ is the unique set-theoretic map for which $p \circ h = \varphi$ and $q \circ h = \psi$, and we have to show is that h is a morphism. As remarked in 5.6, for this it suffices to remark that each of the induced maps $h^{-1}(U_i \times V_j) \to U_i \times V_j$ is a morphism, and this is true because we have already proved that $U_i \times V_j$ is a product in the category of pre-varieties.

§2. *Algebraic varieties.*

5.8. If X is a pre-variety we denote by $\Delta\colon X \to X \times X$ the diagonal morphism, given by $P \mapsto (P, P)$. It is the unique morphism such that $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X = \mathrm{pr}_2 \circ \Delta$.

The image $\Delta(X)$ is in general not closed in $X \times X$. The standard example showing this is the "affine line with doubled origin"; it works as follows. Start with two copies of the affine line, say $Y_1 = Y_2 = \mathbb{A}^1$. Let $U_i = Y_i \setminus \{0\}$, for $i = 1, 2$. Now glue the open parts U_1 and U_2, using the identity map $U_1 \to U_2$ as glueing map. (If, instead, we use the map given by $x \mapsto x^{-1}$ we get the projective line.) Let X be the pre-variety obtained in this way.



As a topological space, X is just the union of $\mathbb{A}^1 \setminus \{0\}$ with two points 0_1 and 0_2, equipped with the co-finite topology.

The product pre-variety $X^2 = X \times X$ is obtained by glueing the four affine varieties $Y_i \times Y_j$ for $(i, j) \in \{1, 2\}^2$. (These are four copies of $\mathbb{A}^2$.) To help understanding the result, consider the natural morphism $X \to \mathbb{A}^1$. On the product it gives a morphism $\pi\colon X^2 \to \mathbb{A}^2$. Over the complement of $\bigl(\{0\} \times \mathbb{A}^1\bigr) \cup \bigl(\mathbb{A}^1 \times \{0\}\bigr)$ this morphism is an isomorphism. The x-axis and y-axis are doubled in X^2, but above the origin in $\mathbb{A}^2$ we have four points $(0_i, 0_j)$.

Away from the origin, $\Delta(X) \subset X^2$ is the usual diagonal; further two of the four origins are in $\Delta(X)$, namely $(0_1, 0_1)$ and $(0_2, 0_2)$. However, all four origins are in the closure of $\Delta(X)$.

5.9. Definition. An *algebraic variety over* k is a pre-variety $(X, \mathscr{O}_X)$ over k for which the diagonal $\Delta(X) \subset X \times X$ is closed.

We shall usually drop the adjective "algebraic" and just talk about varieties. We denote by Var_k the category of algebraic varieties over k.

Note that the requirement that the diagonal is closed is a global property of X, not a local property. In the example in 5.8, every point has an open neighbourhood isomorphic to the affine line, so the X considered there is locally a variety, but globally not.

5.10. Remark. A topological space X is a Hausdorff space if and only if the diagonal $\Delta(X) \subset X \times X$ is closed. The condition in Definition 5.9 is therefore an algebraic analogue of the Hausdorff property, even though varieties (other than single points) are not Hausdorff spaces.

The closedness of the diagonal can also be stated in terms of morphisms. Namely, if X is a pre-variety then the diagonal $\Delta(X)$ is closed (and hence: X is a variety) if and only if for any pre-variety T and morphisms f, $g\colon T \to X$ the set

$$E(f, g) = \bigl\{t \in T \bigm| f(t) = g(t)\bigr\}$$

is closed in T. Indeed, the "only if" follows from the remark that the pair (f, g) defines a morphism $T \to X^2$ and that $E(f, g)$ is the inverse image of $\Delta(X)$, while for the "if" we take $T = X \times X$ and remark that $E(\mathrm{pr}_1, \mathrm{pr}_2) = \Delta(X)$.

5.11. We shall now extend to the general setting of varieties some notions that we have already encountered for quasi-affine varieties. We begin by defining subvarieties and immersions.

If X is a variety and $U \subset X$ is a non-empty subset, we have already remarked in 5.5 that U is again a pre-variety. But $\Delta(U) = (U \times U) \cap \Delta(X)$; hence U is in fact a variety. We say that U, with its structure of a variety, is an *open subvariety* of X.

Next consider an irreducible closed subset $Y \subset X$. Writing $i\colon Y \hookrightarrow X$, let $\mathscr{O}_Y$ be the image of the natural homomorphism of sheaves $i^{-1}\mathscr{O}_X \to \mathscr{F}_Y$. Concretely, if $U \subset Y$ is open and $f\colon U \to k$ is a function, f is a section in $\mathscr{O}_Y(U)$ if and only if for every $P \in U$ there exists an open $V \subset X$ containing P and a regular function $\tilde{f}\colon V \to k$ such that $f|_{Y\cap V} = \tilde{f}|_{Y\cap V}$. If X is affine, it follows from Exercise 4.4 that $(Y, \mathscr{O}_Y)$ gives the usual structure of an affine variety on Y. Hence, back to the general case, Y is a pre-variety. But $\Delta(Y) = (Y \times Y) \cap \Delta(X)$; so Y is in fact a variety. We say that Y, with its structure of a variety, is a *closed subvariety* of X.

Finally, by a *subvariety* of X we mean a locally closed subvariety, i.e., an open subvariety of a closed subvariety of X.

A morphism of varieties $Y \to X$ is called an immersion (resp. open immersion, resp. closed immersion) if it induces an isomorphism from Y to a subvariety (resp. open subvariety, resp. closed subvariety) of X.

5.12. Definition. The function field $k(X)$ of a variety X is the direct limit of the k-algebras $\mathscr{O}_X(U)$, where U runs through the set of non-empty open subsets of X.

If U is any non-empty open subset of X, viewed as an open subvariety, we have $k(U) = k(X)$.

§3. *Birational geometry.*

5.13. Definition. Let X and Y be varieties. Then a *rational map* $\varphi\colon X \dashrightarrow Y$ is an equivalence class of pairs (U, φ_U) where U is a non-empty open subset of X and $\varphi_U\colon U \to Y$ is a morphism, and where two such pairs (U, φ_U) and (V, φ_V) are equivalent if φ_U and φ_V agree on $U \cap V$.

5.14. Remarks. (i) By what was remarked in 5.10, in order for (U, φ_U) and (V, φ_V) to be equivalent, it suffices that φ_U and φ_V are the same on some non-empty open subset of $U \cap V$, because any such set is dense in $U \cap V$.

(ii) For a rational map $\varphi\colon X \dashrightarrow Y$ there is always a maximal open subset $U \subset X$, called the domain of definition of φ, on which φ is defined as a morphism. After all, if (U, φ_U) and (V, φ_V) are equivalent, φ_U and φ_V define a morphism $\varphi\colon (U \cup V) \to Y$. As varieties are noetherian, this process of enlarging the domain cannot go on indefinitely and therefore gives us a maximal open subset on which φ is a morphism.

(iii) Suppose the rational map $\varphi\colon X \dashrightarrow Y$ is represented by the pair (U, φ_U). If $\varphi_U(U) \subset Y$ is dense in Y then for every non-empty open $V \subset U$ the image $\varphi_U(V)$ is dense in Y, too. (Use that V is dense in U.)

5.15. Definition. A rational map $\varphi\colon X \dashrightarrow Y$ is *dominant* if for some representative (U, φ_U) the image $\varphi_U(U)$ is dense in Y.

By Remark 5.14(iii), this is in fact independent of the chosen representative.

Observe that dominant rational maps can be composed: If $\varphi\colon X \dashrightarrow Y$ is represented by (U, φ_U) and $\psi\colon Y \dashrightarrow Z$ is represented by (V, ψ_V) then $\varphi_U^{-1}(V)$ is a non-empty open subset of X and $\psi \circ \varphi\colon X \dashrightarrow Z$ is the rational map represented by $\bigl(\varphi_U^{-1}(V), \psi_V \circ \varphi_U\bigr)$. One checks without difficulty that this does not depend on the chosen representatives. Also, $\psi \circ \varphi$ is again dominant. Hence we obtain a category $\mathsf{Var}_k^{\mathrm{birat}}$ with as objects algebraic varieties and as morphisms dominant rational maps. The isomorphisms in this category are called birational maps:

5.16. Definition. (i) A dominant rational map $\varphi\colon X \dashrightarrow Y$ is called a *birational map* if there exists a rational map $\psi\colon Y \dashrightarrow X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ as rational maps.

(ii) Two varieties are said to be *birationally equivalent* if there exists a birational map between them.

It readily follows from the definitions that X and Y are birationally equivalent if and only if there exist non-empty open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$ as varieties. See Exercise 5.4.

5.17. Example. The varieties $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent because they both have $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ as an open subset. However, it can be shown that there are no (everywhere defined) morphisms between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ that are birational, in either direction.

A variety that is rationally equivalent to $\mathbb{A}^n$ for some n is called a *rational variety*. Examples of such include projective spaces, conics in $\mathbb{P}^2$, and products of such. The rationalty means that there is an open subset that we can identify with an open part of $\mathbb{A}^n$ and on which we therefore have some kind of "nice coordinates". But the rational varieties form a rather special class. For instance, we shall see later that a (non-singular) curve $C \subset \mathbb{P}^2$ defined by an irreducible homogeneous polynomial of degree d is rational only for $d = 1$ or $d = 2$. Similarly, a (non-singular) surface $S \subset \mathbb{P}^3$ defined by an irreducible polynomial of degree d is rational only for $d \leqslant 3$.

5.18. Example. Let X be a variety. Just as morphisms $X \to \mathbb{A}^1$ are the same as regular functions $X \to k$ (see Exercise 5.1), rational maps $X \dashrightarrow \mathbb{A}^1$ are the same as elements of the function field $k(X)$.

5.19. Let $\varphi\colon X \dashrightarrow Y$ be a dominant rational map, represented by a pair (U, φ_U). If $f = [V, f]$ is an element of the function field $k(Y)$, the pair $\bigl(\varphi_U^{-1}(V), f \circ \varphi_U\bigr)$ defines an element $\varphi^*(f) \in k(X)$, which is independent of choices. The map $\varphi^*\colon k(Y) \to k(X)$ is a homomorphism of k-algebras.

5.20. Proposition.
(i) *If X and Y are varieties, the map $\varphi \mapsto \varphi^*$ gives a bijection*

$$\left\{ \begin{matrix} \text{dominant rational maps} \\ \varphi\colon X \dashrightarrow Y \end{matrix} \right\} \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Alg}_k}\bigl(k(Y), k(X)\bigr)\,.$$

(ii) *The contravariant functor*

$$\mathsf{Var}_k^{\mathrm{birat}} \longrightarrow \begin{pmatrix} \text{finitely generated} \\ \text{field extensions } k \subset K \end{pmatrix}$$

that sends a variety X to its function field $k(X)$ and sends a dominant rational map $\varphi\colon X \dashrightarrow Y$ to the induced homomorphism $\varphi^*\colon k(Y) \to k(X)$, is an anti-equivalence of categories.

Proof. (i) Let $\alpha\colon k(Y) \to k(X)$ be a homomorphism of k-algebras. Our goal is to associate to α a dominant rational map $\varphi\colon X \dashrightarrow Y$ such that $\alpha = \varphi^*$. In order to achieve this we may assume that X and Y are both affine. (Replacing a variety by an affine open subset does not change the function field.) Write the coordinate rings of Y in the form $A(Y) = k[y_1,\ldots,y_n]/I$, and write $\alpha(y_i \bmod I) = g_i/h_i$ for some g_i, $h_i \in A(X)$ with $h_i \neq 0$. (Recall that $k(X)$ is the fraction field of $A(X)$.) Let $h = h_1 \cdots h_n$, which is a non-zero element of $A(X)$. Then $D(h)$ is a non-empty affine open subset of X with coordinate ring $A(X)\bigl[1/h\bigr] \subset k(X)$ and h therefore restricts to an injective homomorphism $A(Y) \to A\bigl(D(h)\bigr)$. This gives us a dominant morphism $\varphi\colon D(f) \to Y$, which represents a dominant rational map $\varphi\colon X \dashrightarrow Y$. (Use Exercise 5.5.) By construction, $\alpha = \varphi^*$. If we start with a dominant rational map φ and apply the construction to the homomorphism $\alpha = \varphi^*$, we find back the rational map φ.

(ii) The only thing left to prove is that any finitely generated field extension $k \subset K$ is the function field of a variety over k. This is clear, because the finite generation of K means that we can find elements $x_1,\ldots,x_n \in K$ such that K is the fraction field of its k-subalgebra generated by the x_i. This k-subalgebra is a ring of the form $A = k[x_1,\ldots,x_n]/I$ for some ideal $I \subset k[x_1,\ldots,x_n]$, and because A is a subring of a field it is a domain, which means that I is a prime ideal. Hence $A = A(X)$ for some affine variety X and $K = \mathrm{Frac}(A) = k(X)$. □

5.21. Proposition. *Any variety X is birationally equivalent to a hypersurface in some projective space.*

For the proof, which is based on some results in the theory of fields, we refer to HAG, Chap. I, Proposition 4.9.

§4. *Grassmannians.*

5.22. Definition. Let V be a k-vector space of finite dimension d. Given an integer n, we define the *Grassmann variety* $\mathrm{Grass}(n,V)$ to be the set of n-dimensional linear subspaces of V.

Of course, at this stage $\mathrm{Grass}(n,V)$ is only a set and the structure of a variety is still to be defined. Note that $\mathbb{P}(V) = \mathrm{Grass}(1,V)$, so at least for $n = 1$ we already know what to expect. Further note that the group $\mathrm{GL}(V)$ acts transitively on $\mathrm{Grass}(n,V)$. The structure of a variety will be such that this action

$$\mathrm{GL}(V) \times \mathrm{Grass}(n,V) \to \mathrm{Grass}(n,V) \tag{1}$$

is a morphism of varieties. This then makes $\mathrm{Grass}(n,V)$ an example of what is called a *homogeneous variety*, which is a variety on which an algebraic group acts transitively. (Note that if $H \subset \mathrm{GL}(V)$ is the stabilizer of some point $W \in \mathrm{Grass}(n, V)$, we get an identification of $\mathrm{Grass}(n, V)$ with the quotient $\mathrm{GL}(V)/H$. This representation depends, however, on the choice of a base point W and therefore somewhat obscures the high degree of symmetry that $\mathrm{Grass}(n, V)$ possesses.)

5.23. In order to define the structure of an algebraic variety on $\mathrm{Grass}(n, V)$ we follow the method outlined in 5.6. Of course, $\mathrm{Grass}(n, V)$ is interesting only for $0 < n < d$, so in what follows we assume this.

Given a subspace $B \subset V$ of dimension $d - n$ (so: codimension n), let $U_B \subset \mathrm{Grass}(n, V)$ be the subset of $W \in \mathrm{Grass}(n, V)$ for which $W \cap B = (0)$. As the first step in the construction, we are going to give these sets U_B the structure of an affine variety.

For this, fix B, and choose a complementary subspace A, so that we have a direct sum decomposition $V = A \oplus B$. (In fact, A is just any element of U_B.) The vector space $\mathrm{Hom}_k(A, B)$ has dimension $n(d-n)$ and, like any vector space, it has a natural structure of an affine variety. Define a map

$$\Phi = \Phi_{A,B} \colon \mathrm{Hom}_k(A, B) \to \mathrm{Grass}(n, V)$$

by sending a linear map $f \colon A \to B$ to its graph $\Gamma_f \subset V = A \oplus B$, which is an n-dimensional linear subspace. (In other words, $\Gamma_f \subset V$ is the image of the map $A \to V$ given by $a \mapsto a + f(a)$.) The map Φ gives a bijection $\mathrm{Hom}_k(A, B) \xrightarrow{\sim} U_B$. Indeed, if $W \in U_B$ then the projection map $\mathrm{pr}_A \colon W \to A$ is an isomorphism, and if ψ is the inverse then $W = \Gamma_{(\mathrm{pr}_B \circ \psi)}$. Via the bijection Φ we get the structure of an affine variety on U_B. This structure is independent of the chosen complement A, for if A' is another choice of a complementary subspace, a little (not completely trivial) calculation shows that the composition

$$\mathrm{Hom}_k(A, B) \xrightarrow{\Phi_{A,B}} U_b \xrightarrow{\Phi^{-1}_{A',B}} \mathrm{Hom}_k(A', B)$$

(which we already know to be a bijection) is a linear map and is therefore an isomorphism of affine varieties.

5.24. The next step is to show that the structures of affine varieties on the sets U_B are compatible. So we consider two subspaces $B_1, B_2 \subset V$, both of dimension $d - n$ and look at the associated subsets $U_i = U_{B_i} \subset \mathrm{Grass}(n, V)$. Then we claim that the intersection U_{12} is open in U_1 and U_2 and that the two induced structures of an affine variety are the same. For the details we refer to Exercise 5.6.

Let us note that a finite number of sets U_B suffice to cover $\mathrm{Grass}(n, V)$. To see this, choose a k-basis $\{e_1, \ldots, e_d\}$ for V. Any $W \in \mathrm{Grass}(n, V)$ is the span of the column vectors (with respect to the chosen basis) of a $d \times n$ matrix M of full rank. Then there is a subset $I \subset \{1, \ldots, d\}$ of n elements such that the corresponding n rows of M form an invertible $n \times n$ matrix; if B is the span of the remaining $d - n$ base vectors, W lies in $U_B \subset \mathrm{Grass}(n, V)$.

Follow the method outlined in 5.6 we now have the structure of a pre-variety on $\mathrm{Grass}(n, V)$. Note that the irreducibility follows from Exercise 5.2(ii). Note that the $\mathrm{GL}(V)$-action of (1) is indeed a morphism of algebraic varieties; for any $B \subset V$ as above the action of $g \in \mathrm{GL}(V)$ restricts to an isomorphism $U_B \xrightarrow{\sim} U_{gB}$.

5.25. The final step is to show that $\mathrm{Grass}(n, V)$ is in fact a projective variety. This is done via the *Plücker embedding*

$$\mathrm{pl}\colon \mathrm{Grass}(n, V) \to \mathbb{P}(\wedge^n V)$$

that sends $W \in \mathrm{Grass}(n, V)$ to the line $\wedge^n W$ in $\wedge^n V$. It is easily seen from the way we have defined the structure of a pre-variety on $\mathrm{Grass}(n, V)$ that this map is indeed a morphism. (If we choose a basis in a convenient way, as in Exercise 5.6, the map $\mathrm{pl} \circ \Phi\colon \mathrm{Hom}_k(A, B) \to \mathbb{P}(\wedge^n V)$ can easily be made explicit.)

It can be shown that pl gives an isomorphism of $\mathrm{Grass}(n, V)$ with a closed subvariety of $\mathbb{P}(\wedge^n V)$. We shall not give full details; a reference is Lecture 6 in Harris's book [5]. In particular, the projectivity implies that $\mathrm{Grass}(n, V)$ is a variety, rather than only a pre-variety. Note that Grassmannians are rational varieties, since the open subsets U_B are just affine spaces.

5.26. Remarks. (i) If $W \subset V$ is an n-dimensional linear subspace, $(V/W)^\vee$ is a $(d-n)$-dimensional linear subspace of the dual space $V^\vee$. The map $W \mapsto (V/W)^\vee$ defines an isomorphism $\mathrm{Grass}(n, V) \xrightarrow{\sim} \mathrm{Grass}(d-n, V^\vee)$. As an example, the space $\mathrm{Grass}(d-1, V)$ of hyperplanes in V is isomorphic to the projective space $\mathbb{P}(V^\vee) = \mathrm{Grass}(1, V^\vee)$. We write $\check{\mathbb{P}}(V) = \mathrm{Grass}(d-1, V)$; it is called the *dual projective space* of V.

(ii) The Grassmannian $\mathrm{Grass}(n, V)$ is also the set of $(n-1)$-dimensional linear subspaces of $\mathbb{P}(V)$. If this is the perspective we want to take, we use the notation $\mathbb{G}\bigl(n-1, \mathbb{P}(V)\bigr)$. Thus, for instance, $\mathbb{G}(1, \mathbb{P}^n) = \mathrm{Grass}(2, k^{n+1})$ is the variety of lines in $\mathbb{P}^n$.

(iii) There is no standard notation for Grassmannians in the literature. Especially, the shift in perspective mentioned in the previous point is a source of notational confusion. In what follows we shall write $\mathrm{Grass}(n, d)$ for $\mathrm{Grass}(n, k^d)$.

5.27. Example. The first example of a Grassmannian that is not a projective space is the variety $\mathrm{Grass}(2, 4) = \mathbb{G}(1, \mathbb{P}^3)$ of lines in $\mathbb{P}^3$. The space $\wedge^2 k^4$ is 6-dimensional, so the Plücker embedding realizes $\mathrm{Grass}(2, 4)$ as a subvariety of $\mathbb{P}^5$. It is convenient here to use X_{ij} with $0 \leqslant i < j \leqslant 2$ as coordinates on $\mathbb{P}^5$. Then $\mathrm{Grass}(2, 4) \subset \mathbb{P}^5$ is isomorphic to the quadric given by $X_{01}X_{23} - X_{02}X_{13} + X_{03}X_{12} = 0$. The map sends the point of $\mathrm{Grass}(2, 4)$ corresponding to the plane spanned by two vectors (a_0, a_1, a_2, a_3) and (b_0, b_1, b_2, b_3) to

$$\bigl(a_0b_1 - a_1b_0 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0 : a_1b_2 - a_2b_1 : a_1b_3 - a_3b_1 : a_2b_3 - a_3b_2\bigr)\,.$$

In general, $\mathrm{Grass}(n, V) \subset \mathbb{P}(\wedge^n V)$ can always be defined by quadratic equations, but the number of equations grows rapidly for large n and $d - n$.

Exercises for Chapter 5.

Exercise 5.1. Let X be an algebraic variety and Y an affine variety. Show that there is a natural bijection

$$\mathrm{Hom}_{\mathsf{Var}_k}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Alg}_k}\bigl(A(Y), \mathscr{O}(X)\bigr)\,.$$

In particular, morphisms $X \to \mathbb{A}^1$ are "the same" as regular functions $X \to k$.

Exercise 5.2. (i) If $(X, \mathscr{O}_X)$ is a pre-variety, show that X is a noetherian topological space.

(ii) Let $(X, \mathscr{O}_X)$ be a k-space that can be covered by finitely many open subspaces U_i that are affine varieties, such that all intersections $U_i \cap U_j$ are non-empty. Show that X is irreducible and is therefore a pre-variety.

Exercise 5.3. Let X and Y be varieties over k.

(i) Prove that the product $X \times Y$ (in the category of pre-varieties) is again a variety.

(ii) If $\varphi\colon X \to Y$ is a morphism, prove that its graph $\Gamma_\varphi \subset X \times Y$ is closed.

(iii) If U and V are affine open subsets of X, show that $U \cap V$ is again affine.

Exercise 5.4. Prove that two varieties X and Y are birationally equivalent if and only if there are non-empty open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$.

Exercise 5.5. Let $\varphi\colon X \to Y$ be a morphism of varieties over k.

(i) Show that φ is dominant if and only if $\varphi^*\colon \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$ is an injective homomorphism of sheaves.

(ii) Assume Y is affine. Consider the homomorphism $\varphi^*\colon A(Y) \to \mathscr{O}(X)$. (Cf. Exercise 5.1.) Show that under the bjection of Proposition 1.24, the closure of $\varphi(X)$ in Y corresponds with the prime ideal $\mathrm{Ker}(\varphi^*) \subset A(Y)$.

Exercise 5.6. As in 5.24, we consider a k-vector space V of dimension d and two subspaces B_1, B_2, both of codimension n. Choose ordered bases $\{e_1, \ldots, e_d\}$ and $\{f_1, \ldots, f_d\}$ for V such that $B_1 = \mathrm{Span}(e_{n+1}, \ldots, e_d)$ and $B_2 = \mathrm{Span}(f_{n+1}, \ldots, f_d)$. As complementary subspaces we choose $A_1 = \mathrm{Span}(e_1, \ldots, e_n)$ and $A_2 = \mathrm{Span}(f_1, \ldots, f_n)$. Via the chosen bases we identify the spaces $\mathrm{Hom}_k(A_i, B_i)$ with the space $M_{(d-n),n}$ of $(d-n) \times n$ matrices. Finally, write $U_i = U_{B_i}$ and let

$$\Phi_i\colon M_{(d-n),n} \xrightarrow{\sim} U_i \subset \mathrm{Grass}(n, V)$$

be the isomorphisms of 5.23.

(i) For $N \in M_{(d-n),n}$, show that $\Phi_1(N) \in \mathrm{Grass}(n, V)$ is the span of the columns of the $d \times n$ matrix

$$\begin{pmatrix} \mathbf{1} \\ N \end{pmatrix}$$

where $\mathbf{1}$ is the identity matrix of size $n \times n$ and where the columns are viewed as vectors in V with respect to the basis $\{e_1, \ldots, e_d\}$.

(ii) Let $Q \in \mathrm{GL}_d(k)$ be the matrix such that the ith column of Q gives the expression of e_i as a vector with respect to the basis $\{f_1, \ldots, f_d\}$. Write Q as a block matrix

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$$

where Q_1 is of size $n \times n$. (So Q_2 has size $n \times (d-n)$, etc.) Show that $\Phi_1^{-1}(U_{12}) \subset M_{(d-n),n}$ is the subset of matrices N for which $Q_1 + Q_2 N$ is invertible. Also show that this is a non-empty Zariski open subset of $M_{(d-n),n}$.

(iii) Show that the composition

$$\Phi_1^{-1}(U_{12}) \xrightarrow{\Phi_1} U_{12} \xrightarrow{\Phi_2^{-1}} \Phi_2^{-1}(U_{12})$$

is the map given by $N \mapsto (Q_3 + Q_4 N)(Q_1 + Q_2 N)^{-1}$ and conclude that this composition is an isomorphism of quasi-affine varieties.

Exercise 5.7. *The Segre embedding.* Given non-negative integers m and n, let $r = mn+m+n$, so that $(m+1)(n+1) = (r+1)$. The goal of this exercise is to discuss the *Segre embedding*, which is an embedding of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^r$. On $\mathbb{P}^r$ we shall use the projective coordinates Z_{ij} with $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, sorted alphabetically. The Segre embedding is then the map $s\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^r$ that sends a point $P = \big((a_0 : \cdots : a_m), (b_0 : \cdots, b_n)\big)$ to the point with coordinates a_ib_j; so

$$s(P) = \big(a_0b_0 : \cdots : a_0b_n : a_1b_0 : \cdots : a_1b_n : \cdots : a_mb_0 : \cdots : a_mb_n\big)\,.$$

(i) Prove that s is a morphism.

(ii) Consider the homomorphism of k-algebras $k[Z_{ij}] \to k[X_0, \ldots, X_m, Y_0, \ldots, Y_n]$ that sends Z_{ij} to X_iY_j. Let $\mathfrak{p} \subset k[Z_{ij}]$ be the kernel. Prove that $\mathfrak{p}$ is a homogeneous prime ideal and that s gives an isomorphism $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\sim} \mathscr{Z}(\mathfrak{p}) \subset \mathbb{P}^r$. Can you give explicit generators for $\mathfrak{p}$?

(iii) Prove that the product of projective varieties is again projective.

Exercise 5.8. Let $\varphi\colon X \dashrightarrow Y$ be a rational map of varieties.

(i) If φ is represented by (U, φ_U) we can consider the Zariski closure of $\varphi_U(U)$ in Y, which is a closed subvariety $Z \subset Y$. Show that this subvariety does not depend on the chosen representative (U, φ_U). We call Z the *closed image* of φ. (In particular this definition of course also applies to morphisms $X \to Y$.)

(ii) If φ is represented by (U, φ_U) we can consider the graph of φ_U inside $X \times Y$. Let $\overline{\Gamma}_\varphi \subset X \times Y$ be the closure. Show that $\overline{\Gamma}_\varphi$ is again independent of the chosen representative (U, φ_U).

(iii) With $\overline{\Gamma}_\varphi$ as in (ii), consider the projection $p\colon \overline{\Gamma}_\varphi \to X$. Show that the domain of definition of φ, as in Remark 5.14(ii), is the largest open subset $U \subset X$ such that $p\colon p^{-1}(U) \to U$ is an isomorphism.

CHAPTER 6

Dimensions, tangent spaces, and singularities

The notion of dimension is an important concept about which we have a clear geometric intuition. Upon closer inspection, however, it turns out that a correct treatment of this notion requires quite a bit of non-trivial commutative algebra. As we have chosen not to develop this in detail in these notes, this has as consequence that several basic results in this chapter are left unproven—we merely point out that our assertions are translations of results in algebra to the geometric setting. (The most important of these algebraic results are collected in the section on commutative algebra, without proofs.)

§1. *The dimension of algebraic varieties.*

6.1. Definition. The *dimension* $\dim(X)$ of a non-empty topological space X is the supremum of the integers r for which there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r$$

of closed irreducible subsets of X, with the understanding that $\dim(X) = \infty$ if there exist arbitrarily long such chains.

It should be realized that this definition of the dimension is very much oriented towards algebraic geometry. In other branches of mathematics, one finds different definitions of the dimension of a variety or a space.

6.2. The only 0-dimensional variety is a single point. (Why?) A variety of dimension 1 is called a *curve*, one of dimension 2 a *surface*. After that we have *threefolds*, *fourfolds*, etc.

6.3. Example. Let $Y \subset \mathbb{A}^n$ be an affine variety with coordinate ring $A(Y)$. Then $\dim(Y) = \mathrm{Kdim}\big(A(Y)\big)$. This is immediate from the definitions, together with Proposition 1.24. It therefore follows from A3.2 that $\dim(\mathbb{A}^n) = n$.

If $Z \subset Y$ are varieties, the *codimension* of Z in Y is by definition the difference in dimensions: $\mathrm{codim}_Y(Z) = \dim(Y) - \dim(Z)$. If Y is affine and $\mathfrak{p} \subset A(Y)$ is the prime ideal that corresponds with Z, it follows from Theorem A3.5 that the codimension $\mathrm{codim}_Y(Z)$ equals the height of $\mathfrak{p}$.

6.4. Proposition. *If $Z \subset Y$ are affine varieties and*

$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = Y$$

is any maximal chain of closed irreducible subvarieties, we have $r = \dim(Y) - \dim(Z)$.

To avoid any confusion, let us note that the maximality of the chain means that it can not be refined to a longer chain. A priori, this does not imply that it has maximal length among all possible chains between Z and Y.

Proof. This is the geometric translation of Proposition A3.5(ii). $\square$

6.5. Proposition. *Let* Y *be a variety. Then* $\dim(Y) < \infty$ *and if* U *is a non-empty subset of* Y*, we have* $\dim(U) = \dim(Y)$.

Proof. We first prove the proposition in case Y is affine. The finiteness of the dimension then follows from A3.5. Let $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = U$ be a chain of closed irreducible subsets of U. Then $\{P\} = \overline{Z}_0 \subset \overline{Z}_1 \subset \cdots \subset \overline{Z}_r = Y$ is a chain of closed irreducible subsets of Y, and all inclusions in this chain are again strict because $Z_i = U \cap \overline{Z}_i$. Hence $r \leqslant \dim(Y)$ and it follows that $\dim(U) \leqslant \dim(Y) < \infty$. Hence we can choose the chain $Z_\bullet$ to be maximal, with $r = \dim(U)$. Note that $Z_0 = \{P\}$ for some $P \in U$ and $Z_r = U$. It then follows from Exercise 1.5(i) that the chain $\overline{Z}_\bullet$ is again maximal. (Use that $\overline{Z}_i \cap U$ is non-empty, because it contains P.) By Proposition 6.4 it follows that $\dim(U) = r = \dim(Y)$.

Next we consider a general variety Y. By definition, it can be covered by finitely many affine varieties, say $Y = V_1 \cup \cdots \cup V_N$. By what we have just proven, $\dim(V_i) = \dim(V_{ij}) = \dim(V_j)$ for all i and j. So all V_i have the same dimension; say $\dim(V_i) = d$. If $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = U$ is a chain of closed irreducible subsets of Y, choose a point $P \in Z_0$ and an index j with $P \in V_j$. Then the $(Z_i \cap V_j)$ form a chain of closed irreducible subsets of V_j and the inclusion in this chain are strict. Hence $r \leqslant d$ and it follows that $\dim(Y) \leqslant d < \infty$. On the other hand, there exists a chain $Z_\bullet$ of length d in V_1 and then $\overline{Z}_\bullet$ is a chain of length d in Y; so in fact $\dim(Y) = d$. This shows that for any affine open $V \subset Y$ we have $\dim(V) = \dim(Y)$, and since the affines form a basis for the topology it follows that for any non-empty open $U \subset Y$ we have $\dim(U) = \dim(Y)$. $\square$

6.6. Corollary. *For any variety* Y *we have* $\dim(Y) = \mathrm{trdeg}_k\big(k(Y)\big)$.

Proof. Take any affine $U \subset Y$ and use Theorem A3.5(iii). $\square$

6.7. Example. The Grassmann variety $\mathrm{Grass}(n, V)$ has dimension $n(d-n)$, where $d = \dim(V)$. Indeed, with notation as in 5.23 the open subset $U_B \subset \mathrm{Grass}(n, V)$ is isomorphic to $\mathrm{Hom}_k(A, B)$, which is an affine space of dimension $\dim(A) \cdot \dim(B) = n(d - n)$.

6.8. Proposition.

(i) *Let* $Y \subset \mathbb{A}^n$ *be an affine variety. If the ideal* $\mathscr{I}(Y) \subset k[x_1, \ldots, x_n]$ *can be generated by* r *elements then* $\dim(Y) \geqslant n - r$.

(ii) *Let* $f \in k[x_1, \ldots, x_n]$ *be a non-constant polynomial. Then all irreducible components of* $\mathscr{Z}(f)$ *have dimension* $n - 1$.

Proof. Part (i) follows from Theorems A3.5 and A3.4, using that $\dim(\mathbb{A}^n) = n$.

For (ii), write $f = g_1^{m_1} g_2^{m_2} \cdots g_r^{m_r}$ where $g_1, \ldots, g_r$ are mutually coprime irreducible polynomials. (Recall that $k[x_1, \ldots, x_n]$ is a UFD.) The irreducible components of $\mathscr{Z}(f)$ are the sets $\mathscr{Z}(g_i)$. (Cf. Exercise 1.11.) Now use (i) and note that $\mathrm{ht}\big((g_i)\big) > 0$ because $(0) \subset (g_i)$ is a non-trivial chain of prime ideals. $\square$

6.9. It follows from Proposition 6.8 that the affine varieties $Y \subset \mathbb{A}^n$ of dimension $n - 1$ are precisely the varieties of the form $\mathscr{Z}(f)$ with $f \in k[x_1, \ldots, x_n]$ an irreducible polynomial. We call such varieties *hypersurfaces*.

The conclusion in 6.8(ii) is very special to the case of hypersurfaces. If $Y \subset \mathbb{A}^n$ is an affine variety of codimension $r > 1$, this does certainly not imply, in general, that the ideal $\mathscr{I}(Y)$ can be generated by r elements. (It follows from 6.8(i) that we need at least r generators.) For example, if $C = \{(t^3, t^4, t^5) \mid t \in k\}$ is the curve in $\mathbb{A}^3$ that appears in Exercise 1.8, it can be shown that $\mathscr{I}(C) \subset k[x, y, z]$ cannot be generated by two elements.

6.10. Proposition.

(i) *(Affine dimension theorem) Let X, $Y \subset \mathbb{A}^n$ be affine varieties of codimension r and s, respectively. Then every irreducible component of $X \cap Y$ has codimension at most $r + s$.*

(ii) *(Projective dimension theorem) Let X, $Y \subset \mathbb{P}^n$ be projective varieties of codimension r and s, respectively. Then every irreducible component of $X \cap Y$ has codimension at most $r + s$ and if $r + s \leqslant n$ the intersection $X \cap Y$ is non-empty.*

Sketch of the proof. To prove (i) one first does the special case when Y is a hypersurface, say $Y = \mathscr{Z}(f)$ for an irreducible $f \in k[x_1, \ldots, x_n]$. If $\overline{f}$ is the image of f in the coordinate ring $A(X)$, the irreducible components of $X \cap Y$ correspond to the minimal prime ideals of $A(X)$ containing $\overline{f}$. By Krull's height theorem, see Theorem A3.4, these prime ideals have height $\leqslant 1$ in $A(X)$, which means that the components of $X \cap Y$ have codimension at most $r + 1$.

For the general case, use that $X \cap Y$ is the intersection of the subvariety $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}$ with the diagonal $\Delta \subset \mathbb{A}^n \times \mathbb{A}^n$. As this diagonal is an intersection of n hypersurfaces (namely those given by $x_i = y_i$ for $i = 1, \ldots, n$), repeated application of the special case gives the result.

In (ii), the dimension estimate is obtained by covering $\mathbb{P}^n$ by affine opens and using Proposition 6.5. For the non-emptiness of $X \cap Y$ in case $r + s \leqslant n$, look at the cones $C(X)$ and $C(Y)$ in $\mathbb{A}^{n+1}$, which again have codimensions r and s. The intersection $C(X) \cap C(Y)$ is non-empty, as it contains O. Part (i) then tells us that every component of this intersection has codimension at most n, and therefore has dimension at least 1 ! In particular, $C(X) \cap C(Y)$ contains a point other than O, which means that $X \cap Y$ is non-empty. □

§2. *The tangent space at a point.*

6.11. Definition. Let X be a variety and $P \in X$. The the *tangent space of X at the point P*, notation $T_{X,P}$, is the k-vector space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$, where $\mathfrak{m}$ is the (unique) maximal ideal of $\mathscr{O}_{X,P}$ and $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ denotes the k-linear dual of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$.

Note that $\mathscr{O}_{X,P}/\mathfrak{m} \xrightarrow{\sim} k$ via evaluation of functions at P; so $\mathfrak{m}/\mathfrak{m}^2$ is indeed a k-vector space. Further note that $\mathscr{O}_{U,P} = \mathscr{O}_{X,P}$ for any open $U \subset X$ containing P; this enables us to restrict our attention to affine varieties.

6.12. Example. Let $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$, and let $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ be the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Then $\mathfrak{m}/\mathfrak{m}^2 \cong k^n$. The classes $x_i - a_i \bmod \mathfrak{m}^2$ form a basis. Consequently, $T_{\mathbb{A}^n,P} \cong k^n$.

We shall write $\{\xi_1, \ldots, \xi_n\}$ for the basis of $T_{\mathbb{A}^n,P}$ that is dual to the given basis of $\mathfrak{m}/\mathfrak{m}^2$. Concretely, $\xi_i \in T_{\mathbb{A}^n,P}$ is the element with $\xi_i(x_j - a_j \bmod \mathfrak{m}^2) = \delta_{ij}$. (Kronecker delta.)

Of course, we should like to think of tangent vectors geometrically—something like an "infinitesimal vector" pointing in a certain direction. The idea behind our definition of the tangent space is that such a "geometric" tangent vector corresponds to the operation on functions given by taking partial derivatives in the direction of the tangent vector. While less intuitive, such an operation is much easier to deal with algebraically. In fact, if we imagine we are taking partial derivatives of functions in a certain tangent direction τ, it makes sense to restrict our attention to functions that vanish at P (the derivative cannot give us information about the value at P anyway), and then the Leibniz rule of derivations $\partial_\tau(fg) = f(P)\partial_\tau g + g(P)\partial_\tau f$ implies that elements in $\mathfrak{m}^2$ go to zero; hence the derivative gives us a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, which is just an element of $T_{X,P}$ as we have defined it.

6.13. If $\varphi\colon X \to Y$ is a morphism, $P \in X$ and $Q = \varphi(P)$, we have an induced local homomorphism $\varphi^*\colon \mathscr{O}_{Y,Q} \to \mathscr{O}_{X,P}$. (Cf. Exercise 2.4.) As φ^* maps $\mathfrak{m}_Q$ into $\mathfrak{m}_P$ and therefore also maps $\mathfrak{m}_Q^2$ into $\mathfrak{m}_P^2$, we get a k-linear map $\mathfrak{m}_Q/\mathfrak{m}_Q^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2$. The k-linear dual of this map is a k-linear map $T\varphi\colon T_{X,P} \to T_{Y,Q}$, called the tangent map of φ at P. If the context requires it, we may write $T\varphi_P$ instead of $T\varphi$.

6.14. Proposition. *Let $X \subset \mathbb{A}^n$ be an affine variety, defined by the ideal $\mathscr{I}(X) = (f_1, \dots, f_r)$. Write $i\colon X \to \mathbb{A}^n$ for the inclusion map, let $P \in X$, and let $\{\xi_1, \dots, \xi_n\}$ be the k-basis of $T_{\mathbb{A}^n,P}$ as in* Example 6.12. *Then $Ti\colon T_{X,P} \to T_{\mathbb{A}^n,P}$ gives an isomorphism of $T_{X,P}$ with the k-linear subspace of $T_{\mathbb{A}^n,P} = k\xi_1 \oplus \cdots \oplus k\xi_n$ given by the linear equations*

$$\frac{\partial f_i}{\partial x_1}(P)\cdot \xi_1 + \cdots + \frac{\partial f_i}{\partial x_n}(P)\cdot \xi_n = 0\,,$$

for $i = 1, \dots, r$.

Proof. We have $\mathscr{O}_{X,P} = \mathscr{O}_{\mathbb{A}^n,P}/I$, where I is the ideal of $\mathscr{O}_{\mathbb{A}^n,P}$ generated by $f_1, \dots, f_r$. Writing $\overline{\mathfrak{m}} = \mathfrak{m}/I$ for the maximal ideal, $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \cong \mathfrak{m}/(I + \mathfrak{m}^2)$, which is the quotient of $\mathfrak{m}/\mathfrak{m}^2$ modulo the subspace generated by the image of I. Dually, we find that $T_{X,P}$ is the subspace of $T_{\mathbb{A}^n,P}$ consisting of the linear maps $\mathfrak{m}/\mathfrak{m}^2 \to k$ that vanish on the image of I. But

$$f_i \equiv \frac{\partial f_i}{\partial x_1}(P)\cdot(x_1 - a_1) + \cdots + \frac{\partial f_i}{\partial x_n}(P)\cdot(x_n - a_n) \mod \mathfrak{m}^2\,,$$

and this gives the equations for $T_{X,P}$ as stated. □

6.15. Proposition. *Let X be a variety. Then $\dim_k(T_{X,P}) \geqslant \dim(X)$ for any $P \in X$.*

Proof. Without loss of generality we may assume X is affine. Then the assertion is just A3.3, now stated in a geometric context. □

6.16. Definition. A point $P \in X$ is called a *regular, or non-singular, point* if $\dim_k(T_{X,P}) = \dim(X)$. If $\dim(T_{X,P}) > \dim(X)$ then P is called a *singular point*.

The variety X is called *regular, or non-singular*, if all points of X are regular.

An affine variety X is regular if and only if its coordinate ring is a regular ring in the sense of Definition A3.6. Similarly, X is regular at a point P if and only if $\mathscr{O}_{X,P}$ is a regular local ring.

6.17. Example. Consider the curve $C \subset \mathbb{A}^2$ given by $y^2 = x^3 - 1$. According to Proposition 6.14, the tangent space at a point $P = (a, b)$ is the linear subspace of k^2 (with basis $\{\xi_1, \xi_2\}$) given by the equation $3a^2 \cdot \xi_1 - 2b \cdot \xi_2 = 0$. The only singular points of C are therefore the points $P = (a, b)$ with $3a^2 = 2b = 0$. If $\mathrm{char}(k) \notin \{2, 3\}$ the only possibility is $a = b = 0$ but as $(0, 0) \notin C$ we find that C is regular. If $\mathrm{char}(k) = 2$ or 3, however, C has a (unique) singular point, namely $(1, 0)$.

6.18. Proposition. *Let X be a quasi-affine variety. Then the set* $\mathrm{Sing}(X) \subset X$ *of singular points is a proper closed subset of X.*

Proof. Without loss of generality we may assume X is affine, say $X = \mathscr{Z}(f_1, \ldots, f_r) \subset \mathbb{A}^n$. If $\dim(X) = n - s$, a point $P \in X$ is singular if and only if the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(P) & \frac{\partial f_1}{\partial x_2}(P) & \cdots & \frac{\partial f_1}{\partial x_n}(P) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1}(P) & \frac{\partial f_r}{\partial x_2}(P) & \cdots & \frac{\partial f_r}{\partial x_n}(P) \end{pmatrix}$$

has rank $< s$. This conditions means that all $s \times s$ minors of the matrix are 0. But any such minor is a polynomial in the coordinates $a_1, \ldots, a_n$ of P; so indeed $\mathrm{Sing}(X)$ is closed.

Next we prove that $\mathrm{Sing}(X)$ cannot be all of X. By Proposition 5.21, it suffices to prove this if X is a hypersurface in $\mathbb{A}^n$, so $X = \mathscr{Z}(f) \subset \mathbb{A}^n$ for some irreducible polynomial $f \in k[x_1, \ldots, x_n]$. The singular points are then the points $P \in X$ at which all partial derivatives $\partial f / \partial x_i$ vanish. If $X = \mathrm{Sing}(X)$ we get

$$\mathscr{Z}(f) \subset \mathscr{Z}(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$$

which implies that for every index i some power $(\partial f/\partial x_i)^n$ lies in the ideal (f). Now $k[x_1, \ldots, x_n]$ is a UFD, f is irreducible, and $\deg(\partial f/\partial x_i) < \deg(f)$ for all i; so the only possibility is that $\partial f/\partial x_i = 0$ for all i. This implies that $\mathrm{char}(k) = p > 0$ and that f lies in the subring $k[x_1^p, \ldots, x_n^p] \subset k[x_1, \ldots, x_n]$. As $k = \overline{k}$, any such f is the pth power of another polynomial; so we get a contradiction with the assumption that f is irreducible. □

§3. *Blowing up.*

6.19. Definition. The *blowing-up* of $\mathbb{A}^n$ at the origin $O = (0, \ldots, 0)$ is the closed subvariety $B \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ given by

$$B = \Big\{ \big((a_1, \ldots, a_n), (b_1 : \cdots : b_n)\big) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \;\Big|\; a_i b_j = a_j b_i \quad \text{for all } i \text{ and } j \Big\}.$$

(Note the unusual indexing of the coordinates on the factor $\mathbb{P}^{n-1}$.)

Geometrically, if we interpret $\mathbb{P}^{n-1}$ as the variety of lines $L \subset \mathbb{A}^n$ through O, the definition of B can be rewritten as

$$B = \big\{ (P, L) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \;\big|\; P \in L \big\}.$$

As such, it is an example of an *incidence variety:* inside the variety of all pairs (point, line) we consider the subvariety given by the condition that the point lies on the line.

6.20. Let us justify the claim that the blowing-up B is a closed subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. We use $(x_1, \ldots, x_n)$ as coordinates on $\mathbb{A}^n$ and $(Y_1 : \cdots : Y_n)$ as homogeneous coordinates on $\mathbb{P}^{n-1}$. For $i \in \{1, \ldots, n\}$ let $U_i \subset \mathbb{P}^{n_1}$ be the affine open subset given by $Y_i \neq 0$. Then $U_i \xrightarrow{\sim} \mathbb{A}^{n-1}$ via $(b_1 : \cdots : b_n) \mapsto (b_1/b_i : \cdots : b_{i-1}/b_i : b_{i+1}/b_i : \cdots : b_n/b_i)$. Using $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$ as coordinates on U_i, we find that $B \cap (\mathbb{A}^n \times U_i)$ is the closed subvariety of $\mathbb{A}^n \times U_i \cong \mathbb{A}^n \times \mathbb{A}^{n-1}$ given by the equations $x_j = x_i y_j$ for all j. Hence,

$$B \cap (\mathbb{A}^n \times U_i) \xrightarrow{\sim} \mathbb{A}^n$$

via the isomorphism given by

$$\big((a_1, \ldots, a_n), (b_1 : \cdots : b_n)\big) \mapsto \Big(\frac{a_i b_1}{b_i}, \ldots, \frac{a_i b_{i-1}}{b_i}, a_i, \frac{a_i b_{i+1}}{b_i}, \ldots, \frac{a_i b_n}{b_i}\Big).$$

As $B \cap (\mathbb{A}^n \times U_i) \cap (\mathbb{A}^n \times U_j)$ is non-empty for all i, j, it follows from Exercise 5.2 that B is irreducible.

We denote by $\pi\colon B \to \mathbb{A}^n$ the projection map. Observe that this map is an isomorphism away from the fiber over O:

$$\pi\colon B \setminus \pi^{-1}\{O\} \xrightarrow{\sim} \mathbb{A}^n \setminus \{O\}.$$

The fiber $E = \pi^{-1}\{O\}$ is the whole space $\{O\} \times \mathbb{P}^{n-1}$, which, as already remarked, should be thought of as the variety of lines in $\mathbb{A}^n$ passing through O. This fiber is called the *exceptional fiber*.

6.21. Definition. Let $X \subset \mathbb{A}^n$ be a closed subvariety with $O \in X$, and let $\pi\colon B \to \mathbb{A}^n$ be the blowing-up of $\mathbb{A}^n$ at O. Then the blowing-up of X at O is the subvariety $\tilde{X} \subset B$ obtained as the closure of $\pi^{-1}\big(X \setminus \{O\}\big)$ inside B.

Note that π induces an isomorphism $\pi^{-1}\big(X \setminus \{O\}\big) \xrightarrow{\sim} X \setminus \{O\}$. In particular, $\tilde{X}$ is irreducible (because it is the closure of $\pi^{-1}\big(X \setminus \{O\}\big)$) and is therefore a closed subvariety of B.

The idea of a blowing-up is that we are changing X at the origin, leaving $X \setminus \{O\}$ unchanged, and that the blowing-up $\tilde{X}$ is usually "less singular" than the original variety X.

6.22. Example. Let $X \subset \mathbb{A}^2$ be the affine curve given by $y^2 = x^2(x+1)$. The origin O is the unique singular point of X. Using $(t : u)$ as homogeneous coordinates on $\mathbb{P}^1$, the blowing-up $B \subset \mathbb{A}^2 \times \mathbb{P}^1$ is given by the equation $xu = yt$. We compute the blowing-up $\tilde{X}$ separately on two affine charts.

First chart: $t \neq 0$. We can scale to $t = 1$; then we are working inside the variety $\mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$ with coordinates (x, y, u) and B is given by the equation $y = xu$. So $B \cong \mathbb{A}^2$ and (x, u) is a system of coordinates; the projection map $B \to \mathbb{A}^2$ is in these coordinates given by $(x, u) \mapsto (x, xu)$.

Substituting the relation $y = xu$ into the equation for X we get $x^2u^2 = x^2(x+1)$. This means that $\pi^{-1}(X)$ on this chart has two irreducible components: a component E_1 formed by the points with $x = 0$ (and u arbitrary), and an irreducible component given by the relation $u^2 = x + 1$. However, the first irreducible component lies entirely over the origin. So we find that $\tilde{X}$, which is defined to be the closure of $\pi^{-1}\big(X \setminus \{O\}\big)$ intersects this chart in the subvariety $\tilde{X}_1 \subset \mathbb{A}^2$ given by $u^2 = x + 1$.

Second chart: $u \neq 0$. We can scale to $u = 1$; then we are working inside the variety $\mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$ with coordinates (x, y, t) and B is given by the equation $x = yt$. So $B \cong \mathbb{A}^2$ and (y, t) is a system of coordinates; the projection map $B \to \mathbb{A}^2$ is in these coordinates given by $(y, t) \mapsto (yt, y)$.

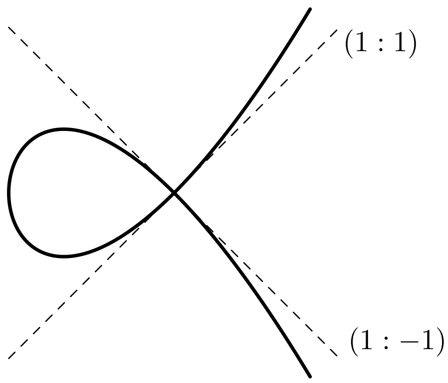
Substituting the relation $x = yt$ into the equation for X we get $y^2 = y^2 t^2 (yt + 1)$. This means that $\pi^{-1}(X)$ on this chart has two irreducible components: a component E_2 formed by the points with $y = 0$ (and t arbitrary), and an irreducible component given by the relation $1 = t^2(yt + 1)$. But E_2 lies over the origin; so $\tilde{X}$ intersects this chart in the subvariety $\tilde{X}_2 \subset \mathbb{A}^2$ in the subvariety given by $1 = t^2(yt + 1)$.

The blowing-up $\tilde{X}$ is obtained by glueing the two affine varieties $\tilde{X}_1$ and $\tilde{X}_2$ that we have found. The glueing corresponds to the change of coordinates $(y, t) = (xu, 1/u)$. Note that after glueing, the components E_1 and E_2 give the exceptional fiber $E \cong \mathbb{P}^1$. Further, in this example the inclusions $\tilde{X}_i \hookrightarrow \tilde{X}$ are in fact isomorphisms; it should be noted, though, that this is not something one can expect in general.

Next we note that $\tilde{X}_1$ and $\tilde{X}_2$ are non-singular; hence $\tilde{X}$ is non-singular. In fact, $\tilde{X} \cong \mathbb{A}^1 = \mathbb{P}^1 \setminus \{(0 : 1)\}$ via the projection map $B \to \mathbb{P}^1$.

The full inverse image $\pi^{-1}(X)$ is the union of $\tilde{X}$ and E. Since we are only interested in $\tilde{X}$, we see why we have defined it as the closure of $\pi^{-1}\big(X \setminus \{O\}\big)$; in this way we get rid of the extra component E.

In this example, we also nicely see the interpretation of $E \cong \mathbb{P}^1$ as the space of lines in $\mathbb{A}^2$ through O: the blowing-up $\tilde{X}$ intersects E in two points, namely $(1 : 1)$ and $(1 : -1)$; these correspond precisely to the two tangent directions of X at the origin.



6.23. Example. It is of course not true that blowing up the singular point always results in a non-singular variety. In simple examples, however, the singularities can be resolved by a finite sequence of blow-ups. To illustrate this, assume $\mathrm{char}(k) = 0$, take integers $m > n \geqslant 2$ and consider the affine curve $X \subset \mathbb{A}^2$ given by $y^2 = x^m + x^n$. Calculating $\tilde{X}$ as in the previous example, we find that the piece $\tilde{X}_2$ on the second chart is given by $1 = y^{m-2}t^m + y^{n-2}t^n$, and this is a non-singular curve. On the first chart we get as new equation $u^2 = x^{m-2} + x^{n-2}$. For $n \geqslant 4$ this is still singular, but we recognize that the new equation is of the same form as the one we started with, only with smaller exponents. So we may repeat the procedure, blowing up $\tilde{X}_1$ in the singular point. After $\lfloor n/2 \rfloor$ blow-ups we obtain a non-singular curve.

6.24. If X is a variety with singular locus $S = \mathrm{Sing}(X)$, a *resolution of singularities* is a variety X' with a surjective morphism $\pi\colon X' \to X$ such that X' is non-singular and π is an isomorphism over the regular locus $X \setminus \mathrm{Sing}(X)$. (This means we modify only the singular part of X.) In fact, we should require π to be a proper morphism—this notion will be defined later. Resolution of singularities is very important as a tool in Algebraic Geometry.

A famous theorem of Hironaka says that if $\mathrm{char}(k) = 0$ there always exists a resolution of singularities, in any dimension. (What Hironaka proved is actually stronger.) In positive characteristic this is not known! For varieties of dimension $\leqslant 3$, resolution of singularities is known in arbitrary characteristic.

For some applications, it turns out that something a little weaker than resolution of the singularities is good enough. In 1995 the Dutch mathematician A.J. de Jong proved a beautiful result about so-called *alterations*, valid in any characteristic. This result has found many applications.

6.25. Example. As a final example, let us blow up a surface singularity. Assuming $\mathrm{char}(k) \neq 2$, the affine surface $X \subset \mathbb{A}^3$ given by $z^2 = x^2 + y^3$ has a unique singularity at $O = (0, 0, 0)$. Using $(t : u : v)$ as homogeneous coordinates on $\mathbb{P}^2$, consider $B \subset \mathbb{A}^3 \times \mathbb{P}^2$ given by

$$xu = yt\,, \quad xv = zt\,, \quad yv = zu\,.$$

To calculate the blowing-up $\tilde{X}$, we work on three different charts:

First chart: $t \neq 0$. Scaling to $t = 1$ we work inside $\mathbb{A}^3 \times \mathbb{A}^2$ and $B \cong \mathbb{A}^3$ is the subvariety given by $y = xu$ and $z = xv$. Substituting these into the equation of X we get $x^2v^2 = x^2 + x^3u^3$. The points with $x = 0$ lie over the singular point O. Hence the blowing-up of X intersects this chart in the surface $\tilde{X}_1$ given (inside $\mathbb{A}^3$ with coordinates x, u, v) by $v^2 = 1 + xu^3$. This is a non-singular surface.

Second chart: $u \neq 0$. Now we get the relations $x = yt$ and $z = yv$. We work inside $\mathbb{A}^3$ with coordinates (y, t, v) and find that $\tilde{X}_2$ is given by $v^2 = t^2 + y$. This is a non-singular surface.

Third chart: $v \neq 0$. Now we get the relations $x = zt$ and $y = zu$. We work inside $\mathbb{A}^3$ with coordinates (z, t, u) and find that $\tilde{X}_3$ is given by $1 = t^2 + zu^3$. Again this is a non-singular surface.

The blowing-up is obtained by glueing the three affine pieces $\tilde{X}_i$. We find that $\tilde{X}$ intersects the exceptional fiber $E \cong \mathbb{P}^2$ in the two lines given by $v = \pm t$.

Exercises for Chapter 6.

Exercise 6.1. Assume $\mathrm{char}(k) = 0$. Determine the dimension of the affine variety $X \subset \mathbb{A}^4$ given by the equations

$$xy - z^2 = x^2w^3 - y^6 = 0\,.$$

and find its singular points. Calculate the dimension of $T_{X,P}$ at the singular points.

Same for the variety given by

$$x^2w^2 - y^2z^3 = x^2yw - z^4 = y^3 - wz = 0\,.$$

(You don't need to prove that these are indeed varieties.)

Exercise 6.2. Let X and Y be varieties. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projection maps. If $A \in X$ and $B \in Y$, prove that the map

$$Tp_{(A,B)} \oplus Tq_{(A,B)}\colon T_{X\times Y,(A,B)} \to T_{X,A} \oplus T_{Y,B}$$

is an isomorphism. Deduce that $(A,B) \in X \times Y$ is non-singular if and only if $A \in X$ and $B \in Y$ are both non-singular points.

Exercise 6.3. Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be affine varieties with coordinate rings $A(X) = k[x_1,\dots,x_m]/(f_1,\dots,f_r)$ and $A(Y) = k[y_1,\dots,y_n]/(g_1,\dots,g_s)$. Let $\varphi\colon X \to Y$ be a morphism, $P = (a_1,\dots,a_m) \in X$ and $Q = \varphi(P) \in Y$. If $\varphi^*\colon A(Y) \to A(X)$ sends the class of y_i to the class of the polynomial $h_i \in k[x_1,\dots,x_m]$, give an explicit description of the tangent map $T\varphi\colon T_{X,P} \to T_{Y,Q}$, using the description of the tangent spaces given in Proposition 6.14.

Exercise 6.4. (i) If $F \in k[X_0,\dots,X_n]$ is homogeneous of degree d, prove Euler's formula $d \cdot F = \sum_{i=0}^n X_i \cdot \partial F/\partial X_i$.

(ii) Assume F is irreducible and let $X = \mathscr{Z}(F)$ be the hypersurface in $\mathbb{P}^n$ defined by F. Prove that $\mathrm{Sing}(X) = \mathscr{Z}\big(F, \partial F/\partial X_0,\dots,\partial F/\partial X_n\big)$.

(iii) If $\mathrm{char}(k) \nmid d$, show that $\mathrm{Sing}(X) = \mathscr{Z}\big(\partial F/\partial X_0,\dots,\partial F/\partial X_n\big)$. Give an explicit example that shows that this is not true, in general, if $\mathrm{char}(k)$ divides d.

Exercise 6.5. (i) Let F be a homogenous polynomial in $k[X_0,\dots,X_n]$ of positive degree. Let

$$S = \mathscr{Z}\Big(F, \frac{\partial F}{\partial X_0},\dots,\frac{\partial F}{\partial X_0}\Big)\,.$$

If F is reducible, show that S has an irreducible component of dimension at least $n-2$.

(ii) Assuming $\mathrm{char}(k) = 0$, determine S if

$$\begin{aligned} F = {} & X_0X_1^2 + 2X_0X_1X_3 + X_0X_2^2 + 2X_0X_3^2 + X_0X_4^2 \\ & +2X_0X_2X_3 + 2X_0X_2X_4 + 2X_0X_3X_4 + X_1^3 + 3X_1^2X_2 + 3X_1X_2^2 + X_2^3 \end{aligned}$$

and conclude that F is irreducible. [In this example, $X = \mathscr{Z}(F)$ is a threefold whose singular locus $S = \mathrm{Sing}(X)$ consists of three curves in X; we say that X has non-isolated singularities.]

Exercise 6.6. Let V be a k-vector space of finite dimension and consider the Grassmannian variety $\mathrm{Grass}(n,V)$ for some integer n. If $W \in \mathrm{Grass}(n,V)$, prove that the tangent space of $\mathrm{Grass}(n,V)$ at W is canonically isomorphic to the vector space $\mathrm{Hom}_k(W, V/W)$, in such a way that for $g \in \mathrm{GL}(V)$ we have a commutative diagram

$$\begin{array}{ccc} T_{\mathrm{Grass},W} & \xrightarrow{\sim} & \mathrm{Hom}_k(W,V/W) \\ {\scriptstyle TL_g}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_g} \\ T_{\mathrm{Grass},g(W)} & \xrightarrow{\sim} & \mathrm{Hom}_k\big(g(W),V/g(W)\big) \end{array}$$

Here $L_g\colon \mathrm{Grass}(n,V) \to \mathrm{Grass}(n,V)$ is the left action of g, i.e., the morphism $W \mapsto g(W)$, and $\varphi_g\colon \mathrm{Hom}_k(W,V/W) \xrightarrow{\sim} \mathrm{Hom}_k\big(g(W),V/g(W)\big)$ is the linear isomorphism given by $f \mapsto gfg^{-1}$.

COMMUTATIVE ALGEBRA 3

Some dimension theory

A3.1. Definition. Let R be a commutative ring with $1 \neq 0$.

(i) If $\mathfrak{p} \subset R$ is a prime ideal, the *height* of $\mathfrak{p}$, notation $\mathrm{ht}(\mathfrak{p})$, is the supremum of the integers r for which there exists a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$$

ending in $\mathfrak{p}$, with the understanding that $\mathrm{ht}(\mathfrak{p}) = \infty$ if there exist arbitrarily long such chains.

(ii) The *Krull dimension* of R, notation $\mathrm{Kdim}(R)$, is the supremum of the integers r for which there exists a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \,,$$

or, what is the same, the supremum of the heights of the prime ideals in R.

A3.2. Example. If R is noetherian, $\mathrm{Kdim}\big(R[x]\big) = \mathrm{Kdim}(R) + 1$. Caution: if R is not noetherian this is no longer true, in general.

By induction on n it follows that, for k a field, $\mathrm{Kdim}\big(k[x_1, \ldots, x_n]\big) = n$.

A3.3. Proposition. *Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring. Then*

$$\mathrm{ht}(\mathfrak{m}) = \mathrm{Kdim}(R) \leqslant \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) < \infty \,.$$

Caution: in general, a noetherian ring need not have finite Krull dimension. (Though it is hard to construct examples.)

A3.4. Theorem. *Let R be a noetherian ring.*

(i) *If $a \in R$ is not a unit or a zero divisor, every minimal prime ideal containing a has height* 1.

(ii) *Let $I = (a_1, \ldots, a_r)$ be an ideal generated by r elements. If $\mathfrak{p}$ is a minimal prime ideal containing I we have $\mathrm{ht}(\mathfrak{p}) \leqslant r$.*

This result is referred to as Krull's height theorem; part (i) goes under the name Krull's Hauptidealsatz.

A3.5. Theorem. *Let k be a field and R an affine k-algebra; this means that R is a finitely generated k-algebra without zero divisors.*

(i) *The Krull dimension of R is finite, and for any prime ideal $\mathfrak{p} \subset R$ we have the relation*

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{Kdim}(R/\mathfrak{p}) = \mathrm{Kdim}(R) \,.$$

(ii) *If $\mathfrak{q} \subset \mathfrak{p}$ are prime ideals of R and*

$$\mathfrak{q} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$$

is any maximal chain of prime ideals from $\mathfrak{q}$ *to* $\mathfrak{p}$*, we have* $r = \mathrm{ht}(\mathfrak{p}) - \mathrm{ht}(\mathfrak{q})$.

(iii) *We have* $\mathrm{Kdim}(R) = \mathrm{trdeg}_k\bigl(\mathrm{Frac}(R)\bigr)$*, the transcendence degree of the fraction field* $\mathrm{Frac}(R)$ *as an extension of* k.

The assertion in (ii) tells us that *any* maximal chain between two prime ideals correctly measures the difference in heights. Rings in which this holds are called *catenary rings*. Note that the relation in (i) is actually an immediate consequence of (ii).

A3.6. Definition. (i) A noetherian local ring $(R, \mathfrak{m}, k)$ is said to be *regular* if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \mathrm{Kdim}(R)$.

(ii) A *regular ring* is a noetherian ring R such that for every maximal ideal $\mathfrak{m} \subset R$ the localization $R_\mathfrak{m}$ is a regular local ring.

A3.7. Remarks. (i) In (i), note that $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is the minimal number of generators of $\mathfrak{m}$. Indeed, if $\mathfrak{m}$ can be generated by d elements then it is clear that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leqslant d$. Conversely, let $d = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and let $a_1, \ldots, a_d \in \mathfrak{m}$ be elements whose classes modulo $\mathfrak{m}^2$ form a k-basis. Then $\mathfrak{m} = (a_1, \ldots, a_d) + \mathfrak{m}^2$ and by Nakayama's Lemma it follows that $\mathfrak{m} = (a_1, \ldots, a_d)$.

(ii) Let $(R, \mathfrak{m})$ be a regular local ring. A result of Serre says that the localizations $R_\mathfrak{p}$, for $\mathfrak{p} \subset R$ a prime ideal, are again regular. Hence, the regularity of a ring R is equivalent to the requirement that all localizations $R_\mathfrak{p}$ be regular.

CHAPTER 7

Proper morphisms and complete varieties

§1. *Definition and basic properties of proper morphisms.*

7.1. Definition. (i) A morphism of varieties $\varphi\colon X \to Y$ is called a *proper morphism* if for all varieties Z the morphism $\varphi \times \mathrm{id}_Z\colon (X \times Z) \to (Y \times Z)$ is a closed map.

(ii) A variety X is said to be *complete* if the morphism $X \to \{\mathrm{pt}\}$ is proper.

Thus, X is complete if for all varieties Z the projection morphism $X \times Z \to Z$ is a closed map.

7.2. Remark. To motivate these definitions, let us remark that a topological space X is quasi-compact if and only if for all spaces Z the projection map $X \times Z \to Z$ is a closed map. A proof can be found in Bourbaki [2], Chap. 1, §10. Thus, completeness is the algebro-geometric analogue of compactness, just as in our definition of a variety the requirement that the diagonal is closed is an analogue of the Hausdorff condition in topology. Let us, once again, point out that the topology on a product variety is not the product topology (except when one of the factors is a point).

7.3. Example. Of course, for any variety X the map $X \to \{\mathrm{pt}\}$ is closed. So the whole point in the definition of completeness is that we require closedness of the projection $X \times Z \to Z$ for any variety Z. As an example, let us see that the affine line $\mathbb{A}^1$ is not complete. (It is hoped that this agrees with your geometric intuition about whether $\mathbb{A}^1$ ought to be "compact".) So we take $X = \mathbb{A}^1$. For Z we also take the affine line; so the projection map we consider is the map $\mathbb{A}^2 \to \mathbb{A}^1$ given by $(a, b) \mapsto b$. This is not a cosed map: $\mathscr{Z}(xy-1)$ is a closed set in $\mathbb{A}^2$ whose projection is $\mathbb{A}^1 \setminus \{0\}$, which is not closed in $\mathbb{A}^1$.

The following proposition gives some elementary properties of proper maps that follow directly from the definitions. We leave the proof to the reader.

7.4. Proposition.
(i) *If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are proper, $\psi \circ \varphi$ is proper.*
(ii) *Closed immersions are proper.*
(iii) *If $\varphi_1\colon X_1 \to Y_1$ and $\varphi_2\colon X_2 \to Y_2$ are proper, $(\varphi_1 \times \varphi_2)\colon (X_1 \times X_2) \to (Y_1 \times Y_2)$ is proper.*

The next proposition gives a partial converse of (i).

7.5. Proposition. *Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be morphisms of varieties.*
(i) *If $\psi \circ \varphi$ is proper then φ is proper.*
(ii) *If $\psi \circ \varphi$ is proper and φ is surjective, ψ is proper.*

Proof. (i) Let C be a closed subset of $X \times W$, for some W. The image of C under the morphism $X \times W \to X \times Y \times W$ given by $(x, w) \mapsto \big(x, \varphi(x), w\big)$ is closed, because this image equals $\mathrm{pr}_{XW}^{-1}(C) \cap \mathrm{pr}_{XY}^{-1}(\Gamma_\varphi)$, where $\mathrm{pr}_?$ denotes the projection onto the indicated factors and $\Gamma_\varphi \subset X \times Y$ is the graph of φ. (By Exercise 5.3(ii) this graph is closed.)

The properness of the morphism $\psi\varphi$ implies that $(\psi\varphi) \times \mathrm{id}_{Y\times W}\colon X \times Y \times W \to Z \times Y \times W$ is closed. Hence the image of C under the morphism $F\colon X \times W \to Z \times Y \times W$ given by $(x, w) \mapsto \big(\psi\varphi(x), \varphi(x), w\big)$ is closed. Now observe that $(\varphi \times \mathrm{id}_W)\big(C\big)$ is the pre-image of $F(C)$ under the map $Y \times W \to Z \times Y \times W$ given by $(y, w) \mapsto \big(\psi(y), y, w\big)$.

(ii) For $D \subset Y \times W$ we have $(\psi \times \mathrm{id}_W)\big(D\big) = (\psi\varphi \times \mathrm{id}_W)\big((\varphi \times \mathrm{id}_W)^{-1}(D)\big)$, because φ is surjective. Because $\psi\varphi \times \mathrm{id}_W$ is closed, it follows that $\psi \times \mathrm{id}_W$ is a closed map, too. □

7.6. Corollary. *If X is a complete variety, any morphism $\varphi\colon X \to Y$ is proper and the image of φ is a complete subvariety of Y.*

Proof. Apply Proposition 7.5 with $Z = \{\mathrm{pt}\}$. □

7.7. Corollary. *If X is a complete variety, $\mathscr{O}_X(X) = k$; that is, the only regular functions $X \to k$ are the constant functions.*

Proof. Let $f\colon X \to k$ be a regular function. It defines a morphism $f\colon X \to \mathbb{A}^1$. The image of f is a closed and complete subvariety of $\mathbb{A}^1$. As $\mathbb{A}^1$ is not complete, this implies that $f(X)$ is a single point. □

§2. *Completeness versus projectivity.*

7.8. Theorem. *For all n, the projective space $\mathbb{P}^n$ is complete.*

Proof. We have to show that for any variety Y the projection map $\mathrm{pr}_Y\colon \mathbb{P}^n \times Y \to Y$ is a closed map. It suffices to prove this for affine varieties Y, so from now on we assume that Y is affine with coordinate ring A. We consider the grading on the ring $S = A[X_0, \ldots, X_n]$ for which the elements of A have degree 0 and the variables X_i all have degree 1. Let $S_m \subset S$ be the subspace of homogeneous polynomials of degree m (together with 0); then $S = \oplus_{m\geqslant 0} S_m$. A homogeneous element $f \in S_m$ has a well-defined zero locus $\mathscr{Z}(f) \subset \mathbb{P}^n \times Y$.

Let $Z \subset \mathbb{P}^n \times Y$ be a nonempty closed subset. Define the homogeneous ideal of Z to be $I = \oplus_{m\geqslant 0} I_m$, where

$$I_m = \big\{f \in S_m \;\big|\; Z \subset \mathscr{Z}(f)\big\}\,.$$

One checks without any trouble that I is indeed a homogeneous ideal of $A[X_0, \ldots, X_n]$. Further, $Z = \mathscr{Z}(I) = \cap_{m\geqslant 0} \cap_{f\in I_m} \mathscr{Z}(f)$.

Let $P \in Y \setminus \mathrm{pr}_Y(Z)$, and let $\mathfrak{m} \subset A$ be the corresponding maximal ideal. We claim that there exists an integer $M > 0$ such that $X_i^M \in I + \mathfrak{m}A[X_0, \ldots, X_n]$ for all $i = 0, \ldots, n$. Before proving this, let us show how this implies the theorem. Assuming the claim is true, there is an integer $N > 0$ such that any monomial of degree N is contained in $I + \mathfrak{m}A[X_0, \ldots, X_n]$. (Concretely, we can take $N = (n+1)M$.) So $S_N = I_N + \mathfrak{m}S_N$. By Nakayama's Lemma (use (ii) of Corollary A1.12) it follows that there is an element $f \in 1 + \mathfrak{m} \subset A$ such that $f \cdot S_N \subset I_N$. In particular, $f \cdot X_i^N \in I$ for all i, and this implies that $f \in I$. Then the basic open subset $D(f) \subset Y$ contains P and is disjoint from $\mathrm{pr}_Y(Z)$. This proves that $\mathrm{pr}_Y(Z) \subset Y$ is closed.

It remains to prove the claim. Fix an index i and let $U_i \subset \mathbb{P}^n$ be the corresponding standard open subset. Then $U_i \times Y \cong \mathbb{A}^n \times Y$ is affine with coordinate ring $A[y_1, \ldots, y_n]$. The closed subset $U_i \times \{P\}$ corresponds to the ideal $\mathfrak{m}A[y_1,\ldots,y_n]$. If $F_1,\ldots,F_r \in A[X_0,\ldots,X_n]$ are homogeneous generators of the ideal I then the closed subset $Z \cap (U_i \times Y) \subset U_i \times Y$ is the zero locus of the ideal $\mathrm{dehom}_i(I)$ generated by the "dehomogenized" polynomials $\mathrm{dehom}_i(F_j) = F_j(y_1,\ldots,y_{i-1},1,y_i,\ldots,y_n)$. The assumption that $P \notin \mathrm{pr}_Y(Z)$ implies that the closed subsets $U_i \times \{P\}$ and $Z \cap (U_i \times Y)$ are disjoint. On ideals this means that the radical of the ideal

$$\mathrm{dehom}_i(I) + \mathfrak{m}A[y_1,\ldots,y_n]$$

is the whole ring $A[y_1,\ldots,y_n]$. Hence $1 \in \mathrm{dehom}_i(I) + \mathfrak{m}A[y_1,\ldots,y_n]$. Clearing denominators, it follows that there is a positive integer M_i such that $X_i^{M_i} \in I + \mathfrak{m}A[X_0,\ldots,X_n]$, and taking $M = \max\{M_0,\ldots,M_n\}$ then gives the claim. □

7.9. Corollary. *Any projective variety is complete.*

7.10. Remarks. (i) One thing we have used in the proof, and that we shall continue to use, is the following. If Y is an affine variety with coordinate ring A then there is an inclusion-reversing bijection between the set of closed subvarieties of $\mathbb{P}^n \times Y$ and the set of homogeneous prime ideals in $A[X_0,\ldots,X_n]$ that do not contain the ideal $(X_0,\ldots,X_n)$. This is an extension of the statements in Exercise 3.2.

(ii) Let $Z \subset \mathbb{P}^n \times Y$ be a closed subvariety that corresponds with the homogeneous prime ideal $\mathfrak{p} \subset A[X_0,\ldots,X_n]$. Then the image of Z in Y is a closed subvariety of Y. It corresponds with the prime ideal $(\mathfrak{p} \cap A) \subset A$. (Make sure you understand this.) The fact that $\mathrm{pr}_Y(Z) \subset Y$ is closed means, algebraically, that for every maximal ideal $\mathfrak{m} \subset A$ that contains $\mathfrak{p} \cap A$, there exists a homogeneous maximal ideal $\mathfrak{n} \subset A[X_0,\ldots,X_n]$ with $\mathfrak{m} = \mathfrak{n} \cap A$.

If $\mathfrak{p}$ is generated by homogeneous polynomials F_j, to determine equations for $\mathrm{pr}_Y(Z)$ we have to find generators for $\mathfrak{p} \cap A$. This is called "elimination theory", as it means we have to find polynomials in the F_j in which the variables X_i have all been eliminated. HAG, Chapter I, Theorem 5.7A gives a purely algebraic statement that is equivalent to Theorem 7.8.

7.11. Example. Fix an integer $d > 1$. Write the coordinate ring of $\mathbb{A}^{d+1}$ as $A = k[c_0,\ldots,c_d]$, where the c_i are variables. Then

$$F = c_0X^d + c_1X^{d-1}Y + \cdots + c_{d-1}XY^{d-1} + c_dY^d$$

is a homogeneous polynomial in $A[X,Y]$. Let $Z \subset \mathbb{P}^1 \times \mathbb{A}^{d+1}$ be the zero locus of F. Next, let $S \subset Z$ be the zero locus of the ideal $(F, \partial F/\partial X, \partial F/\partial Y)$. By completeness of $\mathbb{P}^1$, the image of S under the projection map $\mathbb{P}^1 \times \mathbb{A}^{d+1} \to \mathbb{A}^{d+1}$ is a closed subset of $\mathbb{A}^{d+1}$. Hence the intersection of this image with the hyperplane $\mathbb{A}^d \subset \mathbb{A}^{d+1}$ given by $c_0 = 1$ is a closed subset $D \subset \mathbb{A}^d$.

Let $a = (1, a_1,\ldots,a_d)$ be a point of this hyperplane $\mathbb{A}^d$. Write $F_a = X^d + a_1X^{d-1}Y + \cdots + a_dY^d$ for the corresponding homogeneous polynomial and $f_a = x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d$ for the corresponding monic inhomogeneous polynomial. Now one checks that $a \in D$ if and only if f_a and $\partial f_a/\partial x$ have a common zero, which means precisely that f_a has a double root. The conclusion, therefore, is that if we consider monic polynomials f_a of a given degree d, there are polynomials $g_1,\ldots,g_r$ in $k[c_1,\ldots,c_d]$ (namely the polynomials that define the closed subset $D \subset \mathbb{A}^d$) such that

$$f_a \text{ has a double root} \quad \Leftrightarrow \quad g_1(a) = \cdots = g_r(a) = 0\,.$$

But, in fact, in this particular example we know this, and we even know we can take $r = 1$. Indeed,

$$f_a \text{ has a double root} \quad \Leftrightarrow \quad \mathrm{discr}(f_a) = 0\,,$$

and the discriminant $\mathrm{discr}(f_a)$ is a polynomial in the coefficients a_i of f.

7.12. Example. We now consider the analogue of the previous example with several variables. Fix integers $n \geqslant 2$ and $d \geqslant 1$. Let $\mathscr{M}$ denote the set of all $(n+1)$-tuples $(m_0, \ldots, m_n)$ with $m_0 + \cdots + m_n = d$. To $\mu \in \mathscr{M}$ corresponds the monomial $X^\mu := X_0^{m_0} X_1^{m_1} \cdots X_n^{m_n}$ of degree d. In this way the affine space $\mathbb{A}^{\mathscr{M}} \cong \mathbb{A}^{\binom{d+n}{n}}$ parametrizes the homogeneous polynomials in $k[X_0, \ldots, X_n]$ of degree d. Write the coordinate ring of this affine space as $A = k[c_\mu; \mu \in \mathscr{M}]$, and we have a universal polynomial

$$F = \sum_{\mu \in \mathscr{M}} c_\mu X^\mu$$

in $A[X_0, \ldots, X_n]$.

If $a = (a_\mu)_{\mu \in \mathscr{M}}$ is a point of $\mathbb{A}^{\mathscr{M}}$ and $F_a = \sum_{\mu \in \mathscr{M}} a_\mu X^\mu$ is the corresponding homogeneous polynomial, it follows from Exercises 6.4 and 7.2 that the zero set of the ideal $(F_a, \partial F_a/\partial X_0, \ldots, \partial F_a/\partial X_n)$ is empty if and only if the hypersurface $\mathscr{Z}(F_a) \subset \mathbb{P}^n$ defined by F_a is irreducible and non-singular. (We use that $n \geqslant 2$.) Now the theorem tells us that the locus of $a \in \mathbb{A}^{\mathscr{M}}$ for which $\mathscr{Z}(F_a)$ is irreducible and non-singular is Zariski-open; in other words: there are polynomial conditions on the coefficients a_μ that express that F_a is either reducible or defines a singular hypersurface. (In this discussion we make the convention that we view the zero polynomial as a reducible polynomial.)

To make this precise, let $Z = \mathscr{Z}(F) \subset \mathbb{P}^n \times \mathbb{A}^{\mathscr{M}}$, and let $S \subset Z$ be the zero locus of the ideal $(F, \partial F/\partial X_0, \ldots, \partial F/\partial X_n)$. The image of S in $\mathbb{A}^{\mathscr{M}}$ under the projection map pr: $\mathbb{P}^n \times \mathbb{A}^{\mathscr{M}} \to \mathbb{A}^{\mathscr{M}}$ is a closed subvariety. Let $g_1, \ldots, g_r \in k[c_\mu; \mu \in \mathscr{M}]$ be generators for the ideal of $\mathrm{pr}(S)$. Then the conclusion is that

$$\mathscr{Z}(F_a, \partial F_a/\partial X_0, \ldots, \partial F_a/\partial X_n) \neq \emptyset \quad \Leftrightarrow \quad g_1(a) = \cdots = g_r(a) = 0\,.$$

What is true, but lies deeper, is that in this example again a *single* polynomial condition on the coefficients a_μ suffices: there exists a discriminant polynomial $\mathrm{discr} \in k[c_\mu; \mu \in \mathscr{M}]$ such that $\mathscr{Z}(F_a) \subset \mathbb{P}^n$ is reducible or singular if and only if $\mathrm{discr}(a) = 0$.

Let $U = D(\mathrm{discr}) \subset \mathbb{A}^{\mathscr{M}}$ be the open subset given by the condition $\mathrm{discr} \neq 0$. The intersection $Z \cap (\mathbb{P}^n \times U)$ may be viewed as a family of non-singular hypersurfaces of degree d, parametrized by U. Note, however, that a given non-singular hypersurface occurs in this family infinitely often. Indeed, if $a = (a_\mu)_{\mu \in \mathscr{M}}$ is a point of $\mathbb{A}^{\mathscr{M}}$ and $\lambda \in k^*$ then $\lambda \cdot a = (\lambda a_\mu)$ corresponds to the same hypersurface. (We are just rescaling the equation.) Writing $s + 1 = \#\mathscr{M} = \binom{d+n}{n}$, it is therefore more natural to work with the projective space $\mathbb{P}^s$ of non-zero homogeneous polynomials of degree d up to scalar multiplication. There is then a Zariski open subset $V \subset \mathbb{P}^s$ corresponding to the non-singular hypersurfaces in $\mathbb{P}^n$ of degree d and a universal family $H \subset \mathbb{P}^n \times V$ of non-singular hypersurfaces.

7.13. Theorem. (Chow's lemma) *Let X be a complete variety. Then there exists a projective variety Y and a surjective morphism $Y \to X$ that is a birational map.*

For a proof we refer to MRB, Chapter I, §10. It can be shown that every complete curve is projective, and that every complete non-singular surface is projective. But for $d \geqslant 3$ there exist non-singular complete varieties of dimension d that are not projective.

§3. *Some classical constructions in projective geometry.*

7.14. As a first application of the techniques we now have at our disposal, let us construct the *join* of two projective varieties. The situation here is that we are given two disjoint projective varieties X, $Y \subset \mathbb{P}^n$; their join is then given by

$$J(X,Y) = \bigcup_{P \in X, Q \in Y} \overline{PQ}$$

where $\overline{PQ}$ is the line through P and Q. We claim that $J(X,Y)$ is again a projective variety.

To prove that $J(X,Y)$ is closed in $\mathbb{P}^n$, consider the Grassmannian $\mathbb{G}(1,\mathbb{P}^n) = \mathrm{Grass}(2,n{+}1)$ of lines in $\mathbb{P}^n$. (See Remark 5.26(ii).) Then we have a morphism $X \times Y \to \mathbb{G}(1,\mathbb{P}^n)$ given by $(P,Q) \mapsto \overline{PQ}$. As $X \times Y$ is complete (see Exercise 7.1) the image $Z \subset \mathbb{G}(1,\mathbb{P}^n)$ of this morphism is closed.

Next we consider the incidence variety $\mathbb{I} \subset \mathbb{P}^n \times \mathbb{G}(1,\mathbb{P}^n)$, given by

$$\mathbb{I} = \bigl\{(P,\ell) \in \mathbb{P}^n \times \mathbb{G}(1,\mathbb{P}^n) \bigm| P \in \ell\bigr\}\,.$$

This is a closed subvariety of $\mathbb{P}^n \times \mathbb{G}(1,\mathbb{P}^n)$, and we have the two projections $\mathrm{pr}_1\colon \mathbb{I} \to \mathbb{P}^n$ and $\mathrm{pr}_2\colon \mathbb{I} \to \mathbb{G}(1,\mathbb{P}^n)$. Now we observe that

$$J(X,Y) = \mathrm{pr}_1\bigl(\mathrm{pr}_2^{-1}(Z)\bigr)\,,$$

and because the Grassmannian $\mathbb{G}(1,\mathbb{P}^n)$ is a projective, and hence complete, variety, the first projection map is closed. Hence $J(X,Y)$ is closed in $\mathbb{P}^n$.

To prove that $J(X,Y)$ is irreducible, we make a small detour. Let $q\colon \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the natural morphism. Given a non-empty $V \subset \mathbb{P}^n$ we define the *cone over* V, notation $\mathrm{C}(V)$, to be the closure of $q^{-1}(V)$ inside $\mathbb{A}^{n+1}$, or, what is the same, $q^{-1}(V) \cup \{O\}$. If V is the zero set of some homogeneous polynomials $F_1,\ldots,F_r$, the cone $\mathrm{C}(V) \subset \mathbb{A}^{n+1}$ is the zero set of these same polynomials but where we now take the zero locus inside $\mathbb{A}^{n+1}$. Further, V is irreducible if and only if $\mathrm{C}(V)$ is irreducible. (Cf. Exercise 3.2.)

The reason that it is convenient to consider cones is the following. If $P = (a_0 : \cdots : a_n)$ and $Q = (b_0 : \cdots : b_n)$ are distinct points, the line $\overline{PQ}$ consists of all points of the form

$$(\lambda a_0 + \mu b_0 : \cdots : \lambda a_n + \mu b_n)$$

for $(\lambda : \mu) \in \mathbb{P}^1$. So we are tempted to describe the join $J(X,Y)$ simply as the image of a morphism $X \times Y \times \mathbb{P}^1 \to \mathbb{P}^n$ given by

$$\Bigl((a_0 : \cdots : a_n), (b_0 : \cdots : b_n), (\lambda : \mu)\Bigr) \quad \overset{?}{\mapsto} \quad (\lambda a_0 + \mu b_0 : \cdots : \lambda a_n + \mu b_n)\,.$$

This, however, is not a well-defined map! Instead, what we can do is to consider the map $\mathrm{C}(X) \times \mathrm{C}(Y) \times \mathbb{A}^2 \to \mathbb{A}^{n+1}$ given by

$$\Big((a_0, \ldots, a_n), (b_0, \ldots, b_n), (\lambda, \mu)\Big) \mapsto (\lambda a_0 + \mu b_0, \ldots, \lambda a_n + \mu b_n)\,.$$

The image is precisely the cone over $J(X, Y)$, and as this image is irreducible (being the image of an irreducible topological space), this proves that $J(X, Y)$ is irreducible.

7.15. In the construction of the join $J(X, Y)$ we have used the assumption that X and Y are disjoint. It is, however, of interest to extend the construction to more general cases. In fact, a classical example of a join concerns the case where $X = Y$; the resulting variety is called the *secant variety* of X. We shall now discuss it in more detail.

So, let a projective variety $X \subset \mathbb{P}^n$ be given, with $\dim(X) > 0$. If P and Q are distinct points of X they determine a line $\overline{PQ}$. This gives us a rational map $X \times X \dashrightarrow \mathbb{G}(1, \mathbb{P}^n)$ whose domain of definition contains $(X \times X) \setminus \Delta(X)$. Define $\mathscr{S} \subset \mathbb{G}(1, \mathbb{P}^n)$ to be the closure of the image of this rational map. (Cf. Exercise 5.8.)

The *secant variety of* X is the closed subvariety $\mathrm{Sec}(X) \subset \mathbb{P}^n$ obtained as the union of the lines in $\mathscr{S}$. Using the incidence variety $\mathbb{I}$ introduced in 7.14 with its projections $\mathrm{pr}_1 \colon \mathbb{I} \to \mathbb{P}^n$ and $\mathrm{pr}_2 \colon \mathbb{I} \to \mathbb{G}(1, \mathbb{P}^n)$, this is

$$\mathrm{Sec}(X) = \mathrm{pr}_1\big(\mathrm{pr}_2^{-1}(\mathscr{S})\big)\,.$$

Similar to how we argued for joins, the closedness of $\mathrm{Sec}(X)$ follows from the completeness of the Grassmannian: $\mathscr{S}$ is closed by construction, hence $\mathrm{pr}_2^{-1}(\mathscr{S})$ is closed, and by completeness of $\mathbb{G}(1, \mathbb{P}^n)$ the first projection pr_1 is a closed map. The irreducibility of $\mathrm{Sec}(X)$ is a little more subtle; we shall not prove it in these notes.

An interesting point is that the secant variety is, in general, bigger than the union of the lines $\overline{PQ}$ for P, $Q \in X$ distinct. So a natural question is how much we have added. For non-singular varieties we shall give a precise answer in Proposition 7.18.

7.16. Continuing with a projective variety $X \subset \mathbb{P}^n$, we have defined for any $P \in X$ the tangent space $T_{X,P}$ as an abstract vector space. In a more geometric approach, we could also think of the tangent space as a linear subspace of $\mathbb{P}^n$ that is tangent to X at the point P. While this is a less intrinsic notion (it depends on the chosen projective embedding of X), this projective tangent space is a natural object from a geometric perspective.

To make this precise, let $F_1, \ldots, F_r \in k[X_0, \ldots, X_n]$ be homogeneous generators for the ideal of X. If $P = (a_0 : \cdots : a_n) \in X$, we define the *projective tangent space* to X at P (with respect to the given projective embedding) to be the linear subspace $\mathbb{T}_{X,P} \subset \mathbb{P}^n$ defined by the linear equations

$$\frac{\partial F_j}{\partial X_0}(P) \cdot X_0 + \cdots + \frac{\partial F_j}{\partial X_n}(P) \cdot X_n\,, \qquad \text{for } j = 1, \ldots, r.$$

Caution: there is an abuse of notation here. As remarked before, if G is a homogeneous polynomial, G has no well-defined value at P. So what we mean in our definition of $\mathbb{T}_{X,P}$ is that we fix some $(a_0, \ldots, a_n) \in k^{n+1} \setminus \{O\}$ such that $P = (a_0 : \cdots : a_n)$; then we evaluate the polynomials $\partial F_j/\partial X_j$ on these coordinates. The equations we get depend on the chosen representation of P, but only up to scaling by a nonzero factor; hence the space $\mathbb{T}_{X,P}$ defined by these equations is independent of choices. Further note that $P \in \mathbb{T}_{X,P}$ because of Euler's identity. (See Exercise 6.4.)

To understand the relation with the Zariski tangent space, we may simplify calculations by assuming that $P = (1 : 0 : \cdots : 0)$. (Recall that $\mathrm{PGL}_{n+1}(k)$ acts transitively on $\mathbb{P}^n$.) Let $f_j = \mathrm{dehom}_0(F_j) \in k[y_1, \ldots, y_n]$ be the dehomogenized versions of the polynomials F_j. Then $X \cap U_0$ is the zero locus of the ideal $(f_1, \ldots, f_r)$; further, for all indices j we have

$$\frac{\partial F_j}{\partial X_0}(P) = 0 \quad \text{and} \quad \frac{\partial F_j}{\partial X_i}(P) = \frac{\partial f_j}{\partial y_i}(0, \ldots, 0) \qquad \text{if } i > 0.$$

(Check this!) So we find that $\mathbb{T}_{X,P} \cap U_0$ can be identified with the Zariski tangent space $T_{X,P}$, now viewed as a linear subspace of $U_0 \cong \mathbb{A}^n$. In particular, $\dim(\mathbb{T}_{X,P}) = \dim_k(T_{X,P})$ for all P. Do make sure, though, not to confuse the two notions of a tangent space. Whereas $T_{X,P}$ is an intrinsically defined k-vector space, $\mathbb{T}_{X,P}$ is a linear projective subvariety of $\mathbb{P}^n$; in particular, it is not a vector space.

7.17. Continuing the above discussion, let us now assume $X \subset \mathbb{P}^n$ is a non-singular projective variety of dimension d. In this case we obtain a morphism $t\colon X \to \mathbb{G}(d, \mathbb{P}^n)$, sending $P \in X$ to $\mathbb{T}_{X,P} \in \mathbb{G}(d, \mathbb{P}^n)$. The image is a closed subvariety $t(X) \subset \mathbb{G}(d, \mathbb{P}^n)$. Similar to what we did in 7.14, we have the incidence variety

$$\mathbb{I}_{0,d} \subset \mathbb{P}^n \times \mathbb{G}(d, \mathbb{P}^n)$$

consisting of the pairs (p, V) with $P \in V$. (Cf. Exercise 7.3.) Letting

$$\mathbb{P}^n \xleftarrow{\mathrm{pr}_1} \mathbb{I}_{0,d} \xrightarrow{\mathrm{pr}_2} \mathbb{G}(d, \mathbb{P}^n)$$

denote the projections, we define the *tangential variety* of X to be the closed subvariety $\mathbb{T}(X) \subset \mathbb{P}^n$ given by $\mathbb{T}(X) = \mathrm{pr}_1\bigl(\mathrm{pr}_2^{-1}(t(X))\bigr)$. Thus, $\mathbb{T}(X)$ is the union of all projective tangent spaces $\mathbb{T}_{X,P}$. Similar to the earlier examples, the closedness of $\mathbb{T}(X)$ follows from the projectivity of the Grassmannian. Again, we shall omit the proof that $\mathbb{T}(X)$ is irreducible, as we do not yet have enough theory at our disposal for this.

Using the notions discussed so far, we can say more, at least for non-singular varieties, about which lines other than the honest secant lines are included in the secant variety. The result is that the lines that are added are precisely the lines in $\mathbb{P}^n$ that are tangent to X at some point.

7.18. Proposition. *Let $X \subset \mathbb{P}^n$ be a non-singular projective variety. Then the secant variety* $\mathrm{Sec}(X)$ *is the union of all honest secant lines $\overline{PQ}$, for $P \neq Q \in X$ and the tangential variety* $\mathbb{T}(X)$.

For a proof we refer to Harris's book [5], Prop. 15.10. As an application of these techniques, let us now give a sketch of a classical result about projective embeddings.

7.19. Theorem. *Let X be a non-singular projective variety of dimension d. Then X can be embedded into $\mathbb{P}^{2d+1}$.*

Sketch of the proof. As we assume that X is projective, there exists a closed embedding $X \hookrightarrow \mathbb{P}^N$ for some N. The idea is that for $N > 2d+1$ we may choose a point $R \notin X$ and a hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ not containing R, such that the projection from R gives an embedding $X \hookrightarrow \mathbb{P}^{N-1}$. Continuing in this way we get the result.

What needs to be proven is that for a suitable choice of the point R the projection map $\pi_R\colon X \to \mathbb{P}^{N-1}$ is indeed an embedding. The idea is that we have to choose R not to lie on the secant variety $\mathrm{Sec}(X)$ for the given embedding $X \subset \mathbb{P}^N$. The logic behind this is clear: if $R \notin \mathrm{Sec}(X)$ then π_R is injective (indeed, $\pi_R(P) = \pi_R(Q)$ for $P \neq Q \in X$ just means that $R \in \overline{PQ}$). Further, using that R does not lie on any projective tangent space $\mathbb{T}_{X,P}$ (by Proposition 7.18), one shows that $\pi_R\colon X \to \mathbb{P}^{N-1}$ is an immersion.

What remains is only a dimension count: the secant variety $\mathrm{Sec}(X)$ has dimension at most $2d+1$; so as long as $N > 2d+1$ we can find $R \in \mathbb{P}^N$ not in $\mathrm{Sec}(X)$. □

Exercises for Chapter 7.

Exercise 7.1. Show that a product variety $X \times Y$ is complete if and only if X and Y are both complete.

Exercise 7.2. (i) Let G, $H \in k[X_0, \ldots, X_n]$ be homogeneous polynomials of positive degree. Prove that $\mathscr{Z}(G, H) \neq \emptyset$ if $n \geqslant 2$. [*Hint:* Use some dimension theory.]

(ii) Let $F \in k[X_0, \ldots, X_n]$ with $n \geqslant 2$ be a homogeneous polynomial of degree $d \geqslant 1$. If F is reducible, prove there exist homogeneous polynomials G, H in $k[X_0, \ldots, X_n]$ of positive degree such that $F = G \cdot H$. Conclude that $\mathscr{Z}(F, \partial F/\partial X_0, \ldots, \partial F/\partial X_n)$ is non-empty.

Exercise 7.3. Given integers $0 \leqslant d \leqslant e \leqslant n$, let $\mathbb{I}_{d,e} \subset \mathbb{G}(d, \mathbb{P}^n) \times \mathbb{G}(e, \mathbb{P}^n)$ be given by

$$\mathbb{I}_{d,e} = \big\{ (V, W) \in \mathbb{G}(d, \mathbb{P}^n) \times \mathbb{G}(e, \mathbb{P}^n) \bigm| V \subset W \big\}.$$

(This is the incidence variety of d-planes contained in e-planes in $\mathbb{P}^n$.) Prove that $\mathbb{I}_{d,e}$ is a non-singular closed subvariety of $\mathbb{G}(d, \mathbb{P}^n) \times \mathbb{G}(e, \mathbb{P}^n)$.

Exercise 7.4. Let $C \subset \mathbb{P}^n$ be a non-singular projective curve. Show that the morphism $(C \times C) \setminus \Delta(C) \to \mathbb{G}(1, \mathbb{P}^n)$ sending (P, Q) to $\overline{PQ}$ extends to a morphism $C \times C \to \mathbb{G}(1, \mathbb{P}^n)$, sending a point (P, P) on the diagonal to the projective tangent line $\mathbb{T}_{C,P} \in \mathbb{G}(1, \mathbb{P}^n)$.

Exercise 7.5. (Rigidity Lemma) Let X, Y and Z be varieties, with X complete. Let $f\colon X \times Y \to Z$ be a morphism. Suppose that there is a point $y_0 \in Y$ such that f is constant on $X \times \{y_0\}$. Prove that f factors through the projection $\mathrm{pr}_Y\colon X \times Y \to Y$, i.e., f is constant on every slice $X \times \{y\}$. [*Hint:* choose a point $x_0 \in X$, and define $g\colon Y \to Z$ by $g(y) = f(x_0, y)$. Choose an affine open neighbourhood U of $z_0 = f(x_0, y_0)$, and consider the set $W = \mathrm{pr}_Y\big(f^{-1}(Z \setminus U)\big)$. Prove that W is a proper closed subset of Y, and that f is constant on $X \times \{y\}$ for all $y \in Y \setminus W$. Then conclude that $f = g \circ \mathrm{pr}_Y$.]

Exercise 7.6. (i) Let X, Y be group varieties with identity elements e_X and e_Y. Assume that X is a complete variety. Let $f\colon X \to Y$ be a morphism of varieties. Prove that f is a homomorphism followed by a left multiplication; i.e., prove that there is an $y \in Y$ and a homomorphism $g\colon X \to Y$ such that $f(x) = y * g(x)$ for all $x \in X$, where $*$ is the group law on Y. [*Hint:* take $y = f(e_X)$; then the morphism $g\colon X \to Y$ given by $g(x) = y^{-1} * f(x)$ has the property that $g(e_X) = e_Y$. Rephrase the condition that g be a homomorphism in terms of the constancy of a certain map $X \times X \to Y$. Use the Rigidity Lemma of the previous exercise to prove this constancy.]

(ii) Prove that a complete group variety is commutative.

A complete group variety is called an abelian variety.

CHAPTER 8

Normalization. Applications to curves.

§1. *Finite morphisms.*

8.1. In what follows, we are going to need a criterion to decide if a given variety is affine. If X is any variety and $f \in \mathscr{O}(X)$ is a regular function on X, define $D(f) = X \setminus \mathscr{Z}(f)$. This extends the definition we have seen in 1.25. If the context requires it, we write $D_X(f)$ instead of $D(f)$.

8.2. Lemma. *Let X be a variety and $f \in \mathscr{O}(X)$. Then $\mathscr{O}(X)_f = \mathscr{O}_X\bigl(D(f)\bigr)$.*

Proof. If $U \subset X$ is affine and $f|_U \in A(U)$ denotes the restriction of f to U then $D(f) \cap U = D(f|_U)$. Clearly $\mathscr{O}(X)_f \subset \mathscr{O}_X\bigl(D(f)\bigr)$ as subrings of $k(X)$. To see that they are equal, suppose $g \in \mathscr{O}_X\bigl(D(f)\bigr)$ is a regular function on $D(f)$. If $U \subset X$ is affine then we know that $\mathscr{O}_U\bigl(D(f|_U)\bigr) = A(U)_{f|_U}$. Hence there exists an integer $n = n_U$ (depending on U) such that $(f^n g)|_{U\cap D(f)}$ extends to a regular function on U. As X is quasi-compact, it follows there exists an integer n (the maximum of the n_U for some finite open cover of X) such that $f^n g$ extends to a regular function on X. Hence $g \in \mathscr{O}(X)_f$ and it follows that $\mathscr{O}(X)_f = \mathscr{O}_X\bigl(D(f)\bigr)$. □

8.3. Lemma. *Let A be a k-algebra which is a domain. Suppose there exists $f_1, \dots, f_r \in A$ with $(f_1, \dots, f_r) = A$ and such that the localizations A_{f_i} are all f.g. as k-algebras. Then A is f.g. as k-algebra.*

Proof. The assumption that $(f_1, \dots, f_r) = A$ means that there exist $a_1, \dots, a_r \in A$ with $a_1 f_1 + \cdots + a_r f_r = 1$. Next, choose generators $\{t_{ij}\}_{j=1,\dots,N(i)}$ for A_{f_i} as a k-algebra, and write $t_{ij} = \theta_{ij}/f_i^{m_{ij}}$ with $\theta_{ij} \in A$.

We claim that A is generated, as a k-algebra, by the elements

$$a_1, \dots, a_r,\ f_1, \dots, f_r, \quad \theta_{ij} \quad (i = 1, \dots, r; j = 1, \dots, N(i))\,.$$

To prove this, let $B \subset A$ be the k-subalgebra generated by these elements. Note that the relation $a_1 f_1 + \cdots + a_r f_r = 1$ is a relation in B; so $Bf_1 + \cdots + Bf_r = B$. If M is a positive integer the radical of $(f_1^M, \dots, f_r^M)$ in the ring B clearly contains the f_i and is therefore again the whole ring B. This means there exist elements $h_1, \dots, h_r \in B$ with $h_1 f_1^M + \cdots + h_r f_r^M = 1$.

Now let $g \in A$. Our goal is to show that $g \in B$. We can find a positive integer M and polynomials $P_i \in k[y, x_1, \dots, x_{N(i)}]$ such that, for each i,

$$g = \frac{P_i(f_i, \theta_{i,1}, \dots, \theta_{i,N(i)})}{f_i^M}$$

in Frac(A). This gives

$$\begin{aligned} g &= \bigl(h_1 f_1^M + \cdots + h_r f_r^M\bigr) \cdot g \\ &= h_1 P_1(f_1, \theta_{1,1}, \dots, \theta_{1,N(1)}) + \cdots + h_r P_r(f_r, \theta_{r,1}, \dots, \theta_{r,N(r)}) \end{aligned}$$

and this is an element of B. □

8.4. Proposition. *Let X be a variety. Suppose there exist elements $f_1, \ldots, f_r \in \mathscr{O}(X)$ with $(f_1, \ldots, f_r) = \mathscr{O}(X)$ such that each $D(f_i)$ is affine. Then X is affine.*

Proof. Let $A = \mathscr{O}(X)$. By Lemma 8.2 the rings A_{f_i} are affine k-algebras. By Lemma 8.3 it follows that A is an affine algebra. Hence there exists an affine variety Y, unique up to isomorphism, with coordinate ring A. The homomorphism (in fact, isomorphism) $A(Y) \to \mathscr{O}(X)$ corresponds to a morphism $\varphi\colon X \to Y$. (See Exercise 5.1.) For each f_i we have $D_X(f_i) = \varphi^{-1}\big(D_Y(f_i)\big)$. Further, φ induces isomorphisms $D_X(f_i) \xrightarrow{\sim} D_Y(f_i)$ because both sides are affine and the corresponding homomorphism on rings is an isomorphism by Proposition 2.20 and Lemma 8.2. Finally, since the ideal generated by the f_i is the whole ring, $Y = \cup_{i=1}^{r} D_Y(f_i)$; hence φ is an isomorphism. □

8.5. Caution. In general, if X is a variety, $\mathscr{O}(X)$ is *not* finitely generated as a k-algebra.

8.6. Proposition. *Let $\varphi\colon X \to Y$ be a morphism of varieties. Then the following two conditions are equivalent.*

(a) *There exists an affine open covering $Y = \cup_{j=1}^{m} V_j$ such that for every index j the inverse image $U_j = \varphi^{-1}(V_j)$ is again affine, and such that $A(U_j)$ is finitely generated as a module over $A(V_j)$.*

(b) *For every affine open $V \subset Y$ the pre-image $U = \varphi^{-1}(V)$ is again affine, and $A(U)$ is finitely generated as a module over $A(V)$.*

Proof. Clearly, (b) ⇒ (a). The proof of the converse proceeds in several steps. For brevity, if $W \subset Y$ is affine, let us say that φ is finite above W if $\varphi^{-1}(W)$ is again affine and $A\big(\varphi^{-1}(W)\big)$ is a f.g. $A(W)$-module.

If φ is finite above W and $f \in A(W) = \mathscr{O}_Y(W)$ then φ is also finite above the basic open subset $D(f) \subset W$. Indeed, if we let $B = A\big(\varphi^{-1}(W)\big)$ we have the homomorphism $\varphi^*\colon A(W) \to B$ and $\varphi^{-1}\big(D(f)\big)$ is the basic open subset $D\big(\varphi^*(f)\big)$ of $\varphi^{-1}(W)$. Further, because B is f.g. as an $A(W)$-module, $B_{\varphi^*(f)}$ is f.g. as an $A(W)_f$-module.

We assume that (a) is true. We take an affine $W \subset Y$. Our goal is to prove that φ is finite above W. Without loss of generality we may assume that $\varphi(X)$ is dense in Y. (If not, replace Y by the closure of $\varphi(X)$.)

We first reduce to the case where $Y = W$. By assumption, for every $w \in W$ there exists an affine open $V \subset Y$ containing w such that φ is finite above V. As the basic open subsets $D_V(h)$, for $h \in A(V)$, form a basis for the topology on V, we may find an $h \in A(V)$ such that $w \in D_V(h) \subset W$. On rings, the inclusion $D_V(h) \hookrightarrow W$ corresponds to a homomorphism $A(W) \hookrightarrow A(V)_h$, which is injective because $D_V(h)$ is dense in W. Next we may choose an $f \in A(W)$ such that $w \in D_W(f) \subset D_V(h)$. Note that $D_W(f)$ is then the same, identifying f with its image in $A\big(D(h)\big) = A(V)_h$, as the basic open $D_V(fh) \subset V$. On rings: $A(W)_f \xrightarrow{\sim} A(V)_{fh}$. In particular, using what we explained above, φ is finite above $D_W(f)$. In this way we see that we can cover W by basic affine open subsets, say $W = D(f_1) \cup \cdots \cup D(f_r)$, such that φ is finite above each $D(f_i)$. This means that (a) holds for the morphism $\varphi^{-1}(W) \to W$.

We now replace Y by W and X by $\varphi^{-1}(W)$. We are then in the situation that Y is affine and has an open covering $Y = D(f_1) \cup \cdots \cup D(f_r)$ such that φ is finite above each $D(f_i)$, and we are done if we show that X is affine and $A(X)$ is f.g. as an $A(Y)$-module. Note that the hypothesis that the open sets $D(f_i)$ cover Y implies that $(f_1, \dots, f_r) = A(Y)$.

By Proposition 8.4, X is affine. Further, for each index i we know that $A(X)_{f_i}$ is integral over $A(Y)_{f_i}$. Let $C \subset A(X)$ be the integral closure of $A(Y)$ in $A(X)$. If $\mathfrak{p} \subset A(Y)$ is a prime ideal, choose an index i such that $f_i \notin \mathfrak{p}$. (Such an index exists because $(f_1, \dots, f_r) = A(Y)$.) Then $C_\mathfrak{p} = \bigl(C_{f_i}\bigr)_\mathfrak{p}$ and $A(X)_\mathfrak{p} = \bigl(A(X)_{f_i}\bigr)_\mathfrak{p}$. By Proposition A4.7, $C_{f_i} \xrightarrow{\sim} A(X)_{f_i}$. It follows that $C_\mathfrak{p} \xrightarrow{\sim} A(X)_\mathfrak{p}$ for every prime ideal $\mathfrak{p} \subset A(Y)$. By Proposition A2.14 this implies that $C = A(X)$, which means that $A(X)$ is integral over $A(Y)$. Finally, by Exercise A4.6 it follows that $A(X)$ is finitely generated as a module over $A(Y)$ and we are done. □

8.7. Definition. A morphism $\varphi\colon X \to Y$ is said to be a *finite morphism* if it satisfies the equivalent conditions (a) and (b) in Proposition 8.6.

8.8. Proposition.

(i) *If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are finite morphisms then $\psi \circ \varphi\colon X \to Z$ is finite, too.*
(ii) *Closed immersions are finite.*
(iii) *If $\varphi\colon X \to Y$ is finite then for any variety Z the morphism $\varphi \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is finite.*

Proof. Part (i) follows from Proposition A4.5, and (ii) is immediate from the definitions. For (iii), first reduce to the case that Z is affine; in that case the claim again readily follows from the definitions. □

8.9. Definition. A morphism $\varphi\colon X \to Y$ is said to be *quasi-finite* if for every $y \in Y$ the fiber $\varphi^{-1}(y)$ is finite.

Quasi-finiteness is weaker than finiteness. For instance, open immersions are quasi-finite but not, in general, finite. The precise relation between the two notions is given by the following result.

8.10. Theorem. *A morphism $\varphi\colon X \to Y$ is finite if and only if it is quasi-finite and proper.*

We shall not give the proof of the "if" statement. For the "only if", assume φ is finite. To see that φ is quasi-finite we may assume Y is affine; the assertion then follows from Exercise A4.6(ii). To see that φ is proper, it suffices, by Proposition 8.8(iii) to show that φ is closed. For this we may again assume that Y is affine, and the assertion is then a direct translation of the Going-Up Theorem A4.11.

8.11. Corollary. *Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be morphisms such that $\psi\varphi\colon X \to Z$ is finite. Then φ is finite, and if φ is surjective, also ψ is finite.*

Proof. Use Proposition 7.5. □

§2. *The normalization of a variety.*

8.12. Definition. A variety X is *normal* if all local rings $\mathscr{O}_{X,P}$, for $P \in X$, are integrally closed.

By Proposition A4.9, X is normal if and only if there is an affine open cover $X = \cup U_i$ such that all coordinate rings $A(U_i) = \mathscr{O}_X(U_i)$ are integrally closed, and if X is normal then for *any* affine open $U \subset X$ the coordinate ring $A(U)$ is integrally closed.

By Theorem A4.14, any non-singular variety is normal.

8.13. Definition. Let Y be a variety with function field $k(Y)$. Let $k(Y) \subset L$ be a finite field extension. Then a *normalization of* Y *in* L is a finite surjective morphism $\pi\colon X \to Y$ such that X is normal and the extension of function fields corresponding to π is the given extension $k(Y) \subset L$.

8.14. Theorem. *Let Y be a variety with function field $k(Y)$. Let $k(Y) \subset L$ be a finite field extension. Then there exists a normalization of Y in L, and this normalization is unique up to unique isomorphism.*

Proof. We start with the uniqueness. For this, suppose $\pi_1\colon X_1 \to Y$ and $\pi_2\colon X_2 \to Y$ are both normalizations of Y in L. The claim is that there is a unique isomorphism $\varphi\colon X_1 \xrightarrow{\sim} X_2$ such that $\pi_2 \circ \varphi = \pi_1$ and such that the induced isomorphism on function fields $k(X_2) \xrightarrow{\sim} k(X_1)$ is the identity on L. If $V \subset Y$ is an affine open with coordinate ring A, the $U_i = \pi_i^{-1}(V)$ are again affine (because the π_i are finite) and the coordinate ring of U_i is the integral closure of A in L. Hence, we find a unique isomorphism $U_1 \cong U_2$ over V that gives the identity map on function fields. It is clear that these isomorphisms glue to give a uniquely determined isomorphism $X_1 \cong X_2$ over Y.

Next we establish the existence. First assume that Y is affine, say with coordinate ring A. Let B be the integral closure of A in L. By Theorem A4.13, B is finitely generated as an A-module. It follows that B is again an affine k-algebra. Hence there exists an affine variety X with coordinate ring B and the given homomorphism $A \to B$ corresponds to a finite morphism $X \to Y$ which is a normalization of Y in L.

For the general case, choose an affine open covering $Y = V_1 \cup \cdots \cup V_m$. Let $\pi_i\colon U_i \to V_i$ be a normalization of V_i in L. Then $\pi_i^{-1}(V_{ij})$ and $\pi_j^{-1}(V_{ij})$ are both normalizations of V_{ij} in L, so by the uniqueness that we have already established, we have unique isomorphisms $\pi_i^{-1}(V_{ij}) \xrightarrow{\sim} \pi_j^{-1}(V_{ij})$ over V_{ij}. Using these, we glue the U_i to a pre-variety X. By (iii) of Exercise 8.1, X is a variety, and by construction X is normal with function field L and $X \to Y$ is finite. $\square$

8.15. Remark. If X is the normalization of Y in L as in the theorem, the morphism $X \to Y$ is proper, because it is finite. (See Theorem 8.10.) Hence, if Y is a complete variety, so is X. It is also true that if Y is projective, X is projective, but this goes beyond what we can prove at this stage.

8.16. Definition. Let Z be an irreducible subvariety of a variety X. Then we define *the local ring of* X *along* Z, notation $\mathscr{O}_{X,Z}$, to be the direct limit of the system of k-algebras $\mathscr{O}_X(U)$ where U is a non-empty open subset of X with $Z \cap U \neq \emptyset$.

8.17. Remarks. (i) If $\overline{Z}$ is the closure of Z we have $Z \cap U \neq \emptyset$ if and only if $\overline{Z} \cap U \neq \emptyset$. Hence $\mathscr{O}_{X,\overline{Z}} = \mathscr{O}_{X,Z}$. For Z a point we recover the local ring at a point defined earlier, for Z dense in X the local ring $\mathscr{O}_{X,Z}$ is the function field $k(X)$.

(ii) If X is affine with coordinate ring A and Z is the closed subvariety corresponding to the prime ideal $\mathfrak{p} \subset A$, we have $\mathscr{O}_{X,Z} = A_{\mathfrak{p}}$.

8.18. Proposition. (i) *If X is an algebraic variety and $Z \subset X$ is a closed irreducible subset then $\mathscr{O}_{X,Z}$ is a regular local ring if and only if Z is not contained in the singular locus* $\mathrm{Sing}(X)$.

(ii) *If X is a normal variety, every irreducible component of* $\mathrm{Sing}(X)$ *has codimension* $\geqslant 2$ *in X.*

We shall not prove this proposition. We include it mainly because part (ii), which is a direct consequence of (i) together with the Algebra fact Theorem A4.14(ii), gives us an important geometric consequence of normality.

§3. *Complete non-singular curves.*

8.19. Proposition. *Let C be a curve. Then C is non-singular if and only if C is normal.*

This is the geometric translation of Corollary A4.21.

8.20. Let C be an affine nonsingular curve with coordinate ring A. Then A is a Dedekind ring. The maximal ideals of A (these are precisely the nonzero prime ideals) correspond with the points of C. If $\mathfrak{p} \subset A$ is the maximal ideal corresponding with $P \in C$, the localization $A_{\mathfrak{p}} = \mathscr{O}_{C,P}$ is a dvr with fraction field $k(C)$. It is the valuation ring of a discrete valuation $v\colon k(C)^* \to \mathbb{Z}$. The interpretation of this valuation is simply that $v(f)$, for $f \in k(C)^*$ is the "order of vanishing" of f at P, with the understanding that a negative order of vanishing, say $v(f) = -n < 0$, means that f has a "pole" of order n at P. So, working with dvr gives a purely algebraic approach to some notions with which we are familiar from complex analysis.

We now return to completeness. The topological intuition is that if a variety X is not complete, there is a "hole" (or "puncture") in it. As we will now discuss, we can detect this by trying to map curves into X. For what follows, it is helpful to remember that the (Zariski) topology on a curve is just the co-finite topology.

8.21. Lemma. *Let C be a curve with function field $K = k(C)$. Let P, $Q \in C$ and suppose that $\mathscr{O}_{C,P} = \mathscr{O}_{C,Q}$ as subrings of K. Then $P = Q$.*

Proof. Choose affine open U containg P and V containing Q. Let A and B be their coordinate rings, $\mathfrak{m} \subset A$ the maximal ideal corresponding to P and $\mathfrak{n} \subset B$ the maximal ideal corresponding to Q. Then $U \cap V$ is again affine, because $U \cap V = \Delta \cap (U \times V)$ in $X \times X$ and the diagonal Δ is closed. (This solves Exercise 5.3(iii).) On rings, $U \times V$ corresponds to $A \otimes_k B$, so the affine ring of $U \cap V$ is a quotient of this. In fact, $\mathscr{O}(U \cap V) \subset K$ is the image of the homomorphism $A \otimes_k B \to K$ given by $a \otimes b \mapsto ab$, or, what is the same, the k-subalgebra of K generated by A and B. By assumption, $A_{\mathfrak{m}} = B_{\mathfrak{n}}$ as subrings of K, so we get: $\mathscr{O}(U \cap V) \subset (A_{\mathfrak{m}} = B_{\mathfrak{n}}) \subset K$. Then

$$\mathfrak{p} = \mathscr{O}(U \cap V) \cap \mathfrak{m}A_{\mathfrak{m}} = \mathscr{O}(U \cap V) \cap \mathfrak{n}B_{\mathfrak{n}}$$

is a maximal ideal of $\mathscr{O}(U \cap V)$ (use that $\mathrm{Kdim}(C) = 1$) corresponding to a point $R \in U \cap V$. Moreover, $A \cap \mathfrak{p} = A \cap \mathfrak{m}A_{\mathfrak{m}} = \mathfrak{m}$ and $B \cap \mathfrak{p} = B \cap \mathfrak{n}B_{\mathfrak{n}} = \mathfrak{n}$, which means that under the inclusions $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ the point R maps to P, respectively Q. □

8.22. Lemma. *If $f\colon X \to Y$ is a birational morphism of curves and Y is non-singular then X is non-singular, too, and f is an open immersion.*

Proof. Because f is birational, $f(X)$ contains an open subset of Y and therefore it itself is open in Y. So we may replace Y by $f(X)$ and assume f is surjective. Our task is then to show f is an isomorphism.

On function fields: $K = k(Y) \cong k(X)$; we take this as an identification. If $P \in X$ maps to $Q \in Y$ we have $\mathscr{O}_{Y,Q} \subset \mathscr{O}_{X,P} \subset K$. But $\mathscr{O}_{Y,Q}$ is a dvr with fraction field K, so we must have $\mathscr{O}_{X,P} = \mathscr{O}_{Y,Q}$. By the previous lemma, above each $Q \in Y$ there is a unique $P \in X$; hence f is a homeomorphism. Moreover, $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism because it's an isomorphism on stalks. □

8.23. Theorem. *Let C be a non-singular curve. Then any rational map $\varphi\colon C \dashrightarrow Y$ to a complete variety Y extends to a morphism $C \to Y$.*

Proof. Choose an open $U \subset C$ on which the map is defined; so we have $\varphi\colon U \to Y$. Let $\Gamma \subset U \times Y$ be the graph of φ and let $\overline{\Gamma} \subset C \times Y$ be its closure. Then the projection $\pi\colon \overline{\Gamma} \to C$ is surjective (because Y is complete) and birational (because it is an isomorphism above U). By the previous lemma π is an isomorphism, and we get the morphism $C \to Y$ by composing π^{-1} with the projection $\overline{\Gamma} \to Y$. □

8.24. Remarks. (i) This theorem in fact gives a characterization of completeness: if every $C \dashrightarrow Y$ extends to a morphism, Y is complete. This is the geometric analogue of the so-called valuative criterion for properness.

(ii) For non-singular varieties X of dimension $\geqslant 2$, it is not true that every rational map to a complete variety extends to a true morphism. For instance, blowing up $\mathbb{A}^2$ in the origin gives a birational map $\pi\colon B \to \mathbb{A}^2$. The inverse of π exists as a rational map $\varphi\colon \mathbb{A}^2 \dashrightarrow B$, defined outside the origin, but this map does not extend to a morphism, as this would give that π is an isomorphism, which is of course not true.

Next we turn to the birational geometry of curves. By a *function field of dimension d* we mean a field K of transcendence degree d over k. Such fields form a category, in which the morphisms are the k-algebra homomorphisms.

8.25. Theorem. (i) *Let C_1 and C_2 be complete nonsingular curves over k. If $\alpha\colon k(C_2) \to k(C_1)$ is a homomorphism of k-algebras, there is a unique morphism $\varphi\colon C_1 \to C_2$ with $\varphi^* = \alpha$.*

(ii) *Let $k \subset K$ be a function field of dimension 1. Then there exists a complete nonsingular curve C such that $k(C) \cong K$ as k-algebras, and C is unique up to isomorphism.*

We could also state the result as an anti-equivalence of catgeories: we have a contravariant functor $C \mapsto k(C)$ from the category of complete nonsingular curves to the category of function fields of dimension 1, and the theorem says that this functor is an anti-equivalence of categories. This may be further combined with Proposition 5.20.

Proof. (i) This follows from (i) of Proposition 5.20 together with Theorem 8.23.

For (ii), start with any curve X with function field K. Take an affine open part $U \subset X$, realize it as a closed subvariety $U \subset \mathbb{A}^n$ for some n, and let $\overline{U} \subset \mathbb{P}^n$ be the projective closure.

Then the normalization of $\overline{U}$ is a complete non-singular curve C with function field K. The uniqueness follows from (i). $\square$

8.26. Remark. Any complete nonsingular curve is in fact projective. The proof of this requires techniques we have not yet discussed.

8.27. The theorem tells us that a complete nonsingular curve is uniquely determined by its function field, or, what amounts to the same by Proposition 5.20, by any representative of its birational equivalence class. So we may start with any curve C_0, possibly non-complete and possibly singular, and then there is a unique complete nonsingular C that is birationally equivalent to C_0.

Thus, algebraic geometers say things like "Let C be the curve (understood: complete and nonsingular) given by the equation $y^2 = x^2 + x^3$ ". What is *not* meant by this is that the curve $C_0 \subset \mathbb{A}^2$ given by that equation is complete and non-singular, because it is not. Instead, we mean that we should take for C the unique complete nonsingular curve that is birationally equivalent to C_0. Of course, to make C explicit may still require some work, but bear in mind that we can sometimes already prove things about C without making it explicit.

8.28. Suppose we do need to make the complete nonsingular curve explicit. How do we do this? Normalization is a great tool from a theoretical perspective, but in practice it is not always so easy to determine the integral closure of a ring. But we can also desingularize a curve by blowing up, which is much more explicit.

Let us continue the example in 8.27. The projective closure of C_0 is the curve $\overline{C}_0 \subset \mathbb{P}^2$ given by the equation $X^3 + X^2Z - Y^2Z$, which is singular only in $(0:0:1)$. To normalize we may try to "guess" a non-trivial element in the function field that is integral over the affine ring $A = k[x,y]/(x^3 + x^2 - y^2)$. If we don't happen to see one, we can also blow up. Indeed, we recognize that this is the same example as in 6.22; as we have seen, the blow-up $\tilde{C}_0$ is the affine curve in $\mathbb{A}^2$ (with coordinates x, u) given by $x + 1 = u^2$ and the map $\tilde{C}_0 \to C_0$ is given by $(x, u) \mapsto (x, xu)$.

The new rational function we have introduced is $u = y/x$. This element is indeed integral over A, as is obvious from the relation $u^2 - x - 1 = 0$ that we have found. So, indeed, blowing-up gives a quick way to find new elements in the function field that are integral over the original coordinate ring.

Finally, the complete nonsingular curve C is obtained by glueing the affine curves $\tilde{C}_0$ and $\overline{C}_0 \setminus \{(0:0:1)\}$. The resulting curve is in this case just the projective closure of $\tilde{C}_0$, i.e., the curve $C \subset \mathbb{P}^2$ given, using $(X:U:Z)$ as homogeneous coordinates, by $XZ + Z^2 - U^2 = 0$.

8.29. Not only complete nonsingular curves are given in an "implicit" form, the same is true for morphisms. As an example, the function "y" on the affine curve C_0 from 8.27 defines a rational map $C_0 \dashrightarrow \mathbb{P}^1$. Theorem 8.25 tells us that this uniquely defines a morphism $C \to \mathbb{P}^1$. As we have used the relation $u = y/x$, this suggests we should look at the map $C \dashrightarrow \mathbb{P}^1$ given by $(X:U:Z) \mapsto (XU:Z^2)$. This morphism is well-defined outside the point $(1:0:0)$. Using the equation $XZ + Z^2 - U^2 = 0$ we find that $(XU:Z^2) = (X^2 + XZ:UZ)$ whenever both sides are defined; hence the morphism $y\colon C \to \mathbb{P}^1$ we are looking for is defined by

$$y(X:U:Z) = \begin{cases} (XU:Z^2) & \text{if } (X:U:Z) \neq (1:0:0); \\ (X^2+XZ:UZ) & \text{if } (X:U:Z) \neq (1:0:-1). \end{cases}$$

8.30. Let $\varphi\colon C \to D$ be a morphism of complete nonsingular curves. Then φ is proper and it is quasi-finite if and only if it is not a constant map. So φ is nonconstant if and only if it is a finite morphism.

We now assume φ is finite. Let $\varphi^*\colon k(D) \to k(C)$ be the induced map on function fields. We define the *degree of* φ to be the degree $\big[k(C):k(D)\big]$ of the function field extension.

The intuition that we shall now make precise is that, given a point $Q \in D$, there are precisely $\deg(\varphi)$ points $P \in C$ with $\varphi(P) = Q$. Of course, this cannot be literally true—just think of the morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto x^n$. (What do we mean here?!) This is a morphism of degree n but the origin $(0:1)$ only has itself as pre-image. Or consider the Frobenius morphism $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ over a field of characteristic p; this morphism is a bijection but it has degree p. As such examples illustrate, we shall have to count pre-images with multiplicity. As we shall do this using some Algebra, we restrict our attention to affine open subsets.

If $V \subset D$ is an affine open subset, its preimage $U = \varphi^{-1}(V) \subset C$ is again affine. On coordinate rings, $B = \mathscr{O}_C(U)$ is the integral closure of $A = \mathscr{O}_D(V)$ in $k(C)$. The rings A and B are Dedekind rings. The nonzero prime ideals in these rings are precisely the maximal ideals, and, as we know, they correspond bijectively to the points of U and V, respectively. If $P \in U$ maps to $Q \in V$ then for the corresponding maximal ideals $\mathfrak{n}_P \subset B$ and $\mathfrak{m}_Q \subset A$ we have $\mathfrak{n}_P \cap A = \mathfrak{m}_Q$.

Next we recall that evaluation of functions at P gives an isomorphism $A/\mathfrak{p} \cong k$. Similarly, $B/\mathfrak{P} \cong k$. (This heavily uses the fact that we are working over an algebraically closed field, through Hilbert's Nullstellensatz.) Therefore, the degree $f(\mathfrak{P})$ of the residue field extension is in our case always 1, and Proposition A4.29 in our case gives that for any $Q \in D$ we have the relation

$$\deg(\varphi) = \sum_{P \mapsto Q} e_P\,,$$

where $e_P = e(\mathfrak{P})$ is the ramification index of the prime ideal $\mathfrak{P}$, also called the ramification index of P. If $e_P > 1$ then we call P a *ramification point*.

As this relation shows, the multiplicities we have to take into account are the ramification indices. We shall later discuss how these indices are related to differential forms.

Note that by Exercise A4.28, the ramification index e_P expresses the local behaviour of φ near the point P. Indeed, a uniformizer of $\mathscr{O}_{C,P}$ is just a function ρ, regular on some open neighbouorhood of P, that vanishes at P with order 1. If we work over $\mathbb{C}$, such a function gives an identification of an *analytically* open neighbourhood of P with an open disc in $\mathbb{C}$ centered at the origin (such that $P \mapsto O$). This is what is called a "coordinate function" or "local coordinate" near P, or a "chart" at P, depending on your preferred point of view. Similarly, a uniformizer π of $\mathscr{O}_{D,Q}$ gives a local coordinate at Q. The ramification index e_P is then the integer e such that $\pi \circ \varphi = (\text{unit}) \cdot \rho^e$ near P. Analytically this means that φ, locally near P, can be described as the map on unit discs in $\mathbb{C}$ given by $z \mapsto z^e$. The theory of dvr gives us a purely algebraic version of this.

Exercises for Chapter 8.

Exercise 8.1. A morphism $\varphi\colon X \to Y$ of pre-varieties is said to be *affine* if for every affine open $V \subset Y$ the inverse image $\varphi^{-1}(V)$ is again affine.

(i) Suppose there exists an open cover $Y = V_1 \cup \cdots \cup V_m$ such that each $\varphi^{-1}(V_i)$ is affine. Prove that φ is affine.

(ii) Show that compositions of affine morphisms are again affine. Show that if $\varphi_1\colon X_1 \to Y_1$ and $\varphi_2\colon X_2 \to Y_2$ are affine, the morphism $(\varphi_1 \times \varphi_2)\colon X_1 \times X_2 \to Y_1 \times Y_2$ is again affine.

(iii) If $\varphi\colon X \to Y$ is an affine morphism of pre-varieties and Y is a variety, prove that X is a variety. [*Hint:* if $Y = \cup_{i=1}^m V_i$ is an open affine cover, let $U_i = \varphi^{-1}(V_i)$. To prove that $\Delta(X)$ is closed, it suffices to show that $\Delta(X) \cap (U_i \times U_j)$ is closed for all i and j. Then use that $\Delta(X) \cap (U_i \times U_j) \cong U_i \cap U_j = \varphi^{-1}(V_i \cap V_j)$.]

Exercise 8.2. Let $\varphi\colon C \to D$ be a morphism of curves. We do not assume the curves are complete, and they are allowed to have singularities.

(i) Prove that φ is dominant if and only if it is quasi-finite.

(ii) Assume φ is dominant, and let $d = \bigl[k(C) : k(D)\bigr]$ be the degree of the function field extension. If D is nonsingular, show that for a given $Q \in D$ there are at most d points $P \in C$ with $\varphi(P) = Q$. Given an example that shows that this is no longer true, in general, if D is singular.

Exercise 8.3. In this exercise we work over an algebraically closed field k of characteristic 0. Let $X \subset \mathbb{A}^4$ be the affine variety given by the equations

$$w^2y - x^2 = 0\,, \quad w^3z - x^3 = 0\,, \quad y^3 - z^2 = 0\,, \quad wz - xy = 0\,.$$

(We use (w, x, y, z) as coordinates on $\mathbb{A}^4$.) You may use without proof that X is a surface.

(i) Show that the origin $O = (0, 0, 0, 0)$ is the only singular point of X.

(ii) Let A be the coordinate ring of X. Find an element $f \in k(X)$ such that f is not in X but f is integral over A.

(iii) Find a nonsingular affine surface $\tilde{X}$ and a proper birational map $\pi\colon \tilde{X} \to X$ such that, writing $U = X \setminus \{O\}$, the induced map $\pi^{-1}(U) \to U$ is an isomorphism. (Such a map π is called a resolution of singularities.)

Exercise 8.4. In this exercise we work (for simplicity) over an algebraically closed field k of characteristic $\neq 2, \neq 3$. Let $f = 4x^3 - ax - b$, where $a, b \in k$ are constants with $a^3 - 27b^2 \neq 0$. Let $E \subset \mathbb{P}^2$ be the projective closure of the affine curve $E_0 \subset \mathbb{A}^2$ given by the equation $y^2 = f(x)$.

(i) Show that E is nonsingular and that the only point in $E \setminus E_0$ is the point $(0 : 1 : 0)$.

(ii) Show that the map $E_0 \mapsto \mathbb{A}^1$ given by $(x, y) \mapsto x$ defines a morphism $\pi\colon E \to \mathbb{P}^1$. Determine the degree of π and describe all ramification points with their ramification indices.

Exercise 8.5. Let $E \subset \mathbb{P}^2$ be as in the previous exercise. We identify the Grassmannian $\mathbb{G}(1, \mathbb{P}^2)$ of lines in $\mathbb{P}^2$ with $\mathbb{P}^2$ (the so-called *dual* of the original projective plane $\mathbb{P}^2$), letting a point $(a_0 : a_1 : a_2) \in \mathbb{P}^2$ correspond with the line $\ell \subset \mathbb{P}^2$ given by the equation $a_0X_0 + a_1X_1 + a_2X_2 = 0$.

(i) Consider the map φ: $E \times E \to \mathbb{G}(1, \mathbb{P}^3)$ that sends a pair (P, Q) with $P \neq Q$ to the line $\overline{PQ}$ and a point (P, P) on the diagonal to the projective tangent line $\mathbb{T}_{E,P} \subset \mathbb{P}^2$. Show that this map is a morphism.

(ii) Consider the closed subset $Z \subset E^3 \times \mathbb{G}(1, \mathbb{P}^2)$ given by

$$Z = \{(P, Q, R, \ell) \in E^3 \times \mathbb{G}(1, \mathbb{P}^2) \mid P, Q, R \in \ell\}.$$

Let $U \subset Z$ be the open subset given by the condition that the three points (P, Q, R) are distinct, and let $I \subset E^3 \times \mathbb{G}(1, \mathbb{P}^2)$ be the closure of U. Let $\Gamma_\varphi \subset E^2 \times \mathbb{G}(1, \mathbb{P}^2)$ be the graph of the morphism φ in part (i). Show that the map $(P, Q, R, \ell) \mapsto (P, Q, \ell)$ gives a well-defined morphism p: $I \to \Gamma_\varphi$ that is finite and birational.

(iii) Show that Γ_φ is nonsingular and that p is an isomorphism. Conclude that we have a well-defined morphism μ: $E \times E \to E$ that sends a pair (P, Q) to a point R such that the three points are on one line, and such that on some open subset of E^2 the three points are distinct.

Remark. Let us also introduce the map ι: $E \to E$ given on E_0 by $(x, y) \mapsto (x, -y)$. Then $m = \iota \circ \mu$: $E^2 \to E$ is a group law on E for which O is the identity element and ι is the inverse. The group variety E obtained in this way is called an *elliptic curve*.

COMMUTATIVE ALGEBRA 4

Integral dependence

A4.1. Definition. Let $A \subset B$ be an extension of rings. An element $b \in B$ is said to be *integral over* A if there exists a monic polynomial $f \in A[x]$ with $f(b) = 0$.

A4.2. Proposition. *Let $A \subset B$ be rings and consider an element $b \in B$. Then the following properties are equivalent:*
(a) *b is integral over A;*
(b) *the subring $A[b] \subset B$ is finitely generated as an A-module;*
(c) *there exists a subring $C \subset B$ containing $A[b]$ such that C is finitely generated as an A-module.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are easy. For (c) $\Rightarrow$ (a), apply Lemma A1.10 to the endomorphism $F\colon C \to C$ given by $F(c) = bc$. We find that there exist $a_1, \dots, a_n \in A$ such that multiplication by $y = b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0$ is the zero endomorphism. In particular, $y = y \cdot 1 = 0$; hence b is integral over A. □

A4.3. Corollary. *In the situation of the proposition, let*

$$C = \left\{ b \in B \;\middle|\; b \text{ is integral over } A \right\}.$$

Then C is a subring of B.

The ring C obtained in this way is called the *integral closure* of A in B. We say that B *is integral over* A, or that $A \subset B$ is an integral extension, if $C = B$, i.e., if every $b \in B$ is integral over A. We say that A *is integrally closed in* B if $A = C$. A domain A is said to be *integrally closed* (without further qualification) if A is integrally closed in its field of fractions.

A4.4. Exercise. Let $A \subset B$ be an integral extension of domains. Let $S = A \setminus \{0\}$, which is a multiplicatively closed subset of B. Show that $S^{-1}B = \mathrm{Frac}(B)$.

A4.5. Proposition. *Let $A \subset B \subset C$ be rings. If B is integral over A and C is integral over B then C is integral over A.*

A4.6. Exercise. (i) Let $A \subset B$ be an extension of rings. Prove that the following two conditions are equivalent:
(a) B is f.g. as an A-module;
(b) B is f.g. as an A-algebra and B is integral over A.

(ii) Let $A \subset B$ be an extension of rings such that B is f.g. as an A-module. Prove that for a prime ideal $\mathfrak{p} \subset A$ there are only finitely many prime ideals $\mathfrak{q} \subset B$ with $\mathfrak{q} \cap A = \mathfrak{p}$. [*Hint:* reduce to the case that $B \cong A[t]/(f)$ for some monic polynomial f. After that, reduce to the case that A is a field and $\mathfrak{p} = (0)$.]

A4.7. Proposition. *Let $A \subset B$ be rings and let C be the integral closure of A in B. If $S \subset A$ is a multiplicatively closed subset, $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

A4.8. Corollary. *If a domain A is integrally closed and $S \subset A$ is a multiplicatively closed subset, $S^{-1}A$ is again integrally closed.*

A4.9. Proposition. *Let A be a domain. Then the following conditions are equivalent:*
(a) *A is integrally closed;*
(b) *$A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p} \subset A$;*
(c) *$A_{\mathfrak{m}}$ is integrally closed for every maximal ideal $\mathfrak{m} \subset A$.*

Note that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) follows from Corollary A4.8. For (c) $\Rightarrow$ (a), let B be the integral closure of A; then we have a homomorphism $h\colon A \to B$ of A-modules and for all maximal ideals $\mathfrak{m}$ of A the induced $h_{\mathfrak{m}}\colon A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is a bijection. By Proposition A2.14 this implies that h is a bijection.

A4.10. Proposition. *Let $A \subset B$ be an integral extension of rings. If $\mathfrak{q}_1 \subset \mathfrak{q}_2$ are prime ideals of B with $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.*

A4.11. Going-up Theorem. *Let $A \subset B$ be an integral extension of rings. Suppose we have prime ideals $\mathfrak{q}_1 \subset B$ and $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset A$ with $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Then there exists a prime ideal $\mathfrak{q}_2 \subset B$ with $\mathfrak{q}_1 \subset \mathfrak{q}_2$ such that $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.*

A4.12. Going-down Theorem. *Let $A \subset B$ be an integral extension of rings with A an integrally closed domain. Suppose we have prime ideals $\mathfrak{q}_2 \subset B$ and $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset A$ with $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$. Then there exists a prime ideal $\mathfrak{q}_1 \subset B$ with $\mathfrak{q}_1 \subset \mathfrak{q}_2$ such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.*

A4.13. Theorem. *Let A be an affine algebra over a field k and* $\mathrm{Frac}(A) \subset L$ *a finite field extension. Let B be the integral closure of A in L. Then B is finitely generated as an A-module.*

A good reference for this is Section 13.3 in Eisenbud's book [3], or Bourbaki [1], Chap V, §3. While we shall not give the details of the proof, let us note that the arguments involved also lead to a proof of Hilbert's Nullstellensatz.

A4.14. Theorem. *Let A be a local domain. Then*

$$A \text{ is regular} \Rightarrow A \text{ is a UFD} \Rightarrow A \text{ is integrally closed}.$$

A4.15. Exercise. Prove the second implication.

A4.16. Definition. Let K be a field. A *discrete valuation* on K is a surjective map

$$v\colon K \to \mathbb{Z} \cup \{\infty\}$$

such that

1. $v(x) = \infty$ if and only if $x = 0$;

2. for nonzero x and y we have $v(xy) = v(x) + v(y)$, so that the map $v\colon K^* \to \mathbb{Z}$ is a group homomorphism;
3. for all $x, y \in K$ we have $v(x+y) \geqslant \inf\{v(x), v(y)\}$.

In practice, sometimes only the homomorphism $v\colon K^* \to \mathbb{Z}$ is given; this is extended to a map on all of K by the convention that $v(0) = \infty$.

A4.17. Examples. (i) Fix a prime number p. A nonzero rational number q can be written in a unique way as $q = p^n \cdot q'$, where $n \in \mathbb{Z}$ and the numerator and denominator of q' are non divisible by p. The number n is called the *p-adic valuation of* q, notation $v_p(q)$. The map $v_p\colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is a valuation.

(ii) Let $\mathbb{C}(t)$ be the field of rational functions in one variable. Choose a point $a \in \mathbb{C}$. A nonzero $f \in \mathbb{C}(t)$ defines a meromorphic function on $\mathbb{C}$, and there is a unique integer n such that f can be expressed, locally near a, in the form $f(z) = (z-a)^n \cdot g(z)$, where g is holomorphic at a with $g(a) \neq 0$. We call this number n the *order of vanishing of f at a*, notation $v_a(f)$. The map $v_a\colon \mathbb{C}(t) \to \mathbb{Z} \cup \{\infty\}$ is a valuation.

A4.18. Exercise. If v is a discrete valuation on the field K, define $A = \{x \in K \mid v(x) \geqslant 0\}$ and $\mathfrak{m} = \{x \in K \mid v(x) > 0\}$. Prove that A is a local subring of K and that $\mathfrak{m}$ is its unique maximal ideal. The subring $A \subset K$ is called the *valuation ring* of the discrete valuation v.

A4.19. Definition. A *discrete valuation ring* (abbreviated to dvr) is a local PID (principal ideal domain) that is not a field.

A4.20. Theorem. *Let $(A, \mathfrak{m})$ be a local domain with fraction field K. Then the following properties are equivalent:*

(a) *A is a dvr;*
(b) *there is a discrete valuation v on K with valuation ring A;*
(c) *A is noetherian and $\mathfrak{m}$ is a nonzero principal ideal;*
(d) *A is noetherian, $\mathrm{Kdim}(A) = 1$ and A is integrally closed.*

Proof. If (a) holds, clearly A is noetherian and $\mathfrak{m}$ is principal, so (c) is true. Next assume (c) holds. An element $\pi \in A$ with $\mathfrak{m} = A \cdot \pi$ is called a *uniformizer*. By assumption, there exists a uniformizer; it is unique up to multiplication by a unit in A. Let $I = \cap_{n \geqslant 0} \mathfrak{m}^n$. Then $\mathfrak{m} \cdot I = I$; hence Nakayama's Lemma gives $I = (0)$. This means that every $a \in A$ can be written as $a = c \cdot \pi^n$ for some $n \geqslant 0$, with $c \in A \setminus \mathfrak{m}$, which implies that $c \in A^*$. The integer n is independent of the choice of a uniformizer; call it $v(a)$. The map v extends to a well-defined map $v\colon K^* \to \mathbb{Z}$ by $v(a/b) = v(a) - v(b)$. One readily checks that this is a valuation and that A is the corresponding valuation ring. So (b) holds.

If (b) holds, there exists an element $\pi \in A$ with $v(\pi) = 1$; choose one. If $x \in A$ has $v(x) = 0$ then $x^{-1} \in K$ has $v(x^{-1}) = 0$; hence $x^{-1} \in A$ and $x \in A^*$. If $x \in K^*$ has $v(x) = n$ then $\pi^{-1} \cdot x$ has $v(\pi^{-1}x) = 0$. It follows that every $x \in K^*$ can be written as $x = c \cdot \pi^n$ for $n = v(x) \in \mathbb{Z}$ and $c \in A^*$. If $I \subset A$ is a non-zero ideal, let $n = \min\{v(a) \mid a \in I\}$. (The minimum exists because $\{v(a) \mid a \in I\}$ is a non-empty subset of $\mathbb{Z}_{\geqslant 0}$.) Then it is immediate from the previous remarks that $I = (\pi^n)$; hence A is a PID and (a) holds.

We have now shown that (a), (b) and (c) are equivalent. If they hold then $\mathfrak{m}$ is a non-zero principal ideal, which implies that $\mathrm{Kdim}(A) = 1$. To see that A is integrally closed, suppose $y \in K$ is integral over A. Then we have a relation

$$y^n = -a_{n-1}y^{n-1} - \cdots - a_1 y - a_0$$

with $a_i \in A$ for all i. If $v(y) = m < 0$ the LHS has valuation mn, whereas the RHS has valuation $\geqslant m(n-1)$; this gives a contradiction. Hence $v(y) \geqslant 0$, which means that $y \in A$.

Finally, suppose (d) holds. Our goal is to show that $\mathfrak{m}$ is a principal ideal, which gives (c). Let $M = \{x \in K \mid x \cdot \mathfrak{m} \subset A\}$, which is an A-submodule of K that contains A. If $0 \neq x \in \mathfrak{m}$ then $M \subset A \cdot x^{-1}$; hence M is a submodule of a f.g. A-module, and because A is noetherian it follows that M is finitely generated.

We first show that $A \subsetneq M$. For this, choose an element $x \in \mathfrak{m}$ and consider the localization A_x. Suppose this is not a field. Then A_x has a non-zero prime ideal. Any such prime ideal is of the form $\mathfrak{p}A_x$ where $\mathfrak{p} \subset A$ is a prime ideal with $x \notin \mathfrak{p}$. (See Proposition A2.9.) But then the chain $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ contradicts the assumption that $\mathrm{Kdim}(A) = 1$. Hence A_x is a field, and as it contains A we have $A_x = K$. This means that every element of K can be written in the form y/x^n for some $y \in A$ and $n \in \mathbb{Z}_{>0}$.

Now fix a nonzero element $z \in \mathfrak{m}$. By the above, if $x \in \mathfrak{m}$ we can write $1/z = y/x^n$; hence $x^n = yz \in (z)$. So for $x \in \mathfrak{m}$, some positive power of x lies in the principal ideal (z). As $\mathfrak{m}$ is finitely generated, it follows that there exists an $N \geqslant 0$ such that $\mathfrak{m}^N \subset (z)$. We take N as small as possible; note that $N \geqslant 1$ because z is not a unit in A. Then there exists an $y \in \mathfrak{m}^{N-1}$ with $y \notin (z)$, and for any such y we have $y \cdot \mathfrak{m} \subset (z)$. But then y/z is an element of M that does not lie in A, proving that $A \subsetneq M$.

Next we consider $\mathfrak{m} \cdot M$, by which we mean the ideal of A generated by all elements xy with $x \in \mathfrak{m}$ and $y \in M$. Then $\mathfrak{m} \subset \mathfrak{m} \cdot M \subset A$, so either $\mathfrak{m} = \mathfrak{m} \cdot M$ or $\mathfrak{m} \cdot M = A$.

So far we have not used the assumption that A is integrally closed. We now use this to deduce that $\mathfrak{m} \cdot M = A$. Indeed, suppose $\mathfrak{m} \cdot M = \mathfrak{m}$. Choose $x \in M \setminus A$. We have $x \cdot \mathfrak{m} \subset \mathfrak{m}$; hence $x^n \cdot \mathfrak{m} \subset \mathfrak{m}$ for all $n \geqslant 0$. But then the subring $A[x] \subset K$ is contained in M, which we have shown to be a finitely generated module, so by Proposition A4.2 x is integral over A, contradicting the assumption that A is integrally closed.

Finally, $\mathfrak{m} \cdot M = A$ means that we have a relation $x_1 y_1 + \cdots + x_r y_r = 1$ with $x_i \in \mathfrak{m}$ and $y_i \in M$. All terms $x_i y_i$ are in A but they are not all in $\mathfrak{m}$. So there exists an index i such that $x_i y_i \in A \setminus \mathfrak{m} = A^*$. Then $z = y_i^{-1} = (x_i y_i)^{-1} \cdot x_i$ is an element of $\mathfrak{m}$. This element generates $\mathfrak{m}$, for if $w \in \mathfrak{m}$ we have $w = (wy_i) \cdot z$ and $wy_i \in A$. □

A4.21. Corollary. *Let A be a local domain with* $\mathrm{Kdim}(A) = 1$ *then*

$$A \text{ is regular} \Leftrightarrow A \text{ is integrally closed} \Leftrightarrow A \text{ is a dvr}\,.$$

Note that the condition that A is regular is equivalent to the condition that the maximal ideal $\mathfrak{m}$ is principal; see Remark A3.7(ii).

A4.22. Definition. A *Dedekind ring* is a noetherian domain A with $\mathrm{Kdim}(A) = 1$ which is integrally closed.

If A is a Dedekind ring and $(0) \neq \mathfrak{p} \subset A$ is a nonzero prime ideal, it follows from Proposition A4.9 and Theorem A4.20 that $A_\mathfrak{p}$ is a dvr. Conversely, if all localizations $A_\mathfrak{p}$, for $\mathfrak{p} \subset A$ a nonzero prime ideal, are dvr then $\mathrm{Kdim}(A) = 1$ and A is Dedekind, again using Proposition A4.9.

A4.23. Examples. (i) Every PID that is not a field is a Dedekind ring. Examples include the rings $\mathbb{Z}$ and $k[t]$.

(ii) Let A be a Dedekind ring with fraction field K. Let $K \subset L$ be a finite field extension and let $B \subset L$ be the integral closure of A in L. Then B is integrally closed and it follows from Proposition A4.10 together with the Going-Up Theorem that $\mathrm{Kdim}(B) = 1$. Hence, if B is noetherian then it is again a Dedekind ring. Sufficient conditions for B to be noetherian are that L/K is a separable extension or that A is an affine k-algebra for some field k.

A4.24. Definition. If A is a Dedekind ring with fraction field K, a *fractional ideal* of A is a finitely generated A-submodule of K.

For instance, every usual ideal $I \subset A$ is also a fractional ideal. Fractional ideals can be multiplied: if $\mathfrak{a}$, $\mathfrak{b} \subset K$ are fractional ideals, then the A-submodule $\mathfrak{a} \cdot \mathfrak{b} \subset K$ generated by all elements xy with $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ is again a fractional ideal. Clearly, A itself (viewed as fractional ideal) is a neutral element for this multiplication.

The submodule $M \subset K$ that appears in the proof of Theorem A4.20 is a fractional ideal. A key point in that proof is that we proved that $\mathfrak{m} \cdot M = A$; thus M is an inverse of $\mathfrak{m}$ for this product structure. The following important result tells us that such inverses always exist, so that the fractional ideals form a group. Moreover, this group turns out to be the free abelian group on the set of maximal ideals. For a proof of the following result we recommend Serre [6], Chapter I.

A4.25. Theorem. *Let A be a Dedekind domain with fraction field K.*

(i) *For every fractional ideal $\mathfrak{a} \subset K$ there is a unique fractional ideal $\mathfrak{a}^{-1} \subset K$ such that $\mathfrak{a} \cdot \mathfrak{a}^{-1} = A$. The fractional ideals of A form an abelian group, with $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a} \cdot \mathfrak{b}$ as group structure, A as identity element and $\mathfrak{a}^{-1}$ as the inverse of $\mathfrak{a}$.*

(ii) *Every fractional ideal $\mathfrak{a}$ can be uniquely written as*

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})} \tag{1}$$

where the product runs over the nonzero prime ideals of A and the exponents $v_\mathfrak{p}(\mathfrak{a})$ are integers, only finitely many of which are nonzero.

If $\mathfrak{p} \subset A$ is a nonzero prime ideal (equivalent: maximal ideal), $A_\mathfrak{p}$ is a dvr with fraction field K. If $\pi_\mathfrak{p} \in A_\mathfrak{p}$ is a uniformizer, the $A_\mathfrak{p}$-fractional ideal $A_\mathfrak{p} \cdot \mathfrak{a} \subset K$ is of the form $\pi^n \cdot A_\mathfrak{p}$ for some uniquely determined integer n. This integer n, called the $\mathfrak{p}$-adic valuation of $\mathfrak{a}$, is precisely the exponent $v_\mathfrak{p}(\mathfrak{a})$ that appears in (1).

A4.26. We now continue Example A4.23(ii); so $K \subset L$ is a finite field extension, B is the integral closure of A in L, and we assume B is finitely generated as an A-module and is therefore again a Dedekind ring. If $\mathfrak{P} \subset B$ is a maximal ideal then $\mathfrak{p} = \mathfrak{P} \cap A$ is a maximal ideal of A and we say that $\mathfrak{P}$ *divides* $\mathfrak{p}$, notation $\mathfrak{P}|\mathfrak{p}$.

If $\mathfrak{p} \subset A$ is a maximal ideal, $\mathfrak{p} \cdot B$ is a fractional ideal of B (in fact, it is an ideal in the usual sense) and we can apply Theorem A4.25(ii); this gives

$$\mathfrak{p}B = \prod_{\mathfrak{P}} \mathfrak{P}^{e(\mathfrak{P})} ,$$

where the product runs over the maximal ideals of B. The exponent $e(\mathfrak{P})$ is called the *ramification index* of $\mathfrak{P}$.

A4.27. Exercise. Show that $e(\mathfrak{P}) = 0$ if $\mathfrak{P}$ does not divide $\mathfrak{p}$.

A4.28. Exercise. Let $\mathfrak{P} \subset B$ be a nonzero prime ideal, $\mathfrak{p} = \mathfrak{P} \cap A$. Then $A_{\mathfrak{p}} \subset B_{\mathfrak{P}}$ (as subrings of L) and if we choose uniformizers $\pi \in A_{\mathfrak{p}}$ and $\rho \in B_{\mathfrak{P}}$, we have $\pi = c \cdot \rho^e$ for some unit $c \in B^*$ and some integer $e \geqslant 1$. Show that $e = e(\mathfrak{P})$.

If $\mathfrak{P}|\mathfrak{p}$ then $\kappa(\mathfrak{P}) = B/\mathfrak{P}$ is a finite field extension of $\kappa(\mathfrak{p}) = A/\mathfrak{p}$. We denote by $f(\mathfrak{P}) = \big[\kappa(\mathfrak{P}) : \kappa(\mathfrak{p})\big]$ the degree of this extension. Note that in our geometric applications we shall always have $f(\mathfrak{P}) = 1$, because we work over an algebraically closed field; see 8.30.

A4.29. Proposition. *Let $A \subset B$ be Dedekind rings as in* Example A4.23*, with $[L : K] = n$. Then for every maximal ideal $\mathfrak{p} \subset A$ we have*

$$\sum_{\mathfrak{P}|\mathfrak{p}} e(\mathfrak{P}) f(\mathfrak{P}) = n .$$

The sum runs over the maximal ideals of B and by Exercise A4.27 we only need to consider the maximal ideals $\mathfrak{P} \subset B$ that divide $\mathfrak{p}$.

CHAPTER 9

Vector bundles and locally free sheaves.

§1. *Vector bundles.*

9.1. Let X be a variety. We are going to define the notion of a vector bundle on X. The idea is that this should be a "bundle of vector spaces" over X. To prepare for the precise definition, suppose we are given a variety E with a morphism $\pi\colon E \to S$. For $U \subset X$ we write $E_U = \pi^{-1}(U)$. In particular, the fiber of π over a point $x \in X$ is denoted by E_x. Further, by $E \times_X E$ we mean the inverse image of the diagonal $\Delta_X \subset X \times X$ under the map $E \times E \to X \times X$, i.e., the closed subset of $E \times E$ consisting of points (P, Q) with $\pi(P) = \pi(Q)$.

Part of the requirements for a vector bundle is that the fibers E_x have a given structure of k-vector spaces, such that:

(i) the map $E \times_X E \to E$ given by $(P, Q) \mapsto P + Q$ is a morphism of varieties;

(ii) if $f \in \mathscr{O}_X(U)$ is a regular function on an open subset $U \subset X$, the map $\lambda_f\colon E_U \to E_U$ given by $P \mapsto f\bigl(\pi(P)\bigr) \cdot P$ is a morphism;

(iii) the map $0\colon X \to E$ sending $x \in X$ to the origin of E_x is a morphism.

A trivial way to obtain a vector bundle is to consider $X \times k^n$ with its projection map to X. The structure of an algebraic variety on $X \times k^n$ is the one we get by identifying k^n with $\mathbb{A}^n$. (In this case we write k^n because we do wish to retain its structure of a vector space.) This vector bundle is called the trivial bundle of rank n on X.

9.2. Definition. A *vector bundle of rank n on X* consists of

1. a variety E with a morphism $\pi\colon E \to X$ and a section $e\colon X \to E$ of π;
2. a given structure of k-vector space on each fiber E_x, with origin $e(x)$.

Further, $E \to X$ is required to be *locally trivial*; by this we mean that for every $x \in X$ there is an open $U \subset X$ containing x and an isomorphism $t\colon E_U \xrightarrow{\sim} U \times k^n$ such that

(a) t is compatible with the projections to U, in the sense that $\mathrm{pr}_U\bigl(t(P)\bigr) = \pi(P)$ for all $P \in E_U$;

(b) for every $x \in X$ the map $t(x)\colon E_x \to \{x\} \times k^n \cong k^n$ is a k-linear isomorphism.

Note that, because of condition (a), t indeed maps the fiber E_x isomorphically to $\{x\} \times k^n$; condition (b) then expresses that this isomorphism should be an isomorphism of vector spaces and not just any isomorphism of affine varieties. Note that these assumptions imply that the fibers E_x are n-dimensional vector spaces.

The section e is called the *zero section* of E. An isomorphism $t\colon E_U \xrightarrow{\sim} U \times k^n$ satisfying (a) and (b) is called a *trivialization* of E over U.

In practice it is usually understood what is the vector space structure on the fibers and we simply say things like "let $E \to X$ be a vector bundle", leaving the remaining data unspecified. In such a case, even the rank may be left unspecified. A vector bundle of rank 1 is called a *line bundle*.

9.3. Definition. (i) If $\pi_E\colon E \to X$ and $\pi_F\colon F \to X$ are vector bundles over X, by a *vector bundle morphism* $\varphi\colon E \to F$ we mean a morphism of varieties such that

(a) φ is compatible with the projections to X, in the sense that $\pi_F \circ \varphi = \pi_E$;

(b) the maps $\varphi_x\colon E_x \to F_x$ induced by φ (grace to assumption (a)) are k-linear maps.

(ii) Let $E \to X$ be a vector bundle of rank n. A *subbundle* of E is a closed subvariety $E' \subset E$ such that for every $x \in X$ the fiber $E'_x \subset E_x$ is a linear subspace of some fixed dimension $m \leqslant n$, and such that E', with these linear structures on the fibers, is a vector bundle of rank m.

9.4. Example. *The universal bundle over the Grassmannian.* Let V be a k-vector space of finite dimension. Let n be a non-negative integer. Over the Grassmannian $\mathrm{Grass}(n,V)$ we have the trivial bundle $V \times \mathrm{Grass}(n,V) \to \mathrm{Grass}(n,V)$.

We define the *universal subbundle* $\mathscr{W} \subset V \times \mathrm{Grass}(n,V)$ by

$$\mathscr{W} = \bigl\{(v,W) \in V \times \mathrm{Grass}(n,V) \bigm| v \in W\bigr\}\,.$$

To see that this is indeed a subbundle, choose a subspace $B \subset V$ of codimension n and a complementary subspace A as in 5.23, and consider the associated open subset $U_B \subset \mathrm{Grass}(n,V)$; then $\mathscr{W} \cap (V \times U_B)$ is isomorphic to

$$\bigl\{(v,f) \in V \times \mathrm{Hom}_k(A,B) \bigm| v \in \Gamma_f\bigr\}\,,$$

which, as bundle over $\mathrm{Hom}_k(A,B) \cong U_B$, is indeed a closed subbundle of the trivial bundle $V \times \mathrm{Hom}_k(A,B)$.

Viewed as a vector bundle of rank n over $\mathrm{Grass}(n,V)$, the bundle $\mathscr{W}$ is also sometimes called the *tautological bundle*, as its fiber over a point $W \in \mathrm{Grass}(n,V)$ is precisely W.

The tautological bundle is not a trivial bundle if $0 < n < \dim(V)$. We shall see this in several concrete examples.

9.5. Example. Take $X = \mathbb{P}^1$. Let $\mathbb{P}^1 = U_0 \cup U_1$ be the standard open cover. We identify U_0 and U_1 in the usual way. Write $U_{01} = U_0 \cap U_1$ viewed as subset of U_0 and $U_{10} = U_0 \cap U_1$ viewed as subset of U_1. Using "x" as a coordinate on U_0 and "y" as a coordinate on U_1, the gluing

$$U_0 \supset U_{01} \xrightarrow{\sim} U_{10} \subset U_1$$

is given by the isomorphism $k[y,y^{-1}] \to k[x,x^{-1}]$ sending y to x^{-1}.

Suppose $\pi\colon E \to \mathbb{P}^1$ is a vector bundle of rank n such that the restrictions $E_0 = E_{U_0}$ and $E_1 = E_{U_1}$ are both trivial. (In fact, this is true for any vector bundle on $\mathbb{P}^1$ but we don't know this yet.) Choose trivializations:

$$\tau_0\colon E_0 \xrightarrow{\sim} \mathbb{A}^1 \times \mathbb{A}^n\,, \qquad \tau_1\colon E_1 \xrightarrow{\sim} \mathbb{A}^1 \times \mathbb{A}^n\,.$$

On the first $\mathbb{A}^1 \times \mathbb{A}^n$ we use $(x,t_1,\dots,t_n)$ as coordinates, on the second $\mathbb{A}^1 \times \mathbb{A}^n$ we use $(y,u_1,\dots,u_n)$. Then we have a gluing map

$$\sigma = \tau_1 \circ \tau_0^{-1}\colon E_{01} \xrightarrow{\sim} E_{10}$$

that is compatible with the projections to the base and fiberwise linear. On rings this means that σ is given by an isomorphism

$$k[y, y^{-1}, u_1, \dots, u_n] \xrightarrow{\sim} k[x, x^{-1}, t_1, \dots, t_n]$$

with $y \mapsto x^{-1}$ and such that the image of u_j is a linear combination

$$A_{1j}t_1 + \dots + A_{nj}t_n$$

with $A_{ij} \in k[x, x^{-1}]$ for all i, j.

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