Tensor products

Let R be a commutative ring. Given R-modules M_1, M_2 and N we say that a map

$$b\colon M_1 \times M_2 \to N$$

is R-bilinear if for all r, $r' \in R$ and module elements m_i, $m_i' \in M_i$ we have

$$b(r \cdot m_1 + r' \cdot m_1', m_2) = r \cdot b(m_1, m_2) + r' \cdot b(m_1', m_2)$$
$$b(m_1, r \cdot m_2 + r' \cdot m_2') = r \cdot b(m_1, m_2) + r' \cdot b(m_1, m_2')\,.$$

The set of all such R-bilinear maps is denoted by $\mathrm{Bilin}_R(M_1 \times M_2, N)$.

Bilinear maps b as above should not be confused with homomorphisms $f\colon M_1 \oplus M_2 \to N$. Indeed, such a homomorphism f satisfies $f(rm_1, rm_2) = r \cdot f(m_1, m_2)$, whereas for an R-bilinear map b we have $b(rm_1, rm_2) = r \cdot b(m_1, rm_2) = r^2 \cdot b(m_1, m_2)$.

If b, $b' \in \mathrm{Bilin}_R(M_1 \times M_2, N)$ and $r \in R$ then also the maps $b + b'$ and $r \cdot b$ are in $\mathrm{Bilin}_R(M_1 \times M_2, N)$; this means that $\mathrm{Bilin}_R(M_1 \times M_2, N)$ itself has a natural structure of an R-module.

If $f_1\colon M_1 \to M_1'$ and $f_2\colon M_2 \to M_2'$ are homomorphisms of R-modules, we have an induced map

$$(f_1 \times f_2)^*\colon \mathrm{Bilin}_R(M_1' \times M_2', N) \to \mathrm{Bilin}_R(M_1 \times M_2, N)$$

given by $b \mapsto b \circ (f_1 \times f_2)$. Hence we obtain a functor

$$\mathrm{Bilin}_R(- \times -, N)\colon {}_R\mathsf{Mod}^{\mathrm{op}} \times {}_R\mathsf{Mod}^{\mathrm{op}} \to {}_R\mathsf{Mod}\,.$$

Similarly, if $g\colon N \to N'$ is a homomorphism, we have an induced map $g_*\colon \mathrm{Bilin}_R(M_1 \times M_2, N) \to \mathrm{Bilin}_R(M_1 \times M_2, N')$ by $b \mapsto g \circ b$, giving a functor

$$\mathrm{Bilin}_R(M_1 \times M_2, -)\colon {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}\,.$$

Theorem 1. Let R be a commutative ring, and let M_1 and M_2 be R-modules. Then the functor $\mathrm{Bilin}_R(M_1 \times M_2, -)$ is co-representable.

In concrete terms this means that there exists an R-module T and an R-bilinear map $\beta\colon M_1 \times M_2 \to T$ with the following *universal property*: for any R-module N the map

$$\mathrm{Hom}_R(T, N) \to \mathrm{Bilin}_R(M_1 \times M_2, N)$$

given by $h \mapsto h \circ \beta$ is a bijection. The pair (T, β) is unique up to unique isomorphism.

The module T is called the *tensor product* of M_1 and M_2 over the ring R and is denoted by $M_1 \otimes_R M_2$. The image of an element $(m_1, m_2) \in M_1 \times M_2$ under the universal map $\beta\colon M_1 \times M_2 \to M_1 \otimes_R M_2$ is denoted by $m_1 \otimes_R m_2$, or if there is no risk of confusion simply by $m_1 \otimes m_2$. The fact that the map β is R-blinear means that we have the relations

$$(r \cdot m_1 + r' \cdot m_1') \otimes m_2 = r \cdot (m_1 \otimes m_2) + r' \cdot (m_1', \otimes m_2)$$
$$m_1 \otimes (r \cdot m_2 + r' \cdot m_2') = r \cdot (m_1 \otimes m_2) + r' \cdot (m_1 \otimes m_2')\,.$$

As a particular case of this we have

$$rm_1 \otimes m_2 = r \cdot (m_1 \otimes m_2) = m_1 \otimes rm_2$$

for all $r \in R$ and $m_i \in M_i$.

At this stage it is not easy to grasp what the tensor product is in given examples. This will become clearer once we have obtained some further results on the structure of the tensor product. Two techniques are relevant here: on one hand we may gain insight in the tensor product by using its universal property; on the other hand, as the next proof shows there is an explicit (but rather abstract!) construction of the tensor product that we may use to deduce information.

Proof of the theorem. We start by considering the free R-module F on the set $M_1 \times M_2$. This means that for every element $(m_1, m_2) \in M_1 \times M_2$ we have a base vector $e_{(m_1,m_2)}$ and that these elements form a basis for F as an R-module. This module F is huge, in general, and it does not have a very interesting structure. We are going to make it more interesting by dividing out relations; this means we will pass from F to a quotient module $T = F/N$ in such a way that the map $(m_1, m_2) \mapsto [e_{(m_1,m_2)}]$ (writing $[x] \in F/N$ for the class of an element $x \in F$) becomes the desired map β from $M_1 \times M_2$ to the tensor product T.

To this end, we define $N \subset F$ to be the R-submodule generated by all the following elements:

— all elements of the form

$$e_{(r \cdot m_1 + r' \cdot m_1', m_2)} - r \cdot e_{(m_1,m_2)} - r' \cdot e_{(m_1', m_2)}\,;$$

— all elements of the form

$$e_{(m_1, r \cdot m_2 + r' \cdot m_2')} - r \cdot e_{(m_1,m_2)} - r' \cdot e_{(m_1, m_2')}\,.$$

If we now write $\beta(m_1, m_2)$ for the class $[e_{(m_1,m_2)}] \in T = F/N$ then we see that the map β: $M_1 \times M_2 \to T$ is R-bilinear by construction. For instance,

$$\beta(r \cdot m_1 + r' \cdot m_1', m_2) = r \cdot \beta(m_1, m_2) + r' \cdot \beta(m_1', m_2)$$

simply because

$$\begin{aligned} &\beta(r \cdot m_1 + r' \cdot m_1', m_2) - r \cdot \beta(m_1, m_2) - r' \cdot \beta(m_1', m_2) \\ &\quad = [e_{(r \cdot m_1 + r' \cdot m_1', m_2)}] - r \cdot [e_{(m_1,m_2)}] - r' \cdot [e_{(m_1', m_2)}] \\ &\quad = [e_{(r \cdot m_1 + r' \cdot m_1', m_2)} - r \cdot e_{(m_1,m_2)} - r' \cdot e_{(m_1', m_2)}] = 0\,, \end{aligned}$$

and in a similar way we find that

$$\beta(m_1, r \cdot m_2 + r' \cdot m_2') = r \cdot \beta(m_1, m_2) + r' \cdot \beta(m_1, m_2')$$

for all r, $r' \in R$, $m_1 \in M_1$ and m_2, $m_2' \in M_2$.

Next we show that (T, β) has the desired universal property. So we give ourselves an R-bilinear map b: $M_1 \times M_2 \to N$ and our task is to show that there exists a unique R-homomorphism g: $T \to N$ such that $b = g \circ \beta$. Note that if such a homomorphism g exists we must have $g[e_{(m_1,m_2)}] = b(m_1, m_2)$ for all $m_1 \in M_1$ and $m_2 \in M_2$, and since the elements $[e_{(m_1,m_2)}]$ generate $T = F/N$ as an R-module this implies that there can be at most one homomorphism g with $b = g \circ \beta$.

To prove the existence of the desired g, we first note that there is a unique homomorphism $\tilde{g}\colon F \to N$ with $\tilde{g}(e_{(m_1,m_2)}) = b(m_1, m_2)$ for all m_1, m_2. (This is really what it means to have a *free* module: you can map the base vectors to any elements you want and then extend R-linearly.) The assumption that b is R-bilinear implies that $\tilde{g}$ is zero on all generators of the submodule $N \subset F$, and hence $\tilde{g}_{|N} = 0$. Therefore $\tilde{g}$ induces a homomorphism $g\colon T \to N$, and by construction we have $b = g \circ \beta$. □

An element of $M_1 \otimes_R M_2$ that is of the form $m_1 \otimes m_2$ is called a *pure tensor*. Note that a scalar multiple of a pure tensor is again pure, since $r \cdot (m_1 \otimes m_2) = rm_1 \otimes m_2 = m_1 \otimes rm_2$. The above proof shows:

Proposition. Every element in $M_1 \otimes_R M_2$ is a finite sum of pure tensors.

In general, not every element of $M_1 \otimes_R M_2$ is a pure tensor. We will see this later in some examples. We will often write the general element in a tensor product $M \otimes_R N$ as $\sum m_i \otimes n_i$, by which we indicate a finite sum of pure tensors. Note that such an expression is by no means unique, since after all we have relations such as $(m + m') \otimes n = m \otimes n + m' \otimes n$.

Functoriality. If $f\colon M \to M'$ and $g\colon N \to N'$ are homomorphisms of R-modules, we have an induced map on tensor products

$$(f \otimes g)\colon (M \otimes_R N) \to (M' \otimes_R N') \,.$$

On elements it is given simply by $\sum_i m_i \otimes n_i \mapsto \sum_i f(m_i) \otimes g(n_i)$. There are a number of rather obvious identities, such as

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1) \,.$$

We may therefore view the tensor product as a functor

$$\otimes\colon {}_R\mathsf{Mod} \times {}_R\mathsf{Mod} \to {}_R\mathsf{Mod} \,.$$

Theorem 2. As before, let R be a commutative ring.

(i) For M, $N \in {}_R\mathsf{Mod}$ we have an isomorphism

$$M \otimes_R N \xrightarrow{\sim} N \otimes_R M \,,$$

given by $\sum m_i \otimes n_i \mapsto \sum n_i \otimes m_i$.

(ii) If $I \subset R$ is an ideal then

$$(R/I) \otimes_R M \xrightarrow{\sim} M/IM \,,$$

where the isomorphism is given by $\sum (r_i \bmod I) \otimes m_i \mapsto \big(\sum r_i m_i\big) \bmod IM$. In particular (taking $I = 0$),

$$R \otimes_R M \xrightarrow{\sim} M \,.$$

(ii) If $\{M_\alpha\}$ is a collection of R-modules then

$$\big(\oplus_\alpha M_\alpha\big) \otimes_R N \xrightarrow{\sim} \oplus_\alpha \big(M_\alpha \otimes_R N\big)\,.$$

Before we prove this, let us draw some consequences and discuss some examples.

Corollary. If $I,\, J \subset R$ are ideals, $(R/I) \otimes_R (R/J) \cong R/(I+J)$.

Proof. The kernel of the natural surjective map $R/J \to R/(I+J)$ is the ideal $I \cdot (R/J) \subset R/J$ generated by the images of the elements in I; hence $(R/J)/I \cdot (R/J) \xrightarrow{\sim} R/(I+J)$. Now apply (ii) of the theorem. ☐

Example. We have $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z}) = 0$. At first sight this may seem surprising. However, if N is a $\mathbb{Z}$-module (i.e., an abelian group) then there are no non-zero bilinear maps b: $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \to N$. Indeed, this follows from the relations

$$b(m,n) = b(3m,n) = 3 \cdot b(m,n) = b(m,3n) = b(m,0) = 0$$

for arbitrary $(m,n) \in (\mathbb{Z}/2\mathbb{Z}) \times_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$. So $\mathrm{Bilin}_{\mathbb{Z}}((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}), -)$ sends every $\mathbb{Z}$-module to zero, and is therefore co-represented by the zero module.

More generally, the Corollary tells us that $(\mathbb{Z}/a\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/b\mathbb{Z}) \cong (\mathbb{Z}/c\mathbb{Z})$ with $c = \gcd(a,b)$; in particular, $(\mathbb{Z}/a\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/b\mathbb{Z}) = 0$ whenever $\gcd(a,b) = 1$. If we combine this with the other properties in Theorem 2, we can already calculate all tensor products $M \otimes_{\mathbb{Z}} N$ for finitely generated abelian groups M and N, or more generally, all $M \otimes_R N$ for finitely generated abelian groups M and N over a principal ideal domain R.

Example. Next let us try to understand tensor products $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$. Here something similar happens as in the previous example: any bilinear map b: $\mathbb{Q} \times (\mathbb{Z}/n\mathbb{Z}) \to N$, for N a $\mathbb{Z}$-module, is zero, because we have

$$b(q,y) = b\big(n \cdot (q/n), y\big) = n \cdot b\big((q/n), y\big) = b\big((q/n), ny\big) = b\big((q/n), 0\big) = 0\,.$$

Hence $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$.

Example. Let $M = R^m$ with basis $e_1, \ldots, e_m$ and $N = R^n$ with basis $f_1, \ldots, f_n$. Application of Theorem 2 gives $R^m \otimes_R R^n \cong R^{mn}$, and the elements $e_i \otimes f_j$ form a basis of $R^m \otimes_R R^n$ as an R-module. Here we can see that not every tensor is pure in general. For instance, take $m = n = 2$. Pure tensors are those that can be written as

$$(a_1e_1 + a_2e_2) \otimes (b_1f_1 + b_2f_2) = a_1b_1 \cdot (e_1 \otimes f_1) + a_1b_2 \cdot (e_1 \otimes f_2) + a_2b_1 \cdot (e_2 \otimes f_1) + a_2b_2 \cdot (e_2 \otimes f_2)\,.$$

It is then a nice exercise to show that an arbitrary tensor

$$c_{1,1} \cdot (e_1 \otimes f_1) + c_{1,2} \cdot (e_1 \otimes f_2) + c_{2,1} \cdot (e_2 \otimes f_1) + c_{2,2} \cdot (e_2 \otimes f_2)$$

is pure if and only if $c_{1,1} \cdot c_{2,2} = c_{1,2} \cdot c_{2,1}$.

Proof of the first two parts of Theorem 2. We first prove (ii), which is the more interesting part. The idea is that we should prove that M/IM co-represents the functor $\mathrm{Bilin}_R\big((R/I) \times M, -\big)$; so we should find isomorphisms

$$\varphi_N\colon \mathrm{Hom}_R(M/IM, N) \xrightarrow{\sim} \mathrm{Bilin}_R\big((R/I) \times M, N\big)$$

that are functorial in N. First, given a homomorphism $f\colon M/IM \to N$, define $b_f\colon (R/I)\times M \to N$ by $b_f\big((r \bmod I), m\big) = f(rm \bmod IM)$. One checks without trouble that this indeed is R-bilinear. Conversely, suppose $b\colon (R/I) \times M \to N$ is R-bilinear. Consider the homomorphism $\tilde{f}_b\colon M \to N$ given by $\tilde{f}_b(m) = b\big((1 \bmod I), m\big)$. If $r \in I$ and $m \in M$ then

$$\tilde{f}_b(rm) = b\big((1 \bmod I), rm\big) = b\big((r \bmod I), m\big) = b\big((0 \bmod I), m\big) = 0\,;$$

hence $\tilde{f}_b$ is zero on IM and drops to a homomorphism $f_b\colon M/IM \to N$. The two constructions $f \mapsto b_f$ and $b \mapsto f_b$ are inverse to each other, and hence we have the desired isomorphism φ_N. Finally, if $g\colon N \to N'$ is a homomorphism, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M/IM, N) & \xrightarrow{g_*} & \mathrm{Hom}_R(M/IM, N') \\ {\scriptstyle\varphi_N}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_{N'}} \\ \mathrm{Bilin}_R\big((R/I) \times M, N\big) & \xrightarrow{g_*} & \mathrm{Bilin}_R\big((R/I) \times M, N'\big) \end{array}$$

is commutative. This means precisely that the maps φ_N define an isomorphism of functors $\varphi\colon \mathrm{Hom}_R(M/IM, -) \xrightarrow{\sim} \mathrm{Bilin}_R\big((R/I) \times M, -\big)$. By the Yoneda lemma and the definition of the tensor product it follows that $(R/I) \otimes_R M \xrightarrow{\sim} M/IM$. To verify that the isomorphism is as claimed, take $N = M/IM$ in the above discussion and let $f = \mathrm{id} \in \mathrm{Hom}_R(M/IM, M/IM)$. The corresponding $b_f\colon (R/I) \times M \to M/IM$ sends $(r \bmod I, m)$ to $(rm \bmod IM)$; hence the isomorphism $(R/I) \otimes_R M \to M/IM$ sends $r \otimes m$ to $rm \bmod IM$.

For (i) we may argue in a similar way; here it is obvious that the functors $\mathrm{Bilin}_R(M \times N, -)$ and $\mathrm{Bilin}_R(N \times M, -)$ are isomorphic. □

Hom-tensor adjunction. Before we turn to the proof of part (iii) of the theorem, let us first discuss a simple but important observation that will help us further. (In computer science this idea is known under the name *currying*.) Namely, suppose $b\colon M_1 \times M_2 \to N$ is R-bilinear. For each $m_1 \in M_1$ the map $b(m_1, -)\colon M_2 \to N$ is then a homomorphism of R-modules, i.e., an element of the R-module $\mathrm{Hom}_R(M_2, N)$. This only uses the linearity of b in the second variable. The linearity of b in the first variable can then be expressed as saying that $m_1 \mapsto b(m_1, -)$ defines a homomorphism of R-modules

$$a\colon M_1 \to \mathrm{Hom}_R(M_2, N)\,.$$

Conversely, if we are given such a homomorphism a then we obtain a bilinear form b by $b(m_1, m_2) = a(m_1)\big(m_2\big)$, and these two constructions are clearly each others inverses. Hence we obtain isomorphisms

$$\psi_N\colon \mathrm{Bilin}_R(M_1 \times M_2, N) \xrightarrow{\sim} \mathrm{Hom}_R\big(M_1, \mathrm{Hom}_R(M_2, N)\big)\,.$$

Using the tensor product we may rewrite this as

$$\psi_N\colon \mathrm{Hom}_R(M_1 \otimes_R M_2, N) \xrightarrow{\sim} \mathrm{Hom}_R\bigl(M_1, \mathrm{Hom}_R(M_2, N)\bigr)\,.$$

Moreover, these isomorphisms are functorial in N, meaning that for every homomorphism $g\colon N \to N'$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M_1 \otimes_R M_2, N) & \xrightarrow{g_*} & \mathrm{Hom}_R(M_1 \otimes_R M_2, N') \\ {\scriptstyle \psi_N}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{N'}} \\ \mathrm{Hom}_R\bigl(M_1, \mathrm{Hom}_R(M_2, N)\bigr) & \xrightarrow{g_*} & \mathrm{Hom}_R\bigl(M_1, \mathrm{Hom}_R(M_2, N')\bigr) \end{array}$$

is commutative. Hence the maps ψ_N define an isomorphism of functors

$$\psi\colon \mathrm{Hom}_R(M_1 \otimes_R M_2, -) \xrightarrow{\sim} \mathrm{Hom}_R\bigl(M_1, \mathrm{Hom}_R(M_2, -)\bigr)\,.$$

(There is also an obvious functoriality with respect to M_1 and M_2, but this we will not further discuss here.)

The isomorphism ψ is known under the name Hom-$\otimes$-adjunction. For now this is only a name; we shall discuss adjunctions of functors later.

Proof of the third part of Theorem 2. Let $\{M_\alpha\}$ be a collection of R-modules. Recall that we have homomorphisms $i_\alpha\colon M_\alpha \to \bigl(\oplus_\alpha M_\alpha\bigr)$, and that $\oplus_\alpha M_\alpha$ co-represents the functor

$$\prod_\alpha \mathrm{Hom}_R(M_\alpha, -)\,.$$

In more detail: if $f\colon \oplus_\alpha M_\alpha \to N$ is a homomorphism then the collection of homomorphisms $f \circ i_\alpha$ defines an element $(f \circ i_\alpha)_\alpha$ of $\prod_\alpha \mathrm{Hom}_R(M_\alpha, N)$, and $f \mapsto (f \circ i_\alpha)_\alpha$ gives an isomorphism of functors

$$\mathrm{Hom}_R(\oplus_\alpha M_\alpha, -) \xrightarrow{\sim} \prod_\alpha \mathrm{Hom}_R(M_\alpha, -)\,.$$

Using Hom-$\otimes$-adjunction we find

$$\begin{aligned} \mathrm{Hom}_R\Bigl(\bigl(\oplus_\alpha M_\alpha\bigr) \otimes N, -\Bigr) &\xrightarrow{\sim} \mathrm{Hom}_R\Bigl(\bigl(\oplus_\alpha M_\alpha\bigr), \mathrm{Hom}_R(N, -)\Bigr) \\ &\xrightarrow{\sim} \prod_\alpha \mathrm{Hom}_R\Bigl(M_\alpha, \mathrm{Hom}_R(N, -)\Bigr) \\ &\xrightarrow{\sim} \prod_\alpha \mathrm{Hom}_R\Bigl(M_\alpha \otimes_R N, -\Bigr) \\ &\xrightarrow{\sim} \mathrm{Hom}_R\Bigl(\oplus_\alpha\, (M_\alpha \otimes_R N), -\Bigr)\,. \end{aligned}$$

By the Yoneda Lemma this gives the assertion. □

Extension of scalars. One situation that frequently occurs is that a tensor product is used to "extend" the ring of coefficients. The basic remark is that, with R a commutative ring as before, if $R \to S$ is a homomorphism of rings and if M is an R-module then $S \otimes_R M$ has a natural structure of a left S-module, by

$$\sigma \cdot \bigl(\sum_i s_i \otimes m_i\bigr) = \sum_i \sigma s_i \otimes m_i$$

(for σ, $s_i \in S$ and $m_i \in M$). If $f\colon M \to N$ is a homomorphism of R-modules, the induced map $(\mathrm{id}_S \otimes f)\colon S \otimes_R M \to S \otimes_R N$ is a homomorphism of S-modules. In this way, we obtain a functor

$$S \otimes_R -\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}\,.$$

Example. Let $K \subset L$ be an extension of fields. If V is a K-vector space of dimension d, then $V_L := L \otimes_K V$ is an L-vector space with $\dim_L(V_L) = d$.

More generally, if M is a free module of rank r over a ring R then $M_S := S \otimes_R M$ is a free S-module of rank r. If $e_1, \ldots, e_r$ is an R-basis for M then the elements $1 \otimes e_i$ form an S-basis for M_S. This explains the meaning of the term "extension of scalars"; note however that $R \to S$ need not be injective, so the word "extension" has to be taken with a grain of salt.

Example. Let $N \subset M$ be an R-submodule. We will see later that $S \otimes_R (M/N)$ is isomorphic to the quotient of $S \otimes_R M$ by the image of the map $S \otimes_R N \to S \otimes_R M$. (Caution: even though $N \hookrightarrow M$, the map $S \otimes_R N \to S \otimes_R M$ is not injective, in general.) This helps to calculate the effect of the functor $S \otimes_R -$ on modules that are given by generators and relations. (Of course we can describe *any* module in this form.) Indeed, suppose we have a module that is of the form F/N, where F is a free R-module with basis some elements e_i (the generators) and $N \subset F$ is an R-submodule generated by some elements n_j (the relations). Then $S \otimes_R (F/N)$ is the quotient of the free S-module F_S generated by elements $1 \otimes e_i$, modulo the submodule generated by the elements $1 \otimes n_j$. In particular, this includes the assertion that if an R-module M is generated by a collection of elements m_i then $S \otimes_R M$ is generated as an S-module by the elements $1 \otimes m_i$.

Here are some concrete examples.

(a) If $I \subset R$ is an ideal then $S \otimes_R (R/I) \cong S/I \cdot S$.

(b) We recover the fact that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$: the above gives that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is the $\mathbb{Q}$-vector space generated by one element $1 \otimes 1$, modulo the $\mathbb{Q}$-subspace generated by $1 \otimes n = n \cdot (1 \otimes 1)$; but this subspace equals the whole space.

(c) If A is a finitely generated abelian group then we know that $A = A_0 \oplus \mathrm{Tors}(A)$, where A_0 is free of finite rank and $\mathrm{Tors}(A) \subset A$ is the submodule of torsion elements. Then $\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q} \otimes_{\mathbb{Z}} A_0$ is a $\mathbb{Q}$-vector space of dimension equal to the rank of A_0.

Tensor products of rings. Recall that if R is a commutative ring, by an R-algebra we mean a ring A (non-commutative in general) together with a homomorphism $a\colon R \to A$, called the structural homomorphism, such that $a(R)$ is contained in the center of A. If $a\colon R \to A$ and $b\colon R \to B$ are R-algebras then by a homomorphism of R-algebras $f\colon A \to B$ we mean a homomorphism of rings such that $f \circ a = b$. In practice it is often clear which structural morphism is meant, and then we omit it from the notation. The R-algebras form a category R-Alg. (Caution: the term "algebra" is also used in a much more general meaning; e.g., Lie algebras are not algebras in the sense considered here. In ring theory the above definition is common, though.)

We can carry the above one step further: if A and B are R-algebras then $A \otimes_R B$ again has the structure of an R-algebra. To define the ring structure, the basic rule is that

$$(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')\,.$$

For arbitrary tensors this leads to

$$\Big(\sum_i a_i \otimes b_i\Big) \cdot \Big(\sum_j a'_j \otimes b'_j\Big) = \Big(\sum_{i,j} (a_i a'_j) \otimes_R (b_i b'_j)\Big) .$$

Associatitivity follows from

$$\begin{aligned}\Big((a \otimes b) \cdot (a' \otimes b')\Big) \cdot (a'' \otimes b'') &= (aa') \otimes (bb') \cdot (a'' \otimes b'') \\ &= (aa'a'') \otimes (bb'b'') \\ &= (a \otimes b) \cdot (a'a'') \otimes (b'b'') = (a \otimes b) \cdot \Big((a' \otimes b') \cdot (a'' \otimes b'')\Big) .\end{aligned}$$

Example. Let $R[x]$ be a polynomial ring in one variable over R. Then for A an R-algebra we have $A \otimes_R R[x] \cong A[x]$. On the level of modules this is clear: $R[x]$ is a free R-module with basis 1, x, $x^2, \ldots$, so if we continue to write x (rather than $1 \otimes x$) for the image of x in $A \otimes_R R[x]$ then the latter is free as an A-module with basis 1, x, $x^2, \ldots$. But from the given recipe for the ring structure it is also clear that we get $A \otimes_R R[x] \cong A[x]$ as rings.

For instance we could take $A = R[x]$, and then we find $R[x] \otimes_R R[x] \cong R[x, y]$. Note that in the answer we are renaming one of the variables x as y; this is necessary because $x \otimes 1$ is not the same as $1 \otimes x$. (You could avoid the problem by taking $A = R[y]$; but then you must realize that rings such as $R[x] \otimes_R R[x]$ really pop up in mathematics, so you'd better learn how to deal with this.)

Iterating this construction we find

$$R[x_1, \ldots, x_m] \otimes_R R[y_1, \ldots, y_n] \cong R[x_1, \ldots, x_m, y_1, \ldots, y_n] .$$

Example. Continuing along similar lines we could also take $B = R[x_1, \ldots, x_m]/I$, where $I = (f_1, \ldots, f_s) \subset R[x_1, \ldots, x_m]$ is the ideal generated by a number of polynomials $f_1, \ldots, f_s$. What comes out is that

$$A \otimes_R R[x_1, \ldots, x_m]/(f_1, \ldots, f_s) \cong A[x_1, \ldots, x_m]/(f_1, \ldots, f_s) ,$$

where in the right hand side the polynomials f_i are viewed as polynomials in $A[x_1, \ldots, x_m]$ via the given ring homomorphism $R \to A$. This notation may at first seem sloppy, but in practice it doesn't lead to confusion. For instance, we get:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}[x]/(x^2 + 1) \cong \mathbb{C} \times \mathbb{C} .$$

As such examples show, it greatly matters over which ground ring we take the tensor product, as for instance $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$, whereas $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ is an uncountable product of copies of the ring $\mathbb{C}$.

Exercise. Let M_1, M_2, M_3 and N be R-modules.

(i) Define what it means for a map $t\colon M_1 \times M_2 \times M_3 \to N$ to be R-trilinear.

(ii) Show that the trilinear maps form an R-module $\mathrm{Trilin}_R(M_1 \times M_2 \times M_3, N)$ and that $N \mapsto \mathrm{Trilin}_R(M_1 \times M_2 \times M_3, N)$ gives rise to a functor $({}_R\mathsf{Mod}^{\mathrm{op}})^3 \to {}_R\mathsf{Mod}$.

(iii) Prove that $M_1 \otimes (M_2 \otimes_R M_3)$ and $(M_1 \otimes_R M_2) \otimes_R M_3$ both co-represent this functor, and conclude that we have a natural isomorphism

$$M_1 \otimes (M_2 \otimes_R M_3) \xrightarrow{\sim} (M_1 \otimes_R M_2) \otimes_R M_3 .$$