CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for February 8 and February 15

These are the exercises for the first two weeks. Hand in Exercises: 7 and 8, to be handed in no later than Monday, February 19 at 13:00.

Names for some categories:

Category	Objects	Morphisms
G-Set	G-Sets (see Exercise 6)	G-equivariant maps
Grp	Groups	Group homomorphisms
Ab	Abelian groups	Group homomorphisms
Ring	Rings	Ring homomorphisms
Pos	Partially ordered sets	Monotone functions
Top	Topological spaces	Continuous maps
Haus	Hausdorff topological spaces	Continuous maps

Exercise 1. Prove that there is a functor Ring $\to$ Grp that sends a ring R to its unit group $R^\times$.

Exercise 2. Show that there does not exist a functor $F\colon$ Grp $\to$ Ab such that $F(G)$ is the center of G for every group G. *Hint*: consider $S_2 \to S_3 \to S_2$.

Exercise 3. Recall that a topological space X is said to be a T_0-space (or Kolmogorov space) if for any $x \neq y$ in X there exists an open U that contains one of x and y but not the other. This is equivalent to the requirement that $x \neq y$ implies $\overline{\{x\}} \neq \overline{\{y\}}$.

(a) Let X be a T_0-space. Prove that the relation $\preceq$ on X defined by $x \preceq y \Leftrightarrow x \in \overline{\{y\}}$ gives a partial ordering on X.

(b) Let Pos be the category of partially ordered sets. Show that there exists a functor F from the category T_0-Top to Pos such that $F(X) = (X, \preceq)$ for all T_0-spaces X.

Exercise 4. Describe products, coproducts, monomorphisms and epimorphisms in the category Pos.

Exercise 5.

(a) Prove that the composition of two monomorphisms is again a monomorphism.

(b) Prove that the composition of two epimorphisms is again an epimorphism.

(c) Let f, g be morphisms such that fg is a monomorphism. Prove that g is a monomorphism.

(d) Let f, g be morphisms such that fg is an epimorphism. Prove that f is an epimorphism.

Exercise 6. Let G be a group. A G-set is a set X together with an action of G on X. A G-equivariant map between G-sets is a map $f\colon X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

(a) Prove that there exists a category G-Set whose objects are the G-sets and whose morphisms are G-equivariant maps.

(b) Do products and coproducts exist in G-Set?

Exercise 7.

(a) Show that the embedding $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category Ring.

(b) An (additively written) abelian group A is called *divisible* if for each $a \in A$ and each $n \in \mathbb{Z}_{>0}$ there exists a $b \in A$ such that $nb = a$. The category of divisible abelian groups (with group homomorphisms as morphisms) is denoted Div. Show that the quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism in Div.

(c) Show that in Top the monomorphisms are the injective continuous maps whereas the epimorphisms are the surjective continuous maps.

(d) Show that in Haus a continuous map is an epimorphism if and only if its image is dense.

Exercise 8.

(a) Let $A, B \in \mathsf{Ab}$. Show that the product group $A \times B$, together with the projection maps, is the categorical product of A and B in Ab.

(b) Let $A, B \in \mathsf{Ab}$. Show that the direct product $A \times B$, together with the inclusion maps $A \to A \times \{e_B\} \subset A \times B$ and $B \to \{e_A\} \times B \subset A \times B$, is also the categorical coproduct of A and B in Ab.

(c) Give examples of abelian groups A and B such that the direct product $A \times B$ is not the categorical coproduct of A and B in the category Grp.