CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for May 17

Exercise 1.

(a) If R is a PID and M is a finitely generated R-module, show that M has a free resolution of length at most 1. By this we mean that M has a free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

with $F_i = 0$ for all $i > 1$.

(b) Let $R = \mathbb{Z}/4\mathbb{Z}$, and consider the R-module $M = R/2R \cong \mathbb{Z}/2\mathbb{Z}$. Show that

$$\cdots \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \longrightarrow 0$$

is a free resolution of M. Also show that M does not admit a free resolution of finite length. [*Hint:* Suppose $F_\bullet$ is a free resolution of M, with quasi-isogeny $\alpha\colon F_\bullet \to M$. Use induction on i to show that $\mathrm{Coker}(F_{i+1} \to F_i)$ contains elements x such that $2x = 0$ and $x \notin 2 \cdot \mathrm{Coker}(F_{i+1} \to F_i)$.]

Exercise 2. Let R be the ring $\mathbb{C}[x, y]/(x^2 - y^3)$, and let $\mathfrak{m} \subset R$ be the maximal ideal generated by (the classes of) x and y. Consider the R-module $R/\mathfrak{m}$. Write down an *explicit* free resolution of M.

Exercise 3. Let n be a integer with $n \geq 2$, and let $K_\bullet$ be the complex of abelian groups given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

with terms $\mathbb{Z}$ in degrees 1 and 0. Let $L_\bullet$ be the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

whose only non-zero term $\mathbb{Z}/n\mathbb{Z}$ is placed in degree 0. Finally, let $f\colon K_\bullet \to L_\bullet$ be the natural morphism of complexes, with $f\colon K_0 \to L_0$ the canonical map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Prove that f is a quasi-isomorphism but that it is not a homotopy-equivalence of complexes.

Exercise 4. Let R be a ring. Let $K_\bullet$, $L_\bullet$ and $M_\bullet$ be complexes of R-modules, and suppose given morphisms of complexes $f\colon K_\bullet \to L_\bullet$ and $g\colon L_\bullet \to M_\bullet$. We say that the resulting sequence

$$(*) \qquad 0 \longrightarrow K_\bullet \xrightarrow{f} L_\bullet \xrightarrow{g} M_\bullet \longrightarrow 0$$

is a short exact sequence of complexes if in every degree i the sequence

$$0 \longrightarrow K_i \xrightarrow{f} L_i \xrightarrow{g} M_i \longrightarrow 0$$

is short exact.

(a) Suppose we have a short exact sequence of complexes as above. Prove that for every $i \in \mathbb{Z}$ we have an induced diagram of R-modules with exact rows

$$\begin{array}{ccccccc} & \operatorname{Coker}(d\colon K_{i+1} \to K_i) & \longrightarrow & \operatorname{Coker}(d\colon L_{i+1} \to L_i) & \longrightarrow & \operatorname{Coker}(d\colon M_{i+1} \to M_i) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \operatorname{Ker}(d\colon K_{i-1} \to K_{i-2}) & \longrightarrow & \operatorname{Ker}(d\colon L_{i-1} \to L_{i-2}) & \longrightarrow & \operatorname{Ker}(d\colon M_{i-1} \to M_{i-2}) & \end{array}$$

in which the vertical maps are induced by the differentials $d\colon K_i \to K_{i-1}$, resp. $d\colon L_i \to L_{i-1}$, resp. $d\colon M_i \to M_{i-1}$.

(b) Prove that the short exact sequence $(*)$ induces a long exact homology sequence

$$\cdots \longrightarrow \mathscr{H}_i(K_\bullet) \xrightarrow{\mathscr{H}_i(f)} \mathscr{H}_i(L_\bullet) \xrightarrow{\mathscr{H}_i(g)} \mathscr{H}_i(M_\bullet) \xrightarrow{\delta_i} \mathscr{H}_{i-1}(K_\bullet) \xrightarrow{\mathscr{H}_{i-1}(f)} \mathscr{H}_{i-1}(L_\bullet) \longrightarrow \cdots$$

for suitable boundary maps δ_i.

Exercise 5. For R a ring, let $\mathsf{C}(R\text{-}\mathsf{Mod})$ be the category of chain complexes of R-modules and $\mathsf{Ho}(R\text{-}\mathsf{Mod})$ the homotopy category of chain complexes (as discussed in the lecture). Let $F\colon \mathsf{C}(R\text{-}\mathsf{Mod}) \to \mathsf{Ho}(R\text{-}\mathsf{Mod})$ be the canonical functor.

(a) Is the functor F faithful? Is it full? Is it essentially surjective?

(b) In general, a functor $G\colon \mathsf{C} \to \mathsf{D}$ between categories is said to be *conservative* if for a morphism $f\colon X \to Y$ in C we have

$$G(f) \text{ is an isomorphism} \implies f \text{ is an isomorphism.}$$

Is the functor F conservative?

(c) Give an example of a ring R and a chain complex $M_\bullet$ that is not isomorphic to the zero object in $\mathsf{C}(R\text{-}\mathsf{Mod})$ but is isomorphic to the zero object in $\mathsf{Ho}(R\text{-}\mathsf{Mod})$.