CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for May 24

Exercise 1. Let $R = k[x, y]$ with k a field.

(a) Consider the maximal ideal $\mathfrak{m} = (x, y) \subset R$ and the R-module $M = R/\mathfrak{m}$. Calculate $\operatorname{Ext}^i_R(M, M)$ for all i.

(b) Calculate $\operatorname{Ext}^i_R\big(R/(x), R/(y)\big)$ for all i.

Exercise 2. Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider $M = R/2R \cong \mathbb{Z}/2\mathbb{Z}$. Calculate $\operatorname{Ext}^i_R(M, M)$ for all i. [Use Exercise 1(b) from last week.]

Exercise 3. Let R be a ring and let

$$0 \longrightarrow M' \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M'' \longrightarrow 0$$

be a short exact sequence of R-modules. Let

$$P'_\bullet \xrightarrow{\alpha'} M' \qquad \text{and} \qquad P''_\bullet \xrightarrow{\alpha''} M''$$

be projective resolutions of M' and M'', respectively. For $i \geq 0$ define $P_i = P'_i \oplus P''_i$, and let $j\colon P'_i \to P_i$ and $\pi\colon P_i \to P''_i$ be the inclusion and projection homomorphisms.

(a) Show that there is a homomorphism $\alpha\colon P_0 = P'_0 \oplus P''_0 \to M$ that makes the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P'_0 & \xrightarrow{\ j\ } & P_0 & \xrightarrow{\ \pi\ } & P''_0 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \alpha'} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \alpha''} & & \\
0 & \longrightarrow & M' & \xrightarrow{\ f\ } & M & \xrightarrow{\ g\ } & M'' & \longrightarrow & 0
\end{array}$$

commutative.

(b) Show that the sequence

$$0 \longrightarrow \operatorname{Ker}(\alpha') \longrightarrow \operatorname{Ker}(\alpha) \longrightarrow \operatorname{Ker}(\alpha'') \longrightarrow 0$$

is exact.

(c) Show that there exists a homomorphism $\delta_1\colon P_1 \to P_0$ that gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P'_1 & \xrightarrow{\ j\ } & P_1 & \xrightarrow{\ \pi\ } & P''_1 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle d'_1} & & \big\downarrow{\scriptstyle \delta_1} & & \big\downarrow{\scriptstyle d''_1} & & \\
0 & \longrightarrow & \operatorname{Ker}(\alpha') & \longrightarrow & \operatorname{Ker}(\alpha) & \longrightarrow & \operatorname{Ker}(\alpha'') & \longrightarrow & 0
\end{array}$$

(d) Iterating the previous step, show that there exist homomorphisms $\delta_i\colon P_i \to P_{i-1}$ that make $P_\bullet$ into a complex such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\bullet & \xrightarrow{j} & P_\bullet & \xrightarrow{\pi} & P''_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows.

Exercise 4. Recall that if G is a group we write $\mathbb{Z}[G]$ for its group ring. The elements of G form a basis of $\mathbb{Z}[G]$ as a $\mathbb{Z}$-module, so that every element of $\mathbb{Z}[G]$ can be written in a unique way as $\sum_{g\in G} n_g \cdot g$ with $n_g \in \mathbb{Z}$ and $n_g = 0$ for almost all g.

If $f\colon G \to H$ is a homomorphism of groups, we have an induced homomorphism of rings $f\colon \mathbb{Z}[G] \to \mathbb{Z}[H]$, given by

$$\sum_{g\in G} n_g \cdot g \;\mapsto\; \sum_{g\in G} n_g \cdot f(g) = \sum_{h\in H} \Big(\sum_{g\in f^{-1}\{h\}} n_g \Big) \cdot h\,.$$

In particular, this allows to view $\mathbb{Z}[H]$ as a module over $\mathbb{Z}[G]$.

(a) For $i \geq 0$, view $F_i := \mathbb{Z}[G^{i+1}]$ as a module over $\mathbb{Z}[G]$ via the diagonal homomorphism $G \to G^{i+1}$ given by $g \mapsto (g, g, \dots, g)$. Prove that F_i is a free $\mathbb{Z}[G]$-module.

(b) For $i \geq 0$, show that the map $d_i\colon \mathbb{Z}[G^{i+1}] \to \mathbb{Z}[G^i]$ given by

$$d_i(g_0, \dots, g_i) = \sum_{j=0}^{i} (-1)^j \cdot (g_0, \dots, g_{j-1}, \hat{g}_j, g_{j+1}, \dots, g_i)$$

is a homomorphism of $\mathbb{Z}[G]$-modules, and that $d_{i-1} \circ d_i = 0$. (Note: $\hat{g}_j$ indicates that we are omitting the element in position j.)

In the rest of the exercise we write $F_\bullet^{\#}$ for the complex

$$\cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0\,.$$

(Note that $\mathbb{Z}[G^0]$ is simply $\mathbb{Z}$ with trivial action of G.)

(c) Define $h_{-1}\colon \mathbb{Z} \to F_0 = \mathbb{Z}[G]$ by $h_{-1}(n) = n$, and for $i \geq 0$ define $h_i\colon F_i \to F_{i+1}$ by $h_i(g_0, \dots, g_i) = (1, g_0, \dots, g_i)$, where $1 \in G$ is the identity element. Show that the maps h_i define a homotopy from the zero map $0\colon F_\bullet^{\#} \to F_\bullet^{\#}$ to the identity map $\mathrm{id}\colon F_\bullet^{\#} \to F_\bullet^{\#}$.

(d) Show that $F_\bullet^\#$ is exact, and conclude that the complex

$$F_\bullet : \qquad \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

together with the homomorphism $d_0 \colon F_\bullet \to \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module. (This resolution will play an important role in our later discussion of group (co)homology.)

Exercise 5. Let R be a ring. Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_\bullet & \longrightarrow & M_\bullet & \longrightarrow & M''_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 0 & \longrightarrow & N'_\bullet & \longrightarrow & N_\bullet & \longrightarrow & N''_\bullet & \longrightarrow & 0 \end{array}$$

in the category $\mathsf{C}(R\text{-}\mathsf{Mod})$ such that the rows are exact. (Recall that a sequence $0 \longrightarrow M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow 0$ of complexes is exact if it is exact in each degree.) Show that if two out of $\{\phi', \phi, \phi''\}$ are quasi-isomorphisms, so is the third. [*Hint:* Use Exercise 4 from last week.]