CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for June 7

Exercise 1. Let R be a ring., and consider an extension

$$\mathscr{E}: \qquad 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

of R-modules. If $h\colon N \to N'$ is a homomorphism, we can form the extension

$$h_*(\mathscr{E}): \qquad 0 \longrightarrow N' \longrightarrow E' \longrightarrow M \longrightarrow 0\,.$$

On the other hand, h induces a homomorphism $h_*\colon \mathrm{Ext}^1_R(M,N) \to \mathrm{Ext}^1_R(M,N')$. Under this homomorphism, the class $[\mathscr{E}] \in \mathrm{Ext}^1_R(M,N)$ is mapped to the class $\bigl[h_*(\mathscr{E})\bigr] \in \mathrm{Ext}^1_R(M,N')$.

(a) In the above situation, if $h\colon N \to N'$ is an isomorphism, show that $E \cong E'$ as R-modules.

In the rest of this exercise we take $R = k[t]$ where k is a field. For $\lambda \in k$, let $M_\lambda = k[t]/(t-\lambda)$ and $M'_\lambda = k[t]/(t-\lambda)^2$. If we describe R-modules as pairs (V,ϕ) then M_λ corresponds with $V = k$ with $\phi = \lambda \cdot \mathrm{id}_k$, and M'_λ corresponds with $V = k^2$ with $\phi = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$.

(b) Show that $\mathrm{Ext}^1_{k[t]}(M_\lambda, M_\lambda) \cong M_\lambda$. (This is only a reminder, we have already done this in much greater generality!)

(c) Let E be a $k[t]$-module that can be obtained as an extension of M_λ by itself. Show that either $E \cong M_\lambda \oplus M_\lambda$ or $E \cong M'_\lambda$.

(d) How many $k[t]$-modules are there, up to isomorphism, that can be obtained as an extension of M'_λ by M_λ ? Give an explicit representative for each isomorphism class.

Exercise 2. Let k be an algebraically closed field, and consider the polynomial ring $R = k[x,y]$. It is a theorem of Hilbert that every maximal ideal of R is of the form $\mathfrak{m} = (x-a, y-b)$ for some a, $b \in k$. (This result is a special instance of *Hilbert's Nullstellensatz*; is is essential here that $k = \overline{k}$.) In the exercise we denote by $M_{(a,b)}$ the R-module $k[x,y]/(x-a,y-b)$. By Hilbert's theorem, these are all possible simple R-modules.

(a) Show that

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y-b \\ -(x-a) \end{pmatrix}} R^2 \xrightarrow{(\,x-a \;\; y-b\,)} R \longrightarrow 0$$

is a free resolution of $M_{(a,b)}$, and use this to show that for an R-module N we have

$$\mathrm{Ext}^1_R(M_{(a,b)}, N) \cong \frac{\bigl\{(n_1,n_2) \in N^2 \bigm| (y-b)\cdot n_1 = (x-a)\cdot n_2\bigr\}}{\bigl\{\bigl((x-a)\cdot n, (y-b)\cdot n\bigr) \bigm| n \in N\bigr\}}\,.$$

In the remainder of this exercise we want to calculate the Ext-groups $\mathrm{Ext}^1_R(M_{(a,b)}, M_{(c,d)})$.

(b) Suppose $c \neq a$. Show that $\mathrm{Ext}^1_R(M_{(a,b)}, M_{(c,d)}) = 0$. [*Hint:* Note that for $m \in M_{(c,d)}$ we have $(x - a) \cdot m = (c - a) \cdot m$, and $(c - a) \in k^*$.]

(c) In a similar way, show that $\mathrm{Ext}^1_R(M_{(a,b)}, M_{(c,d)}) = 0$ if $b \neq d$.

(d) Show that $\mathrm{Ext}^1_R(M_{(a,b)}, M_{(a,b)}) \cong k^2$.

(e) For $(\lambda, \mu) \in k^2$ with $(\lambda, \mu) \neq (0, 0)$, show that the module

$$E_{(\lambda:\mu)} := k[x, y]/(x^2, xy, y^2, \lambda \cdot x + \mu \cdot y)$$

can be obtained as an extension of $M_{(0,0)}$ by itself.

(f) For (λ_1, μ_1) and $(\lambda_2, \mu_2) \in k^2 \setminus \{(0, 0)\}$, show that $E_{(\lambda_1:\mu_1)} \cong E_{(\lambda_2:\mu_2)}$ as $k[x, y]$-modules if and only if there exists a constant $\gamma \in k^*$ such that $(\lambda_1, \mu_1) = (\gamma \cdot \lambda_2, \gamma \cdot \mu_2)$. [*Hint for the "only if":* Suppose $\phi\colon E_{(\lambda_1:\mu_1)} \xrightarrow{\sim} E_{(\lambda_2:\mu_2)}$ is an isomorphism of $k[x, y]$-modules. Then ϕ is completely determined by the class $\xi = \phi(\bar{1})$. Moreover, ξ can be represented by an element of the form $p + qx$ with p, $q \in k$ if $\mu_2 \neq 0$ (respectively $p + qy$ if $\lambda_2 \neq 0$). Now show that we must have $q = 0$.]

(g) An R-module is said to have *length* equal to 2 if it can be obtained as an extension of a simple R-module by another simple R-module. Write down an explicit list of all R-modules of length 2, up to isomorphism. [You will need Exercise 1(a).]

Exercise 3. Let $p < q$ be prime numbers such that p does not divide $q - 1$. Use group cohomology to prove that every group of order pq is isomorphic to $\mathbb{Z}/pq\mathbb{Z}$. [*Hint:* Start by choosing an element $g \in G$ of order q (Cauchy) and note that $\langle g\rangle \subset G$ is normal because its index is the smallest prime number dividing the order of G. Then determine all possible $\mathbb{Z}/p\mathbb{Z}$-module structures on $\langle g\rangle \cong \mathbb{Z}/q\mathbb{Z}$ and for each of thoses calculate $\mathrm{H}^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$.]

Exercise 4. Let $C_2 = \{1, \iota\}$ be the group of order 2.

(a) If A is a C_2-module such that the group underlying A is isomorphic to $V_4 = C_2 \times C_2$ (the Klein group), show that A, *as a C_2-module*, is isomorphic to either V_4 with trivial C_2-action, or to V_4 with C_2-action given by $\iota(a, b) = (b, a)$.

(b) Calculate $\mathrm{H}^2(C_2, V_4)$ for both C_2-module structures in (a).

(c) List all groups of order 8 that can be obtained as an extension of C_2 by V_4 for the *non-trivial* C_2-module structure on V_4.