CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for February 22

Names for some categories:

Category	Objects	Morphisms
Set	Sets	Maps
Grp	Groups	Group homomorphisms
Ring	Rings	Ring homomorphisms
CRing	Commutative rings	Ring homomorphisms
C_G	$\{*\}$	$\mathrm{Hom}(*,*) = G$

Exercise 1. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. Let $A \subset \mathrm{Mor}(\mathsf{D})$ be the image of $F\colon \mathrm{Mor}(\mathsf{C}) \to \mathrm{Mor}(\mathsf{D})$, i.e., the collection of all morphisms in D that are of the form $F(f)$ for some morphism f in C. Give an example that shows that the collection A is in general not closed under composition. (Therefore, the "image" of F is in general not a subcategory of D.)

Exercise 2.

(a) Let $(X, \leq)$ and $(Y, \leq)$ be two posets; we view them as categories. Show that to give a functor $F\colon (X, \leq) \to (Y, \leq)$ is the same as giving an order-preserving map $X \to Y$.

(b) Let G and H be groups, and consider the associated 1-object categories C_G and C_H. Show that to give a functor $\mathsf{C}_G \to \mathsf{C}_H$ is the same as giving a homomorphism of groups $G \to H$.

(c) With $(X, \leq)$ a poset and G a group, describe all functors $\mathsf{C}_G \to (X, \leq)$.

Exercise 3. Fix an integer $n \geq 1$. If R is a commutative ring, we may consider the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices with coefficients in R. Note that if $A \in M_n(R)$ then A is invertible if and only if $\det(A) \in R^\times$. (It is not sufficient to require that $\det(A) \neq 0$!)

(a) Show that $R \mapsto \mathrm{GL}_n(R)$ gives a functor $\mathrm{GL}_n\colon \mathsf{CRing} \to \mathsf{Gr}$.

Recall that there is a functor $U\colon \mathsf{CRing} \to \mathsf{Gr}$ that sends a ring R to its group of units $U(R) = R^\times$.

(b) For R a commutative ring, let $\Phi(R)\colon \mathrm{GL}_n(R) \to R^\times$ be the map that sends a matrix $A \in \mathrm{GL}_n(R)$ to its determinant $\det(A) \in R^\times$. Show that this defines a morphism of functors $\Phi\colon \mathrm{GL}_n \to U$.

Exercise 4. Let R be a commutative ring.

(a) If M and N are R-modules, show that $\mathrm{Hom}_R(M,N)$ naturally has the structure of an R-module, obtained by the rule $(r \cdot f)\big(m\big) = r \cdot f(m)$. (Make sure that you understand the meaning of this formula!)

(b) Show that $\mathrm{End}_R(M)$ has the structure of an R-algebra, with composition of endomorphisms as the product. Also show, by means of a concrete example, that this endomorphism ring is not commutative, in general.

Exercise 5. Let R be a ring.

(a) If M is a left R-module that is generated by a single element, show that there exists a left ideal $I \subset R$ such that $M \cong R/I$ as R-modules.

(b) Prove that an R-module M is simple (meaning: $M \neq 0$ and M has no submodules other than (0) and M itself) if and only if $M \cong R/I$ for a maximal left ideal $I \subset R$.

We may ask if the statements in the previous exercise have an analogue for modules over a non-commutative ring. As an example, let k be a field, and take $R = M_2(k)$, the ring of 2×2 matrices with coefficients in k. Let $M = k^2$, viewed as a left R-module by the usual rule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} .$$

(c) Prove that M is a simple $M_2(k)$-module and that $\mathrm{End}_R(M) \cong k$.

(d) Taking $k = \mathbb{F}_p$, prove that $\mathrm{End}_R(M) = \mathbb{F}_p$ does not admit *any* structure of a left R-module. [*Hint:* Suppose $\mathbb{F}_p$ does have the structure of a module over $R = M_2(\mathbb{F}_p)$. By (a) it is isomorphic to R/I for some left ideal I, and in fact $I = \big\{A \in R \bigm| A \cdot 1 = 0\big\}$. Now take $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and suppose

$$B \cdot 1 = m = \underbrace{1 + \cdots + 1}_{m \text{ terms}} \qquad C \cdot 1 = n = \underbrace{1 + \cdots + 1}_{n \text{ terms}},$$

where of course m and n are determined modulo p. By looking at $B^2 \cdot 1$ and at $C^2 \cdot 1$, show that $m = n = 0$, so that B, $C \in I$. Conclude that $I = M_2(\mathbb{F}_p)$; contradiction.]