CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for March 1

Names for some categories:

Category	Objects	Morphisms
Set	Sets	Maps
G-Set	G-Sets (see problem 1.6)	G-equivariant maps
Grp	Groups	Group homomorphisms
Ring	Rings	Ring homomorphisms
Top	Topological spaces	Continuous maps
Top_*	Pointed topological spaces	Pointed continuous maps
$\mathsf{Fun}(\mathsf{C}, \mathsf{D})$	Functors $\mathsf{C} \to \mathsf{D}$	Natural transformations
C_G	$\{*\}$	$\mathrm{Hom}(*, *) = G$

Exercise 1. In Exercise 2 from last week, we have seen that to give a functor $\mathsf{C}_G \to \mathsf{C}_H$ is the same as giving a homomorphism of groups $f\colon G \to H$.

(a) Let $f\colon G \to H$ be a homomorphism. Under what conditions on f is the corresponding functor $f\colon \mathsf{C}_G \to \mathsf{C}_H$ faithful, resp. full, resp. essentially surjective?

(b) If f_1, $f_2\colon G \to H$ are homomorphisms of groups, both viewed as functors $\mathsf{C}_G \to \mathsf{C}_H$, what does it mean to give a morphism of functors $\Phi\colon f_1 \to f_2$? (You should answer this purely in terms of group theory, of course.)

Exercise 2. Let k be a field. Let p be a prime number.

(a) If p is different from the characteristic of k, prove that there is no group G such that $k[G] \cong k[t]/(t^p)$ as k-algebras.

(b) If $p = \mathrm{char}(k)$, show that $k[t]/(t^p)$ is isomorphic to the group ring of $\mathbb{Z}/p\mathbb{Z}$ over k.

Exercise 3. If C and D are categories, we can form the category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ in which the objects are the functors $\mathsf{C} \to \mathsf{D}$ and the morphisms are the morphisms of functors. If G is a group, show that the category $\mathsf{Fun}(\mathsf{C}_G, \mathsf{Set})$ is isomorphic to the category G-Set. (See Exercise 6 from the exercise sheet for 8–15 February.)

Exercise 4. Let M be a module over a commutative ring R. An element $m \in M$ is called a *torsion element* if there exists an element $r \in R$ with $r \neq 0$ such that $rm = 0$. If R is a domain, prove that

$$\mathrm{Tors}(M) := \{ m \in M \mid m \text{ is a torsion element} \}$$

is an R-submodule of M. Show, by means of a concrete example, that the condition that R is a domain cannot be omitted.

Exercise 5. For each of the following functors, determine whether it is full, and whether it is faithful:

(a) The forgetful functor $\mathsf{Gr} \to \mathsf{Set}$.

(b) The forgetful functor $\mathsf{Top}_* \to \mathsf{Top}$.

(c) The functor $U \colon \mathsf{Ring} \to \mathsf{Gr}$ that sends a ring R to its group of units $U(R) = R^\times$.

Do you know which of these functors is essentially surjective? (If you can't decide for (a) and (c), you may find it interesting to look up the answers.)

Exercise 6. Let $F \colon \mathsf{C} \to \mathsf{D}$ be a functor.

(a) Let $f \colon X \to Y$ be a morphism in C. Prove that

$$f \text{ is an isomorphism} \implies F(f) \colon F(X) \to F(Y) \text{ is an isomorphism.}$$

If F is fully faithful, prove that also the converse holds.

In the rest of the exercise, we assume F is an equivalence of categories.

(b) If A is an initial object in C, show that $F(A)$ is an initial object in D. Likewise with 'initial' replaced by 'terminal'.

(c) If two objects X and Y in C admit a product $X \times Y$ in C, show that $F(X \times Y)$ is a product of $F(X)$ and $F(Y)$ in D. (You should add details about the projection maps yourself.) Likewise with 'product' replaced by 'co-product'.