CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for March 8

Exercise 1.

(a) Look up the definition of a product of two categories.

(b) Let R and S be rings. Prove that the category of $(R \times S)$-modules is equivalent to the product category ${}_R\mathsf{Mod} \times {}_S\mathsf{Mod}$.

Exercise 2. As we have seen in the lecture, if R is a PID then every finitely generated R-module can be written uniquely in the form $M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_k)$ with $r, k \geq 0$, and where $a_1, \ldots, a_k$ are non-zero elements in R that are not units and such that a_i divides a_{i+1}. In particular, if M is torsion-free then it is free of finite rank.

(a) Give an example of a PID R and a torsion-free R-module that is not free as an R-module.

(b) Consider the abelian group $A = \mathbb{Q}/\mathbb{Z}$. Prove that $A = \mathrm{Tors}(A)$ but that A is not isomorphic to any group of the form $\mathbb{Z}/n\mathbb{Z} \oplus A'$ with $n > 1$.

(c) Give an example of a finitely generated torsion-free module M over some domain R such that M is not free.

Exercise 3.

(a) Write the group $(\mathbb{Z}/250\,\mathbb{Z}) \oplus (\mathbb{Z}/7\,\mathbb{Z}) \oplus (\mathbb{Z}/16\,\mathbb{Z}) \oplus (\mathbb{Z}/5040\,\mathbb{Z})$ in standard form.

(b) Let $B \subset \mathbb{Z}^5$ be the subgroup generated by the four elements

$$(2, 4, -5, -2, 1)\,, \quad (2, -1, 0, 3, 4)\,, \quad (8, 12, -8, -6, 16)\,, \quad \text{and} \quad (6, 6, -12, -1, 3)\,.$$

Determine the structure of the abelian group $\mathbb{Z}^5/B$. (I.e., bring this group in standard form.)

Exercise 4. Let k be a field, let V be a finite-dimensional k-vector space, and let $\phi\colon V \to V$ be an endomorphism. As discussed in the lecture, the minimal polynomial P_ϕ^{min} of ϕ divides the characteristic polynomial P_ϕ^{char}. This is equivalent to saying that $P_\phi^{\mathrm{char}}(\phi) = 0$, which is the Cayley-Hamilton theorem. In this exercise we establish another relation between the two polynomials.

(a) Let $f \in k[t]$ be an irreducible monic polynomial. Let $V = k[t]/(f^n)$ for some $n \geq 1$, and let ϕ be the endomorphism given by multiplication by t. Prove that $P_\phi^{\mathrm{min}} = P_\phi^{\mathrm{char}} = f^n$. [*Hint:* Use the Cayley-Hamilton theorem.]

(b) Suppose $V = V_1 \oplus V_2$, and suppose we have endomorphisms ϕ_i of V_i such that $\phi(v_1, v_2) = \big(\phi_1(v_1), \phi_2(v_2)\big)$. Prove that $P_\phi^{\mathrm{char}} = P_{\phi_1}^{\mathrm{char}} \cdot P_{\phi_2}^{\mathrm{char}}$ and that $P_\phi^{\mathrm{min}} = \mathrm{lcm}\big(P_{\phi_1}^{\mathrm{min}}, P_{\phi_2}^{\mathrm{min}}\big)$. (lcm = least common multiple, taken to be monic.)

(c) Prove that for (V, ϕ) as in the beginning, P_ϕ^{char} divides a power of P_ϕ^{min}.

Exercise 5. Let R be a ring. Recall (see Exercise 5 of Feb. 22) that an R-module M is said to be *simple* if $M \neq 0$, and if M has no submodules other than 0 and M itself. An R-module M is said to be *indecomposable* if M is not isomorphic to a direct sum $M_1 \oplus M_2$ with M_1 and M_2 both nonzero.

(a) Prove that if M is simple then M is indecomposable.

(b) Give an example of a module M over some ring such that M is indecomposable but not simple.

(c) If R is a PID, describe all finitely generated R-modules that are indecomposable. Which of these are simple?