CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for March 15

Names for some categories:

Category	Objects	Morphisms
Set	Sets	Maps
Top	Topological spaces	Continuous maps

Exercise 1. Let k be a field. Let $f \in k[t]$ be a non-constant monic polynomial, and let

$$f = g_1^{m_1} \cdot g_2^{m_2} \cdots g_r^{m_r}$$

be its factorization into irreducibles; with this we mean that $g_1, \ldots, g_r$ are distinct monic irreducible polynomials in $k[t]$ and $m_1, \ldots, m_r$ are positive integers.

(a) Prove that $k[t]/(f) \cong k[t]/(g_1^{m_1}) \oplus \cdots \oplus k[t]/(g_r^{m_r})$ as $k[t]$-modules.

(b) Let $\lambda \in k$ and $m \geq 1$, and let $I \subset k[t]$ be the ideal generated by $(t-\lambda)^m$. Multiplication by t gives an endomorphism ϕ of the k-vector space $V = k[t]/I$. Give the matrix of ϕ with respect to the basis $1, (t-\lambda), \ldots, (t-\lambda)^{m-1}$ for V.

If V is a finite dimensional k-vector space and $\phi\colon V \to V$ is an endomorphism, we say that ϕ can be put in Jordan normal form if there exists a basis for V such that the matrix of ϕ with respect to this basis is of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$$

where each A_i is a block matrix that is either of size 1×1 or is of the form

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

for some $\lambda \in k$.

(c) Prove that ϕ can be put in Jordan normal form if and only if its characteristic polynomial $P_\phi^{\mathrm{char}} \in k[t]$ can be written as a product of linear polynomials. (In particular, this proves that over an algebraically closed field every endomorphism of a finite dimensional vector space can be put in Jordan normal form.)

An endomorphism $\phi\colon V \to V$ is said to be *nilpotent* if there exists a positive integer n such that $\phi^n = 0$.

(d) Prove that any nilpotent endomorphism can be put in Jordan normal form.

Exercise 2. Write down an explicit contravariant functor $F\colon \mathsf{Top} \to \mathsf{Set}$ that is *not* representable. Prove that the F in your example is indeed not representable!

Exercise 3. Let pt be a set with 1 element. Let C be a category. Consider the functor $F\colon \mathsf{C} \to \mathsf{Set}$ that sends every object in C to pt and sends every morphism in C to $\mathrm{id}_{\mathrm{pt}}$. Note that F may be viewed both as a covariant and as a contravariant functor!

(a) If we view F as a contravariant functor, show that F is representable if and only if C has a final object.

(b) Formulate and prove a similar statement for F when viewed as a covariant functor.

Exercise 4. Let R be a commutative ring. In what follows, letters like M and N denote R-modules.

(a) Prove that $M_1 \otimes_R M_2 \cong M_2 \otimes_R M_1$.

(b) If $f\colon M \to N$ is a homomorphism, show that the map $b\colon R \times M \to N$ given by $b(r, m) = r \cdot f(m)$ is R-bilinear. Show that $f \mapsto b$ gives an isomorphism of R-modules

$$\mathrm{Hom}_R(M, N) \xrightarrow{\sim} \mathrm{Bilin}(R \times M, N)\,. \tag{1}$$

(c) If $g\colon N_1 \to N_2$ is a homomorphism, show that the isomorphisms found in (b) fit in a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M, N_1) & \xrightarrow{\sim} & \mathrm{Bilin}(R \times M, N_1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(M, N_2) & \xrightarrow{\sim} & \mathrm{Bilin}(R \times M, N_2) \end{array}$$

in which the vertical arrows are given by composing with g. In this situation we say that *the isomorphisms* (1) *are functorial in* N, because what we have proven means that we have isomorphisms of functors $\mathrm{Hom}_R(M, -) \xrightarrow{\sim} \mathrm{Bilin}(R \times M, -)$.

(d) Now prove that $R \otimes_R M \cong M$.

Exercise 5. Let I be an ideal of a commutative ring R. Let M and N be R-modules.

(a) Suppose $b\colon (R/I) \times M \to N$ is a bilinear form. Define $\tilde{f}\colon M \to N$ by $\tilde{f}(m) = b(\bar{1}, m)$. Show that $\tilde{f}$ is a homomorphism of R-modules and that $\tilde{f} = 0$ on the submodule $IM \subset M$. Conclude that $\tilde{f}$ induces a homomorphism $f\colon M/IM \to N$.

(b) Prove that the map $b \mapsto f$ gives an isomorphism of R-modules

$$\operatorname{Bilin}\big((R/I) \times M, N\big) \xrightarrow{\sim} \operatorname{Hom}_R(M/IM, N)$$

which is functorial in N. Conclude that $(R/I) \otimes_R M \cong M/IM$.

(c) If m and n are positive integers, prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})$, where $q = \gcd(m, n)$.