CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for March 22

Names for some categories:

Category	Objects	Morphisms
Set	Sets	Maps
Ab	Abelian groups	Group homomorphisms
Ring	Rings	Ring homomorphisms
Field	Fields	Homomorphisms of fields
Top	Topological spaces	Continuous maps

Exercise 1. Consider the forgetful functors

$$\mathsf{Ab} \to \mathsf{Set}\,, \qquad \mathsf{Ring} \to \mathsf{Set}\,, \qquad \mathsf{Field} \to \mathsf{Set}\,.$$

Prove that the first two of these are co-representable but the third is not.

Exercise 2. For a set X, let $\mathscr{P}(X)$ be its powerset. (The set of all subsets of X.)

(a) For a map $f\colon X \to Y$ of sets, define a map $\mathscr{P}(f)\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ in such a way that we obtain a functor $\mathscr{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$.

(b) Prove that this functor $\mathscr{P}$ is representable. Which set represents it?

Exercise 3. Let R be a commutative ring. Let M_1, M_2 and N be R-modules and let $f\colon M_1 \to M_2$ be a homomorphism of modules.

(a) If f is surjective, show that $(f \otimes \mathrm{id}_N)\colon M_1 \otimes_R N \to M_2 \otimes_R N$ is surjective, too.

(b) Take $R = \mathbb{Z}$ and $M_1 = M_2 = \mathbb{Z}$, and consider the injective homomorphism $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $m \mapsto 3m$. Find an R-module N such that $(f \otimes \mathrm{id}_N)$ is *not* injective.

Exercise 4. Let k be a field. Take $\{e_1, e_2\}$ as a basis of the vector space k^2 and $\{f_1, f_2, f_3\}$ as a basis of k^3. Let $\alpha\colon k^2 \to k^2$ and $\beta\colon k^3 \to k^3$ be linear maps, given by matrices A and B, respectively. What is the matrix of $\alpha \otimes \beta\colon (k^2 \otimes_k k^3) \to (k^2 \otimes_k k^3)$ with respect to the basis $e_i \otimes f_j$, taken in lexicographical ordering? (This matrix is denoted by $A \otimes B$.) Can you generalize to matrices of arbitrary size?

Exercise 5. Write the abelian group

$$\left(\mathbb{Z}^2 \oplus (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/126\mathbb{Z})\right) \otimes_{\mathbb{Z}} \left(\mathbb{Z} \oplus (\mathbb{Z}/45\mathbb{Z}) \oplus (\mathbb{Z}/495\mathbb{Z})\right)$$

in standard form.

Exercise 6. Let R be a subring of a commutative ring S, and let M and N be S-modules. Prove that there exists a surjective homomorphism of R-modules $M \otimes_R N \twoheadrightarrow M \otimes_S N$ with $m \otimes_R n \mapsto m \otimes_S n$ for all $m \in M$ and $n \in N$.

Exercise 7.

(a) Let R be a domain with fraction field K. If M is an R-module, show that every element of $K \otimes_R M$ is a pure tensor (i.e., is of the form $c \otimes m$ with $c \in K$ and $m \in M$).

(b) Show that there is an isomorphism of rings $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$, given by $q_1 \otimes q_2 \mapsto q_1 q_2$.