Intro. to algebraic curves — exercise sheet 2

Deadline: 14.30 Thursday 8 October 2015

These exercises are to be handed in to Johan Commelin (`j.commelin@math.ru.nl`), either in his pigeon hole (Huygens building, opposite to room HG03.708), or electronically. Handing in by email is possible only if you write your solutions using TEX or LATEX; in that case, send the pdf output. You are allowed to collaborate with other students but what you write and hand in should be your own work. If different students hand in the same work, we will not accept their work.

1. Let $X \subset \mathbb{C}^n$ be a compact complex submanifold of $\mathbb{C}^n$. Show that X is discrete. (This is in contrast with the real picture, where every manifold is a submanifold of $\mathbb{R}^n$, for some n.)
2. Let $\lambda \in \mathbb{C}$ be a complex number. Consider the curve C_λ in $\mathbb{C}^2$, defined by
$$z + (1+\lambda)x^2 z = x^3 + \lambda x z^2.$$
Embed $\mathbb{C}^2$ in $\mathbb{P}^2$ via $(x, z) \mapsto (x : 1 : z)$. Let $\bar{C}_\lambda$ be the closure of C_λ in $\mathbb{P}^2$.
   (a) Find a homogenous polynomial F, such that $\bar{C}_\lambda$ is the zero locus of F.
   (b) Describe the points at infinity of C_λ. In other words, describe $\bar{C}_\lambda - C_\lambda$.
   (c) For which λ is $\bar{C}_\lambda$ a non-singular submanifold of $\mathbb{P}^2$?
   For the remainder of the question, assume $\bar{C}_\lambda$ is non-singular.
   (d) Show that the coordinate projections, pr_x and pr_z, as functions $C_\lambda \to \mathbb{C}$, extend to meromorphic functions x and z on $\bar{C}_\lambda$.
   (e) For each of x and z:
      – compute the degree;
      – find the branch points in $\mathbb{P}^1$;
      – find the ramification points in $\bar{C}_\lambda$.
   (f) Calculate $\mathrm{div}(x)$ and $\mathrm{div}(z)$.
3. Let $C \subset \mathbb{P}^2$ be a closed smooth curve defined by a homogeneous polynomial of degree d. Fix a point $p \in C$. Write V for $T_{\mathbb{P}^2,p}$. Let $x \in \mathbb{P}^2 - \{p\}$ be a point. Let $L_x \subset \mathbb{P}^2$ be the line through x and p. Since $L_x \subset \mathbb{P}^2$ is an immersion, $T_{L_x,p}$ can be viewed as a 1-dimensional linear subspace of V. The canonical map
$$\begin{aligned} \mathbb{P}^2 - \{p\} &\to \mathbb{P}(V) \\ x &\mapsto T_{L_x,p} \end{aligned}$$
is holomorphic and surjective.
   (a) Deduce that p induces a holomorphic map $\phi_p \colon C \to \mathbb{P}(V)$. (Warning: think about how to define ϕ_p at p. Hint: show that you can make a suitable choice of coordinates.)
   (b) Compute the degree of ϕ_p.
   (c) Conclude that every closed smooth curve of degree 2 (*i.e.*, $d = 2$) is isomorphic to $\mathbb{P}^1$.