Intro. to algebraic curves — exercise sheet 3

Deadline: 14.30 Thursday 26 November 2015

These exercises are to be handed in to Johan Commelin (`j.commelin@math.ru.nl`), either in his pigeon hole (Huygens building, opposite to room HG03.708), or electronically. Handing in by email is possible only if you write your solutions using TEX or LATEX; in that case, send the pdf output. You are allowed to collaborate with other students but what you write and hand in should be your own work. If different students hand in the same work, we will not accept their work.

1. Let C be a compact Riemann surface of genus g. Show that for all $n \geq g$ the map
$$\begin{aligned} C^n &\longrightarrow \mathrm{Cl}^n \\ (P_i)_{i=1}^n &\longmapsto \sum_{i=1}^n P_i \end{aligned}$$
is surjective. (*N.b.:* The empty product is a point.)
2. Let P be a point on a compact Riemann surface C. Show that there is a meromorphic function on C that is holomorphic on $C - \{P\}$.
3. Let C be a compact Riemann surface of genus g. The smallest degree of a non-constant meromorphic function is called the *gonality* of C. Show that the gonality of C is $\leq g+1$. (Remark: actually, something much better is true: the gonality is bounded from above by $\lfloor (g+3)/2 \rfloor$.)
4. Let C be a compact Riemann surface of genus g. Let D be an effective divisor on C. Prove that $\ell(D) - 1 \leq \deg(D)$. Moreover, prove that equality holds if and only if $D = 0$ or $g = 0$.
5. Let C be a compact Riemann surface. Assume there is a non-constant holomorphic map $\mathbb{P}^1 \to C$. Show that C is isomorphic to $\mathbb{P}^1$.
6. Let $f\colon C_1 \to C_2$ be a non-constant map of compact Riemann surfaces, with genera g_1 and g_2. Assume f is *not* an isomorphism. For which pairs (g_1, g_2) is it possible that $g_1 \leq g_2$? Show that all pairs you list actually occur.
7. Let F_1, F_2, and G be non-constant homogenous polynomials in $\mathbb{C}[X,Y,Z]$ such that $F = F_1 \cdot F_2$ and G have no common irreducible factor. Put $C_i = Z(F_i)$, $C = C_1 + C_2 = Z(F)$, and $D = Z(G)$. Prove that $i(P; C \cdot D) = i(P; C_1 \cdot D) + i(P; C_2 \cdot D)$ for all $P \in \mathbb{P}^2$.

Remark: Possibly with the exception of Exercise 7, we think all problems can be solved within two or three lines!