Intro. to algebraic curves — exercise sheet 5

Deadline: 14.30 Thursday 14 January 2016

These exercises are to be handed in to Johan Commelin (`j.commelin@math.ru.nl`), either in his pigeon hole (Huygens building, opposite to room HG03.708), or electronically. Handing in by email is possible only if you write your solutions using TeX or LaTeX; in that case, send the pdf output. You are allowed to collaborate with other students but what you write and hand in should be your own work. If different students hand in the same work, we will not accept their work.

1. Let C be a compact Riemann surface of genus $g > 2$.
   (a) If C has a g^1_3, show that this g^1_3 is necessarily complete.
   (b) Assume C is not hyperelliptic. If C has a g^1_3, show that this g^1_3 is base-point free.
   [*Remark:* The solution to this exercise can be written in two lines.]
2. Let $C_0 \subset \mathbb{C}^2$ be the curve given by the equation $y^3 = x^5 - 1$. Let C be the normalization of its projective closure, and let $\pi\colon C \to \mathbb{P}^1$ be the morphism corresponding to the meromorphic function x on C. (So on C_0 the map π is given by $(x, y) \mapsto (x : 1)$.)
   Let $\zeta = \exp(2\pi i/5)$, and for $i = 0, \ldots, 4$, let $P_i = (\zeta^i, 0) \in C$.
   (a) Show that there is a unique point $P_\infty \in C$ with $\pi(P_\infty) = (1 : 0)$.
   (b) Show that $g(C) = 4$.
   (c) Calculate the divisors of x, of y and of $\mathrm{d}x$. Using this, write down an explicit basis for the space $H^0(C, \Omega^1)$ of holomorphic 1-forms on C.
   (d) Show that the divisors $3P_i$ for $i \in \{0, 1, 2, 3, 4, \infty\}$ are all linearly equivalent and that $6P_i$ is a canonical divisor.
   (e) Show that C is not hyperelliptic and that the points P_i are Weierstrass points of weight 4 with gap sequence $\{1, 2, 4, 7\}$. [*Hint:* For the first assertion, use information about Weierstrass points and their gap sequences.]
3. Let C be a non-hyperelliptic curve of genus 5. Let $\phi_K\colon C \hookrightarrow \mathbb{P}^4$ be the canonical embedding. Assume C has a g^1_3. By the first exercise, this g^1_3 is automatically complete and base-point free; it corresponds to a non-constant morphism $f\colon C \to \mathbb{P}^1$ of degree 3. If $f^{-1}(Q) = \{P_1, P_2, P_3\}$ for some $Q \in \mathbb{P}^1$, show that the points $\phi_K(P_1)$, $\phi_K(P_2)$ and $\phi_K(P_3)$ are collinear (i.e., they lie on a straight line). [*Hint:* Let $D = P_1 + P_2 + P_3$, so that $|D|$ is the g^1_3. Compute $\ell(K - D)$ and interpret the result.]