

Addendum to the paper [5]

by

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Abstract. In [5] we showed that the Coleman–Oort conjecture can be reduced to three particular cases. Based on a manuscript of Yeung, we here show that one of these case can be eliminated.

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Introduction

In [5] we showed that the Coleman–Oort conjecture can be reduced to three particular cases. In November 2022, Sai Kee Yeung sent me a manuscript in which he explains that one of these cases can be eliminated. (In fact, he claims that two of the three cases can be eliminated; but I think his argument only works in one case.)

The goal of this note is to present this result, which in fact boils down to an immediate application of results of Antonakoudis in [1], a paper that I was not aware of when writing [5]. *The result presented here is entirely due to Yeung.*

1. Quick review of the results of [5], and statement of the sharpening

We work over the complex numbers.

Notation 1.1. Let g be a positive integer.

- A_g is the coarse moduli space of g -dimensional ppav;
- $T_g^\circ \subset A_g$ is the open Torelli locus, $T_g \subset A_g$ is its Zariski closure;
- $HE_g^\circ \subset T_g^\circ$ is the locus of hyperelliptic Jacobians, and $HE_g \subset T_g$ its Zariski closure.

If n is a positive integer, we write $A_{g,[n]}$ for the moduli space of g -dimensional ppav with a symplectic level n structure. For $n \geq 3$ this is a fine moduli scheme. For the other loci, a subscript $[n]$ indicates that we take the pre-image in $A_{g,[n]}$. (So we have $T_{g,[n]}^\circ \subset T_{g,[n]} \subset A_{g,[n]}$, etc.)

The main result of [5] is as follows.

Theorem 1.2. *Let g be an integer with $g \geq 8$. Assume there are infinitely many nonsingular complex curves of genus g whose Jacobians are CM abelian varieties. Then there exists a special subvariety $S \subset A_g$ of positive dimension such that $S \subset T_g$ and $S \cap T_g^\circ \neq \emptyset$, and such that at least one of the following is true:*

- (1) $\dim(S) = 1$,
- (2) $\dim(S) = 2$ and S is complete,
- (3) S is a complete ball quotient.

The following strengthening, based on results of Antonakoudis [1], is due to Yeung.

Theorem 1.3 (Yeung). *With assumptions as in Theorem 1.2, there exists a special subvariety $S \subset T_g$ of positive dimension such that $S \cap T_g^\circ \neq \emptyset$, and such that either (1) or (2) is true.*

2. Proof of Theorem 1.3

2.1. To prove Theorem 1.3, we have to exclude the possibility that there exists a complete ball quotient $S \subset \mathbf{A}_g$ with $\dim(S) \geq 3$ such that $S \subset \mathbf{T}_g$ and S meets $\mathbf{T}_g^\circ$. Moreover, we may assume that S is minimal, in the sense that there are no special $S' \subsetneq S$ of positive dimension. In what follows we assume that we do have such a special subvariety S . Our goal is to derive a contradiction.

The assumption that S is special implies that if we take n sufficiently large, S is the image of a nonsingular subvariety $S_{[n]} \subset \mathbf{T}_{g,[n]} \subset \mathbf{A}_{g,[n]}$. Let $S_{[n]}^\circ = S_{[n]} \cap \mathbf{T}_{g,[n]}^\circ$. By [3], Proposition 6.3, one of the following holds:

- (a) $S_{[n]}^\circ$ is contained in $\mathbf{HE}_{g,[n]}^\circ$;
- (b) $S_{[n]}^\circ$ is disjoint from $\mathbf{HE}_{g,[n]}^\circ$;
- (c) $S_{[n]}^\circ \cap \mathbf{HE}_{g,[n]}^\circ$ is nonempty and has pure codimension 1 in $S_{[n]}^\circ$.

2.2. Let $\mathbf{T}_g^{\text{dec}} = \mathbf{T}_g \setminus \mathbf{T}_g^\circ$ be the locus of decomposable Jacobians. Because all irreducible components of $S^{\text{dec}} = S \cap \mathbf{T}_g^{\text{dec}}$ are again special subvarieties of $\mathbf{A}_g$, the minimality of S implies that S^{dec} is finite.

Suppose that we are in one of the case (a) or (c). Then $S \cap \mathbf{HE}_g^\circ$ is affine (because $\mathbf{HE}_g^\circ$ is affine), and all its irreducible components have dimension ≥ 2 . On the other hand, $S \cap \mathbf{HE}_g$ is a complete variety (because $S \subset \mathbf{T}_g$ is a complete subvariety and $\mathbf{HE}_g \subset \mathbf{T}_g$ is closed), whereas $(S \cap \mathbf{HE}_g) \setminus (S \cap \mathbf{HE}_g^\circ) \subset (S \cap \mathbf{T}_g^{\text{dec}})$ is finite. As this is impossible, this rules out the above options (a) and (c); hence S is disjoint from the hyperelliptic locus in $\mathbf{T}_g^\circ$.

2.3. Let n be an integer with $n \geq 3$. The restriction of the Torelli morphism $j: \mathbf{M}_{g,[n]} \rightarrow \mathbf{A}_{g,[n]}$ to the non-hyperelliptic locus is unramified and finite of degree 2 onto its image.

Let $d = \dim(S)$. By assumption, the universal cover of S is the open ball $\mathbb{B}^d = \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid \sum |z_i|^2 < 1\}$. Let $U \subset S$ be the complement of the finite set S^{dec} . Because S^{dec} has codimension > 2 in S , the universal cover $\tilde{U}$ of U is the complement of a discrete subset in $\mathbb{B}^d$, and because $U \subset \mathbf{T}_g^\circ$ is disjoint from the hyperelliptic locus, we have an induced map $\tilde{U} \rightarrow \text{Teich}_g$ to Teichmüller space.

There exists a totally geodesic $\mathbb{B}^2 \hookrightarrow \mathbb{B}^d$ whose image is contained in $\tilde{U}$. (Use that the intersection of $\mathbb{B}^d$ with any linear subspace of $\mathbb{C}^d$ is isomorphic to $\mathbb{B}^{d-1}$.) Now consider the composition

$$h: \mathbb{B}^2 \hookrightarrow \tilde{U} \rightarrow \text{Teich}_g \rightarrow \mathfrak{H}_g^\pm,$$

and let $\mathfrak{H}_g^+ \subset \mathfrak{H}_g^\pm$ be the connected component that contains the image. By [4], Theorem 4.1, h admits a holomorphic retraction $r: \mathfrak{H}_g \rightarrow \mathbb{B}^2$. Because holomorphic maps are distance-decreasing (better: distance non-increasing) with respect to Kobayashi metrics, it follows that the map $\mathbb{B}^2 \hookrightarrow \text{Teich}_g$ is an isometric embedding with respect to Kobayashi metrics. Note that the Kobayashi metric on $\mathbb{B}^2$ is the Bergmann metric (see ??), and by a result of Royden, the Kobayashi metric on Teich_g is the same as the Teichmüller metric. We now obtain a contradiction with [1], Theorem 1.1.

References

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