1. Documentation for the Magma code

1.1 This document provides details for the Magma computations that are used in the paper *The Tate Conjecture for even dimensional Gushel–Mukai varieties in characteristic* $p \geq 5$ by Lie Fu and Ben Moonen. In everything that follows we are in the situation described in Section 5 of that paper. As described there, we have a matrix E of size 45×5.

A first Magma script, called `SmallRankModP` simply checks that Lemma 5.10 is correct: if p is a prime number with $p \geq 5$ and N is a sub-matrix of E of size 5×5 with $\mathrm{rk}_{\mathbb{Q}}(N) = 5$ then $\mathrm{rk}_{\mathbb{F}_p}(N) \geq 4$. In the code, E is called `EQS` and N is called `M`.

We inspect all 5×5 sub-matrices N of E, we let d be the absolute value of the determinant of N, and if $d > 1$ then for all prime numbers $p \geq 5$ with $\mathrm{rk}_{\mathbb{F}_p}(N) < 4$ we print p, N and the rank.

The output given by Magma is empty, as it should be.

1.2 The next thing we do is to limit the set of prime numbers at which we need to do further work. This is done in a script called `WhichPrimes`. Similar to the above, we inspect all 5×5 sub-matrices of E, we let d be the absolute value of the determinant, and we put all prime divisors of these numbers d in a set. At the end we print this set.

The output given by Magma is $\{2, 3, 5, 7, 11, 13\}$.

1.3 The most important Magma script that we need is `PotentialFamilies`. (The latest version has a date added to the file name.) Let us first outline the structure of the script:

- First we create a matrix `EQS` of size 45×5.
- Next there are functions called `IsValid` and `DoubleEigenval`, which will be discussed below.
- Then we make a couple of definitions for objects `Primes` etc.
- Then comes the main loop, described in more detail below.
- Then there is a function to clean up the output followed by a procedure to print the results in a readable way.
- Finally all relevant output is printed.

In the main loop we carry out what is described in Section 5.12 of the paper. As above, `EQS` is the matrix that in the paper is called E. We inspect all sub-matrices N of E of size 5×5. Such a matrix is relevant for us if $\mathrm{rk}_{\mathbb{Q}}(N) = 5$, and then we keep track of all prime numbers $p \geq 5$ such that $\mathrm{rk}_{\mathbb{F}_p}(N) < 5$. We already know (see above) that in fact $p \in \{5, 7, 11, 13\}$ and $\mathrm{rk}_{\mathbb{F}_p}(N) = 4$. Given N and p, we compute the $A = \mathrm{diag}(-a_1, \ldots, -a_5)$ in $\mathfrak{gl}_{5,k}$ such that $N \cdot (a_1, \ldots, a_5)^{\mathsf{t}} = 0$; this A is unique up to scalars. By Proposition 5.11(4) of the paper, we know that we only need to inspect those cases where not all a_i are distinct, so we check this using `DoubleEigenval`. Then we store the sub-matrix of E that in the paper is called E_A; it consists of all rows X of E for which $X \cdot (a_1, \ldots, a_5)^{\mathsf{t}} = 0$. In the code this matrix is called `QuadricMatrix`. We check if this matrix satisfies the condition in Proposition 5.11(3) of the paper; this is done using the function `IsValid`. If the four-tuple (N, p, A, E_A) passes all tests, we store this as a solution in a set `SolsP`, where `P` is the prime number.

The last part of the code is only about cleaning things up and producing readable output. In the function `Cleanup`, we filter out double occurrences of solutions (rescaling and permutation of the coordinates).

As described in the paper, the output given by Magma consists of four families with $p = 5$, one family with $p = 7$, and no families with $p = 11$ or $p = 13$.