Hand-in Assignment 1

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All vector spaces are over $\mathbb{C}$ and finite-dimensional.

1. Let V be a semisimple representation of a group G.
   (a) Show that every quotient representation of V is again semisimple.
   (b) Show that every subrepresentation of V is also semisimple.

2. Let $V = \mathbb{C}^3$, let ρ be the representation

$$\begin{aligned} \rho\colon \mathbb{G}_m^2 &\to \mathrm{GL}(V) \\ (x, y) &\mapsto \begin{pmatrix} \frac{4xy^2 - x^2}{3y} & \frac{2(x^2 - xy^2)}{3y} & \frac{4(x^2 - xy^2)}{3y} \\ \frac{2(x^2 - xy^2)}{3y} & \frac{2x^2 + xy^2}{3y} & \frac{2(xy^2 - x^2)}{3y} \\ \frac{2(xy^2 - x^2)}{3y} & \frac{x^2 - xy^2}{3y} & \frac{5x^2 - 2xy^2}{3y} \end{pmatrix} \end{aligned}$$

   and let φ be the algebraic homomorphism

$$\begin{aligned} \varphi\colon \mathbb{G}_m^3 &\to \mathbb{G}_m^2 \\ (a, b, c) &\mapsto \left(\frac{a}{b}, \frac{b^2}{c} \right). \end{aligned}$$

   Determine the characters of $\mathbb{G}_m^3$ present in the representation $(V, \rho \circ \varphi)$ and give a basis of each character space.

3. Let $V = \mathbb{C}^2$, considered as the standard representation of $G = \mathrm{SL}_2$. Let W be the representation $V^{\otimes 3}$ of G. We let the group S_3 act on W by permutation of the factors.
   (a) Let $(i\ j)$ be a transposition in S_3. Let W_{ij} be the subspace of W defined as

$$W_{ij} = \{ w \in W : (i\ j)w = -w \}.$$

   Show that W_{ij} is a subrepresentation of W and that $W_{ij} \cong V$ as representations of G.
   (b) Show that $W \cong \mathrm{Sym}^3(V) \oplus V \oplus V$ as representations of G.