Hand-in Assignment 3

Milan Lopuhaä

To be handed in January 9^{th}, 2017

All vector spaces are over $\mathbb{C}$ and finite-dimensional.

1. Let V be the standard representation of $\mathfrak{sl}_2$. Decompose the representation $\bigwedge^2(V^{\otimes 2})$ into irreducible representations.

2. Let $X \in \mathfrak{sl}_3$ be the matrix
$$\begin{pmatrix} -12 & -7 & -8 \\ 4 & 4 & 4 \\ 12 & 6 & 8 \end{pmatrix}.$$
Determine its semisimple part X_{s} and its nilpotent part X_{n}.

3. Let V be a vector space, and let W be a subspace. Define the subgroup $G \subset \mathrm{GL}(V)$ by
$$G = \{g \in \mathrm{GL}(V) : gW = W\}.$$
Show that G is an algebraic subgroup of $\mathrm{GL}(V)$.

4. Let $\mathfrak{h}$ be the diagonal subtorus of $\mathfrak{g} = \mathfrak{o}_4$.
   (a) Determine the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.
   (b) Determine the Killing form on $\mathfrak{h}$.
   (c) Draw the root diagram of $\mathfrak{o}_4$ as an inner product space (i.e. such that the angles and lengths are correct).

5. Consider the Lie algebra $\mathfrak{gl}_2$. Recall that $\mathfrak{gl}_2 = \mathfrak{z} \oplus \mathfrak{sl}_2$, where $\mathfrak{z}$ is its center.
   (a) Let V be any representation of $\mathfrak{gl}_2$, and let $\lambda \in \mathfrak{z}^\vee$. Show that
$$V_\lambda = \{v \in V : Z(v) = \lambda(Z) \cdot v \ \forall Z \in \mathfrak{z}\}$$
   is a $\mathfrak{gl}_2$-subrepresentation of V.
   (b) Suppose V is irreducible. Show that there exists a $\lambda \in \mathfrak{z}^\vee$ such that $Z(v) = \lambda(Z) \cdot v$ for all $Z \in \mathfrak{z}$.

(c) Let λ be as in (b), and define the representation

$$\begin{aligned} \varrho_\lambda \colon \mathfrak{z} &\to \mathfrak{gl}_1 \\ Z &\mapsto \lambda(Z). \end{aligned}$$

Show that $V \cong \varrho_\lambda \boxtimes \varrho$ for some irreducible representation ϱ of $\mathfrak{sl}_2$.

6. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Let R be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix a Weil chamber $\mathcal{C}$, and let R^+ be the corresponding set of positive roots; let Δ be the corresponding basis of positive roots. Let V be a representation of $\mathfrak{g}$, and let ψ be the heighest weight in V. Let $v_0 \in V_\psi \setminus \{0\}$, and let W be the smallest subspace of V such that $v_0 \in W$ and such that $\mathfrak{g}_{-\alpha} W \subset W$ for all $\alpha \in \Delta$.

   (a) Show for any two roots α and β of $\mathfrak{g}$ such that $\alpha + \beta \neq 0$ one has $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. *Hint*: consider the α-string through β in the adjoint representation.

   (b) Show that W is closed under the action of $\mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$.

   (c) Show that W is closed under the action of $\mathfrak{h}$.

   (d) Choose, for every $\alpha \in \Delta$, a nonzero element $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$. For $l \geq 0$, let W_l be the subspace of W generated by the set

   $$\{(X_{-\alpha_1} \circ X_{-\alpha_2} \circ \ldots \circ X_{-\alpha_k})(v_0) : k \leq l, \alpha_i \in \Delta\}.$$

   Show that for every $\alpha \in \Delta$ and every $l \geq 0$ one has $\mathfrak{g}_\alpha W_l \subset W_{l-1}$, where $W_{-1} = 0$. *Hint*: use induction.

   (e) Show that W is closed under the action of $\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$; hence W is a $\mathfrak{g}$-subrepresentation of V.

   (f) Show that ψ is the heighest weight of W, and that W_ψ is one-dimensional.

   (g) Show that W is irreducible as a representation of $\mathfrak{g}$. *Hint*: Let $W = \bigoplus_i U_i$ be a decomposition of W into irreducible subrepresentations. Show that there is exactly one i such that $U_{i,\psi} \neq 0$.