Problems for Representations of Linear Algebraic Groups

Milan Lopuhaä

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All vector spaces are assumed to be over $\mathbb{C}$ and of finite dimension.

1. Let V and W be vector spaces, and let n be a positive integer. Show that $\mathrm{Sym}^n(V \oplus W)$ and
$$\bigoplus_{i=0}^{n} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W)$$
are naturally isomorphic, i.e. give a map between them that does not depend on a choice of basis, and show that it is an isomorphism. *Hint*: Give a map
$$(V \oplus W)^{\otimes n} \to \bigoplus_{i=0}^{n} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W)$$
and show that it is invariant under the action of S_n on the left hand side.

2. Let V be a vector space and let n be a positive integer.
   (a) Let $\mathrm{Inv}^n(V) = \{v \in V^{\otimes n} : \sigma \cdot v = v \ \forall \sigma \in S_n\}$. Show that the composite map
   $$\mathrm{Inv}^n(V) \hookrightarrow V^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n(V)$$
   is an isomorphism.
   (b) Let $\mathrm{Alt}^n(V) = \{v \in V^{\otimes n} : \sigma \cdot v = \epsilon(\sigma) v \ \forall \sigma \in S_n\}$. Show that the composite map
   $$\mathrm{Alt}^n(V) \hookrightarrow V^{\otimes n} \twoheadrightarrow \bigwedge^n V$$
   is an isomorphism.

3. Let V be a vector space.
   (a) Show that the natural map $V^{\otimes 2} \to \mathrm{Sym}^2(V) \oplus \bigwedge^2 V$ is an isomorphism. *Note*: this is not true if the 2 is replaced by any other positive integer!

(b) Let φ be a bilinear form on V. Show that there exist a symmetric φ_1 and an antisymmetric φ_2, such that $\varphi = \varphi_1 + \varphi_2$, and that this decomposition is unique.

4. Let V be a vector space. Choose a basis $(e_1, \ldots, e_n)$ of V. An *algebraic function* on V is a map $f \colon V \to \mathbb{C}$ such that there exists a $P \in \mathbb{C}[X_1, \ldots, X_n]$ so that $f(v) = P(\lambda_1, \ldots, \lambda_n)$ for all $v = \sum_i \lambda_i e_i$ in V. A subset of V is called *algebraic* if it is the zero locus of some set of algebraic functions. Convince yourself that this does not depend on the choice of a basis.

   Now let W be a two-dimensional vector space. An element of $W \otimes W$ is called a *pure tensor* if it is of the form $w \otimes w'$ for some $w, w' \in W$. Show that the pure tensors form a strict algebraic subset of $W \otimes W$.

5. Let V and W be vector spaces. Show that $V^\vee \otimes W$ is naturally isomorphic to $\operatorname{Hom}(V, W)$ (see problem 1 for what we mean by *naturally isomorphic*).

6. Let V be a vector space, and let g be an endomorphism of V.

   (a) Show that g induces an endomorphism $\bigwedge^k g$ of $\bigwedge^k V$ for every nonnegative integer k.

   (b) Now suppose $k = \dim V$. Show that $\bigwedge^k g = (\det g) \cdot \mathrm{id}$ as endomorphisms of the one-dimensional vector space $\bigwedge^k V$.

7. Let V be a vector space, and let φ be a nondegenerate antisymmetric bilinear form on V. Show that the map $\psi \colon V^{\otimes 2} \to \mathbb{C}$ given on pure tensors by $\psi(v_1 \otimes v_2, w_1 \otimes w_2) = \varphi(v_1, w_1)\varphi(v_2, w_2)$ is a nondegenerate bilinear form. Is it symmetric, or antisymmetric? Let $G \subset \mathrm{GL}(V \otimes V)$ be the subgroup given by

$$G = \{g \in \mathrm{GL}(V \otimes V) : \varphi(gx, gy) = \psi(x, y) \ \forall x, y \in V \otimes V\}.$$

   Show furthermore that the natural map $\mathrm{Sp}(V, \varphi) \times \mathrm{Sp}(V, \varphi) \to G$ is neither injective nor surjective.