Problems for Representations of Linear Algebraic Groups

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All vector spaces are over $\mathbb{C}$ and finite-dimensional.

1. Let G_1, G_2 be algebraic groups, and let $f\colon G_1 \to G_2$ be an algebraic homomorphism.
   (a) Let H_1 be an algebraic subgroup of G_1. Show that $f|_{H_1}\colon H_2 \to G_2$ is an algebraic homomorphism.
   (b) Let H_2 be an algebraic subgroup of G_2. Show that $f^{-1}(H_2)$ is an algebraic subgroup of G_1.
2. Let V be a 3-dimensional vector space and let $W = \bigwedge^2 V$; this naturally has a representation of $\mathrm{GL}(V)$. Choose your favourite bases for V and W, so that we may identify $\mathcal{O}(\mathrm{GL}(V)) = \mathbb{C}[X_{ij}, \frac{1}{\det}]$ and $\mathcal{O}(\mathrm{GL}(W)) = \mathbb{C}[Y_{ij}, \frac{1}{\det}]$. Describe the map $\mathcal{O}(\mathrm{GL}(W)) \to \mathcal{O}(\mathrm{GL}(V))$ given by the representation of $\mathrm{GL}(V)$ on W, i.e. give the image of each of the Y_{ij} in terms of the X_{ij}.
3. Let G be a finite subgroup of $\mathrm{GL}(V)$ for some vector space V. Recall from problem 1 of week 1 that G is algebraic.
   (a) Let $g \in G$. Show that there is an $f \in \mathcal{O}(\mathrm{GL}(V))$ such that $f(g) \neq 0$ and $f(g') = 0$ for all $g' \in G \setminus \{g\}$.
   (b) Deduce that every set-theoretic map $G \to \mathbb{C}$ is algebraic.
   (c) Let W be a vector space. Show that every homomorphism of groups $G \to \mathrm{GL}(W)$ is an algebraic representation.
4. Let $\mathbb{G}_{\mathrm{a}}$ be the additive group $(\mathbb{C}, +)$. This can be considered as an algebraic group via the inclusion map

$$\begin{aligned} \mathbb{G}_{\mathrm{a}} &\to \mathrm{GL}_2(\mathbb{C}) \\ x &\mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

   (a) Now let V be a vector space, and let $g \in \mathrm{End}(V)$ be a nilpotent endomorphism. Show that the map

$$\begin{aligned} \mathbb{G}_{\mathrm{a}} &\to \mathrm{GL}(V) \\ x &\mapsto \exp(xg) \end{aligned}$$

is an algebraic representation of $\mathbb{G}_{\mathrm{a}}$ (recall that $\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \cdots$).

(b) Give the map induced map $\mathcal{O}(\mathrm{GL}(V)) \to \mathcal{O}(\mathbb{G}_{\mathrm{a}}) = \mathbb{C}[X]$ in the case that $V = \mathbb{C}^3$ and $g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

5. Let $\mathbb{G}_{\mathrm{m}}$ be the multiplicative group $(\mathbb{C}^\times, \cdot)$, regarded as an algebraic group through its identification with $\mathrm{GL}_1(\mathbb{C})$. Recall that $\mathcal{O}(\mathbb{G}_{\mathrm{m}}) \cong \mathbb{C}[X, X^{-1}]$.

(a) Determine the group $\mathbb{C}[X, X^{-1}]^\times$.

(b) Show that the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{C}\text{-Alg}}(\mathbb{C}[X, X^{-1}], \mathbb{C}[X, X^{-1}]) & \to & \mathbb{C}[X, X^{-1}] \\ \sigma & \mapsto & \sigma(X) \end{array}$$

is injective and that its image is $\mathbb{C}[X, X^{-1}]^\times$.

(c) Let $f\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ be an algebraic homomorphism. Show that there exists an integer n such that $f(z) = z^n$ for all $z \in \mathbb{G}_{\mathrm{m}} = \mathbb{C}^\times$.