Problems for Representations of Linear Algebraic Groups

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1. Show that $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ are not isomorphic as algebraic groups, but their Lie algebras are isomorphic as Lie algebras.

2. Let $f\colon G_1 \to G_2$ be an algebraic homomorphism.

   (a) Show that f induces a homomorphism of Lie algebras $\operatorname{Lie} f\colon \operatorname{Lie} G_1 \to \operatorname{Lie} G_2$.

   (b) Suppose that f is surjective. Proof that $\operatorname{Lie} f$ is surjective. *Hint*: Show that $f^*\colon \mathcal{O}(G_2) \to \mathcal{O}(G_1)$ is injective.

   (c) Consider the map $f\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ given by $f(z) = z^2$. Show that f is not injective, but $\operatorname{Lie} f$ is.

3. Let J_n be the $n \times n$-matrix given by $J_{n,i,j} = \delta_{n+1-i,j}$, i.e. J_n is the matrix with 1 on the antidiagonal and 0 on all other entries. For an $n \times n$-matrix X, define $X^\dagger = JX^TJ$; notice that this is the reflection of X in the antidiagonal.

   (a) Let $V = \mathbb{C}^n$, and let $\varphi\colon V \times V \to \mathbb{C}$ be the symmetric bilinear form given by J_n. Show that

$$\mathfrak{o}_n := \operatorname{Lie} \mathrm{O}(V, \varphi) = \{X \in \mathfrak{gl}(V) : X^\dagger = -X\}.$$

   (b) Let $V = \mathbb{C}^{2n}$, and let $\varphi\colon V \times V \to \mathbb{C}$ be the antisymmetric bilinear form given by $\left(\begin{smallmatrix} 0 & J_n \\ -J_n & 0 \end{smallmatrix}\right)$. Show that

$$\mathfrak{sp}_{2n} := \operatorname{Lie} \mathrm{Sp}(V, \varphi) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(V) : A = -D^\dagger, B = B^\dagger, C = C^\dagger \right\},$$

   where A, B, C, D are $n \times n$-blocks.

4. Show that $\mathfrak{sp}_2 \cong \mathfrak{sl}_2 \cong \mathfrak{o}_3$ as Lie algebras.