Problems for Representations of Linear Algebraic Groups

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1. Let $\mathfrak{g}$ be a semisimple Lie algebra. Show that $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is injective.
2. Let $\mathfrak{g}$ be a Lie algebra. Show that $\mathfrak{g}$ is solvable if and only if there is a chain of Lie algebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_k = 0$ such that each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ and such that each $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.
3. Let $\mathfrak{g}$ be a Lie algebra. Show that $\mathfrak{g}$ is nilpotent if and only if there is a chain of Lie algebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_k = 0$ such that each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ and such that each $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is contained in the centre of $\mathfrak{g}/\mathfrak{g}_{i+1}$.
4. Show that every irreducible representation of a solvable Lie algebra has dimension 1.
5. A Lie algebra $\mathfrak{g}$ is called *simple* if it is noncommutative and its only ideals are 0 and $\mathfrak{g}$.
   (a) Show that a direct sum of simple Lie algebras is semisimple.
   (b) Let $\mathfrak{g}$ be a (not necessarily simple) Lie algebra. Show that its minimal nonzero ideals are either commutative of dimension 1 or simple.
   (c) Now suppose the adjoint representation of $\mathfrak{g}$ is semisimple. Show that there is a set $\mathcal{A}$ of minimal nonzero ideals of $\mathfrak{g}$ such that $\mathfrak{g} \cong \bigoplus_{\mathfrak{a}\in\mathcal{A}} \mathfrak{a}$ as Lie algebras. *Hint*: show that a subrepresentation of the adjoint representation corresponds to an ideal of $\mathfrak{g}$.
   (d) Show that in this case $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and that $\mathfrak{g} \cong \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$.
6. Let $\mathfrak{g}$ be a Lie algebra.
   (a) Suppose that $\mathfrak{g}$ has a faithful irreducible representation. Use Proposition 9.17 of Fulton-Harris to show that $\mathrm{Rad}(\mathfrak{g})$ has dimension at most 1.

(b) Now let n be an integer, and suppose $\mathfrak{g}$ is a Lie-subalgebra of $\mathfrak{sl}_n$ for which the standard representation V of $\mathfrak{sl}_n$ is an irreducible representation of $\mathfrak{g}$; this is the case, for example, for $\mathfrak{o}_n$, for $\mathfrak{sp}_{2k}$ if $n = 2k$, and for $\mathfrak{sl}_n$ itself. Show that $\mathfrak{g}$ is semisimple.