Problems for Representations of Linear Algebraic Groups

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November 1st, 2016

As always, all vector spaces are finite dimensional and over $\mathbb{C}$. If $\mathfrak{g}$ is a Lie algebra and V and W are representations of $\mathfrak{g}$, then $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ is the complex vector space of $\mathfrak{g}$-representation homomorphisms $V \to W$, i.e. the space of linear maps $f \colon V \to W$ such that $f(x(v)) = x(f(v))$ for all $v \in V$ and $x \in \mathfrak{g}$.

1. Let $\mathfrak{g}$ be a semisimple Lie-algebra, and let $\mathfrak{g} = \mathfrak{s}_1 \oplus ... \oplus \mathfrak{s}_n$ be its decomposition into simple factors. Let $J \subset \{1, ..., n\}$. Show that $\bigoplus_{j \in J} \mathfrak{s}_j$ is an ideal of $\mathfrak{g}$, and that every ideal is of this form.

2. For any integer $n \geq 0$, let V_n be the representation $\mathrm{Sym}^n(V)$ of $\mathfrak{sl}_2$, where V is the standard representation.

   (a) Remind yourself, or use induction to show, that $\sum_{i=0}^n i^2 = \frac{1}{2}n^2 + \frac{1}{2}n$ and $\sum_{i=0}^n i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

   (b) Let e_1, e_2 be the standard basis of V, and let $f_1, ..., f_n$ be the basis of V_n given by $f_a = e_1^a \cdot e_2^{n-a}$. Show that the action of the standard elements $X, H, Y \in \mathfrak{sl}_2$ on V_n is given by the following:

$$\begin{aligned} X(f_a) &= a \cdot f_{a+1}; \\ H(f_a) &= (2a - n) \cdot f_a; \\ Y(f_a) &= (n - a) \cdot f_{a-1}. \end{aligned}$$

   (c) Show that with regards to the basis X, H, Y the form B_{V_n} is given by the matrix

$$\frac{n^3 + 3n^2 + 2n}{6} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

3. Let $\mathfrak{g}$ be a Lie algebra.

   (a) Suppose that $\mathfrak{g}$ is semisimple. Let V be a faithful representation of $\mathfrak{g}$. Show that B_V is nondegenerate.

(b) Give an example of a solvable $\mathfrak{g}$ and a faithful representation V of $\mathfrak{g}$ such that B_V is nondegenerate. Why does this not contradict Cartan's criterion?

4. Let $\mathfrak{g}$ be a Lie-algebra, and let V be an irreducible representation of $\mathfrak{g}$. Let W be a semisimple representation of $\mathfrak{g}$ such that all simple subrepresentations of W are isomorphic to V. Show that the map

$$\begin{array}{rcl} V \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(V, W) & \to & W \\ (v, f) & \mapsto & f(v) \end{array}$$

is an isomorphism of vector spaces.

5. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras, and let $\rho_1 \colon \mathfrak{g}_1 \to \mathfrak{gl}(V_1)$ and $\rho_2 \colon \mathfrak{g}_2 \to \mathfrak{gl}(V_2)$ be two representations. Consider the map

$$\begin{array}{rcl} \rho_1 \boxtimes \rho_2 \colon \mathfrak{g}_1 \times \mathfrak{g}_2 & \to & \mathfrak{gl}(V_1 \otimes V_2) \\ (x_1, x_2) & \mapsto & \rho_1(x_1) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \rho_2(x_2); \end{array}$$

This is called the *outer tensor product* of ρ_1 and ρ_2.

(a) Show that $\rho_1 \boxtimes \rho_2$ is a representation of $\mathfrak{g}_1 \times \mathfrak{g}_2$.

(b) Now suppose that V_1 and V_2 are irreducible representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Show that the map

$$\begin{array}{rcl} V_2 & \to & \operatorname{Hom}_{\mathfrak{g}_1}(V_1, V_1 \otimes V_2) \\ w & \mapsto & (v \mapsto v \otimes w) \end{array}$$

is an isomorphism.

(c) Show that the induced isomorphism $V_1 \otimes V_2 \xrightarrow{\sim} V_1 \otimes \operatorname{Hom}_{\mathfrak{g}_1}(V_1, V_1 \otimes V_2)$ is the inverse to the the isomorphism $V_1 \otimes \operatorname{Hom}_{\mathfrak{g}_1}(V_1, V_1 \otimes V_2) \xrightarrow{\sim} V_1 \otimes V_2$ of problem 4.

(d) Show that the map

$$\begin{array}{rcl} \{\text{subspaces of } V_2\} & \to & \{\mathfrak{g}_1\text{-invariant subspaces of } V_1 \otimes V_2\} \\ Y & \mapsto & V_1 \otimes Y \end{array}$$

is a bijection, with the inverse given by sending a $\mathfrak{g}_1$-invariant subspace $X \subset V_1 \otimes V_2$ to the space $\operatorname{Hom}_{\mathfrak{g}_1}(V_1, X)$, regarded as a subspace of $\operatorname{Hom}_{\mathfrak{g}_1}(V_1, V_1 \otimes V_2) \cong V_2$. Hence every $\mathfrak{g}_1$-invariant subspace of $V_1 \otimes V_2$ is of the form $V_1 \otimes Y$ for some $Y \subset V_2$.

(e) Deduce that $\rho_1 \boxtimes \rho_2$ is irreducible as representation of $\mathfrak{g}_1 \times \mathfrak{g}_2$.