Problems for Representations of Linear Algebraic Groups

Milan Lopuhaä

November 8^{th}, 2016

1. Let V and W be two representations of $\mathfrak{sl}_3$, and let $f\colon V \to W$ be a homomorphism of $\mathfrak{sl}_3$-representations. Show that f is a morphism of $\mathfrak{h}^\vee$-graded vector spaces, i.e. for every $\alpha \in \mathfrak{h}^\vee$ the map f sends V_α to W_α.

2. Let V be a representation of $\mathfrak{sl}_2$ or $\mathfrak{sl}_3$. For a weight $\alpha \in \Lambda_{\mathrm{W}}$, let d_α be the dimension of V_α.

   (a) Show that for all $\alpha \in \Lambda_{\mathrm{W}}$ one has
   $$\dim \mathrm{Sym}^2(V)_\alpha = \binom{d_{\frac{1}{2}\alpha}+1}{2} + \sum_{\substack{\beta+\gamma=\alpha \\ \beta\neq\gamma}} d_\beta d_\gamma.$$

   (b) Show that for all $\alpha \in \Lambda_{\mathrm{W}}$ one has
   $$\dim \mathrm{Sym}^3(V)_\alpha = \binom{d_{\frac{1}{3}\alpha}+2}{3} + \sum_{\substack{2\beta+\gamma=\alpha \\ \beta\neq\gamma}} \binom{d_\beta+1}{2} d_\gamma + \sum_{\substack{\beta+\gamma+\delta=\alpha \\ \beta,\gamma,\delta \text{ different}}} d_\beta d_\gamma d_\delta.$$

   (c) Give a formula for $\dim \bigwedge^2(V)_\alpha$.

3. Let V be the standard representation of $\mathfrak{sl}_2$.

   (a) Determine the weights of the representation $W = \mathrm{Sym}^2(\mathrm{Sym}^2(V))$, and decompose W into irreducible representations.

   (b) We can also determine this decomposition in a more algebraic manner. Consider the map
   $$\begin{aligned} f\colon \mathrm{Sym}^2(\mathrm{Sym}^2(V)) &\to \mathrm{Sym}^4(V) \\ (v_1\cdot v_2)\cdot(v_3\cdot v_4) &\mapsto v_1\cdot v_2\cdot v_3\cdot v_4. \end{aligned}$$

   Show that f is a homomorphism of $\mathfrak{sl}_2$-representations, that f is surjective, and that its kernel has dimension 1.

(c) Deduce from the above that $W \cong \mathrm{Sym}^4(V) \oplus \mathbb{C}$ as representations of $\mathfrak{sl}_2$ (where the action on $\mathbb{C}$ is trivial).

4. Let V be the standard representation of $\mathfrak{sl}_3$. Recall the weight diagram of $V^\vee \otimes V = \mathrm{End}(V)$ from the lecture.

   (a) Show that in an irreducible representation of heighest weight $L_1 - L_3$, the weight space of weight 0 is at most two-dimensional; hence $V^\vee \otimes V$ is not irreducible.

   (b) Show that the adjoint representation is irreducible. *Hint*: show that $[E_{3,1}, E_{1,3}]$ and $[E_{3,2}, [E_{2,1}, E_{1,3}]]$ are linearly independent in $\mathfrak{h}$.

   (c) Deduce that $\mathrm{End}(V) \cong \mathbb{C} \oplus \mathrm{ad}$, and that this is a decomposition into irreducible representations.