Problems for Representations of Linear Algebraic Groups

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1. Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{h}$ be a toral subalgebra, and let $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ be the decomposition into character spaces. If $\mathfrak{h} = \mathfrak{g}_0$, show that $\mathfrak{h}$ is maximal. (As mentioned in the lecture, the converse is also true.)

2. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sp}_4$, which we know to be semisimple; see Exercise 6 of October 18th. Let $\mathfrak{h}$ be the diagonal subalgebra, i.e., the set of all elements of the form
$$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}$$
(see Exercise 3b of October 4th).
   (a) Decompose $\mathfrak{g}$ into $\mathfrak{h}$-eigenspaces. Which are the $\alpha \in \mathfrak{h}^*$ that occur? For each such α, give a basis for $\mathfrak{g}_\alpha$.
   (b) Show that $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$.
   (c) Determine the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ and draw the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ on paper such that the inner product of the root system corresponds to the standard inner product of Euclidean space.

3. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a maximal toral subalgebra, $R \subset \mathfrak{h}^*$ the set of roots. For $\alpha \in R$ we have defined an element $H_\alpha \in \mathfrak{h}$. Prove that these elements H_α span $\mathfrak{h}$ as a complex vector space. [*Hint:* Suppose there exists an element $Z \in \mathfrak{h}$ with $B(H_\alpha, Z) = 0$ for all $\alpha \in R$. Show that then $Z \in Z(\mathfrak{g}) = 0$.]

4. Let $\mathfrak{g} = \mathfrak{sl}_n$ with $n \geq 2$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra of diagonal matrices $\mathrm{diag}(a_1, \ldots, a_n)$ with $\sum a_i = 0$. Defining $L_i \in \mathfrak{h}^*$ by
$$L_i\big(\mathrm{diag}(a_1, \ldots, a_n)\big) = a_i\,,$$
this gives
$$\mathfrak{h}^* = \big(\mathbb{C} \cdot L_1 + \cdots + \mathbb{C} \cdot L_n\big)/\mathbb{C} \cdot (L_1 + \cdots + L_n)\,.$$

The set of roots $R \subset \mathfrak{h}^*$ is given by $R = \{L_i - L_j \mid i \neq j\}$, so that the $\mathbb{R}$-linear span of R is the subspace $E \subset \mathfrak{h}^*$ of all $\mathbb{R}$-linear combinations of the vectors $L_i - L_j$. In other words,

$$E = \left\{ \sum c_i L_i \in \mathbb{R} \cdot L_1 + \cdots + \mathbb{R} \cdot L_n \;\middle|\; \sum c_i = 0 \right\}.$$

(Note that this maps *injectively* to $\mathfrak{h}^*$.)

(a) For $n = 2$ (trivial), 3 and 4, verify that the form $(\ ,\ )$ on E is proportional to the standard Euclidean inner product on $E \subset \mathbb{R}^n$. Can you even prove that this is true for arbitrary $n \geq 2$? [*Hint:* As basis for E take the roots $\alpha_i = L_i - L_{i+1}$ for $i = 1, \ldots, n-1$. If $\alpha = L_i - L_j$ is any root, we know that the corresponding element H_α is $H_{ij} = E_{ii} - E_{jj}$. We also know that the element called t_α in the lecture is a multiple of H_α, say $t_\alpha = c \cdot H_\alpha$. Show that the constant c is independent of α. What remains is to calculate the matrix of the Killing form with respect to the basis $H_{\alpha_1}, \ldots, H_{\alpha_{n-1}}$ of $\mathfrak{h}$.]

Let the symmetric group S_n act on E by permutation of the coordinates: $\sigma \in S_n$ sends the class of L_i to the class of $L_{\sigma(i)}$. (Note that this indeed gives a well-defined action on E.)

(b) For $\alpha = L_i - L_j$, show that the reflection $s_\alpha \colon E \to E$ is given by the action of the transposition $(i\ j) \in S_n$. (You may pretend that you have completed (a) for arbitrary n.)

(c) Conclude that the Weyl group W of the root system R is the group S_n.