Problems for Representations of Linear Algebraic Groups

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1. If you have not done so already, do problems 2 and 4 of last week.

2. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a maximal toral subalgebra, V and W two irreducible representations of $\mathfrak{g}$. Let α and β be the heighest weights of V and W, respectively. Let $v \in V$ and $w \in W$ be nonzero elements; let $U \subset V \oplus W$ be the $\mathfrak{g}$-subrepresentation generated by the element (v, w). (In other words, U is the smallest subrepresentation of $V \oplus W$ that contains (v, w).).

   (a) If $\alpha = \beta$ and $v \in V_\alpha$ and $w \in W_\beta = W_\alpha$, show that U is again irreducible with highest weight α. By considering the projections $V \leftarrow U \rightarrow W$, conclude that $V \cong W$ as representations of $\mathfrak{g}$.

   (b) If $\alpha = \beta$ and $v \in V_\gamma$ and $w \in W_\delta$ with $\gamma \neq \delta$, show that $U = V \oplus W$.

   (c) If $\alpha \neq \beta$, show that $U = V \oplus W$ for any choice of (nonzero) $v \in V$ and $w \in W$.

3. Let V be the standard representation of $\mathfrak{g} = \mathfrak{sp}_4$, and let $\mathfrak{h}$ be as in problem 2 of last week.

   (a) Give a decomposition of $\mathfrak{g}$ into root spaces (with respect to $\mathfrak{h}$).

   (b) Give the weights of the following representations of $\mathfrak{g}$ (with respect to $\mathfrak{h}$):

   $$V, \quad V^\vee, \quad \mathrm{Sym}^2(V), \quad \mathrm{ad}\,.$$

   Deduce that $\mathrm{Sym}^2(V) \cong \mathrm{ad}$ as representations of $\mathfrak{g}$.

   (c) Let $\Phi \colon V \xrightarrow{\sim} V^\vee$ be the isomorphism induced by the standard symplectic form φ on V. Show that the image of the composite map of $\mathfrak{g}$-representations

   $$\mathfrak{g} \hookrightarrow \mathrm{End}(V) \xrightarrow{\sim} V^\vee \otimes V \xrightarrow{\Phi^{-1} \otimes \mathrm{id}} V \otimes V$$

   (see problem 5 of September 13^{th}) is equal to $\mathrm{Inv}^2(V)$ (see problem 2 of September 5^{th}). This gives a more constructive proof of the fact that $\mathrm{Sym}^2(V) \cong \mathrm{ad}$.