Representations of Algebraic Groups—instructions for the oral exam.

If your average grade for the three homework assignments is at least a 6, you can finish the course by doing an oral exam. I reserve 45 minutes per candidate. On the course website you will find a table with available time slots on 26 and 27 January. Let me know by email which day and time have your preference, and give me one or two alternatives. The time slots are assigned on a first come, first served basis. If you're unable to take the exam on one of these days, let me know.

The lectures form the basis for what you should know, but I expect from you that you have studied some of the finer details in the literature; see for instance the references [FH] and [H] below. Here are some further details:

General:

— I expect that you will have no trouble reproducing the exact definitions of the most important notions that we have encountered.

— You should know the statements of the main results that we have seen. You need not be able to reproduce all proofs, but you will get a higher grade if you are able to outline the arguments.

— You should be able to describe the weight diagrams of some not-too-complicated representations of Lie algebras such as $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_n$, especially for small values of n. You should also be able to draw conclusions from weight diagrams. For instance, you should be able to explain why a certain representation is reducible, or why two given representations are isomorphic, etc.

— I may ask you about the details of one of the exercises of the hand-in assignments.

Proofs of some results:

As part of the exam, you may be asked to give the proof of one of the following basic results. In each case, the point is not so much that you should be able to recite entire proofs; but you should be able to tell me how some proof works, taking some details for granted, if necessary. Here is the list:

— You should understand the classification of the irreducible representations of $\mathfrak{sl}_2$, where you may assume that the operator called H acts in a semisimple (=diagonalisable) way. References: [FH], § 11.1; [H], Chapter 7; my notes of October 11.

— The theorem that says that, given a semisimple Lie algebra $\mathfrak{g}$ and a maximal toral Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, each root α gives rise to a Lie subalgebra $\mathfrak{s}_\alpha \subset \mathfrak{g}$ that is isomorphic with $\mathfrak{sl}_2$. References for this: [H], Section 8.3, or my notes of November 22.

— Part of the proof of Weyl's theorem which says that any representation of a semisimple Lie algebra $\mathfrak{g}$ is semisimple. Step one of the proof is the following special case: if V is a representation of $\mathfrak{g}$ and $W \subset V$ is a subspace of codimension 1 such that $X(v) \in W$ for all $X \in \mathfrak{g}$ and $v \in V$, then there exists a 1-dimensional $\mathfrak{g}$-submodule $L \subset V$ such that $V = L \oplus W$. The proof of this part you don't need to know; but you should know how the full theorem is deduced from this. References: [FH], § C.2; [H], Section 6.3; or my notes of November 1.

— The proof of the fact that every (algebraic) representation $T \to \mathrm{GL}(V)$ of a torus T is the direct sum of character spaces; so $V = \oplus_{\chi \in X^*(T)} V_\chi$. A minimal version is that you can prove this for $T = \mathbb{G}_{\mathrm{m}}$; if you can do the general case you'll get more points. References: my notes of September 27.

— The proof of the fact that if α and β are roots with $\beta \neq \pm\alpha$, then the roots of the form $\beta + i\alpha$ for $i \in \mathbb{Z}$, form a sequence $\beta - p\alpha, \beta + (1-p)\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ for some p, $q \geqslant 0$; and then $\beta(H_\alpha) = p - q$. References: [H], Section 8.4; my notes of November 22.

If you have questions about what to expect, let me know.

References

[FH] W. Fulton and J. Harris, *Representation theory.*

[H] J. Humphreys, *Introduction to Lie algebras and representations theory*.