CHAPTER 1

Étale cohomology

This chapter summarizes the theory of the étale topology on schemes, culminating in the results on ℓ-adic cohomology that are needed in the construction of Galois representations and in the proof of the Ramanujan–Petersson conjecture. In §1.1 we discuss the basic properties of the étale topology on a scheme, including the concept of a constructible sheaf of sets. The étale fundamental group and cohomological functors are introduced in §1.2, and we use Čech methods to compute some H^1's in terms of π_1's, as in topology. These calculations provide the starting point for the proof of the étale analogue of the topological proper base change theorem. This theorem is discussed in §1.3, where we also explain the étale analogue of homotopy-invariance for the cohomology of local systems and we introduce the vanishing-cycles spectral sequence, Poincaré duality, the Künneth formula, and the comparison isomorphism with topological cohomology over $\mathbf{C}$ (for torsion coefficients).

The adic formalism is developed in §1.4, and it is used to define étale cohomology with ℓ-adic coefficients; we discuss the Künneth isomorphism and Poincaré duality with $\mathbf{Q}_\ell$-coefficients, and extend the comparison isomorphism with topological cohomology to the ℓ-adic case. We conclude in §1.5 by discussing étale cohomology over finite fields, L-functions of ℓ-adic sheaves, and Deligne's purity theorems for the cohomology of ℓ-adic sheaves.

Our aim is to provide an overview of the main constructions and some useful techniques of proof, not to give a complete account of the theory. We generally refer to the literature (especially [**6**], [**9**], and [**15**]) for additional technical details.

1.1. Étale sheaves

We begin with a summary of the definitions of étale and smooth morphisms, and then we define the étale topology on a scheme. There are two categories of sheaves that will be of interest to us: the category of sheaves of sets and the category of sheaves of abelian groups. We initially put more emphasis on the sheaves of sets, as this is the right framework for developing the concept of constructibility.

After setting forth these basics, we analyze the étale topology and sheaf theory on $\operatorname{Spec} k$ for a field k, and we prove that these concepts are a reformulation of the theory of discrete Galois-sets. This leads to an equivalence between étale cohomology of $\operatorname{Spec} k$ and Galois cohomology of k. This special case underlies the relevance of étale cohomology in the construction of Galois representations, and that is why we work it out in detail.

The remainder of this section focuses on basic operations on sheaves of sets and sheaves of abelian groups, such as the three operations of pushforward, pullback, and extension-by-$\emptyset$, as well as additional functors such as limits and stalks. We also explain the rudiements of the theory of constructible sheaves. Aside from the restriction to the noetherian case when we define constructible étale sheaves, we avoid noetherian hypotheses.

1.1.1. Smooth and étale scheme maps. There are several equivalent ways to formulate the definitions of étaleness and smoothness for scheme morphisms. The reader is referred to [**3**, Ch. 1] and [**17**, §17.3–§17.8] for treatments giving much more detail than we do in our summary of the basics. We will give definitions that avoid noetherian conditions, and this requires a replacement for the concept of a locally finite-type morphism:

DEFINITION 1.1.1.1. A map of schemes $f : X \to S$ is *locally of finite presentation* if, for any open affine $\operatorname{Spec} A = U \subseteq S$ and any open affine $\operatorname{Spec} B \subseteq f^{-1}(U)$, there is an A-algebra isomorphism $B \simeq A[T_1, \ldots, T_n]/I$ for a finitely generated ideal I.

This property may be checked with one open affine covering $\{U_i\}$ of S and one open affine covering of each $f^{-1}(U_i)$. Consequently, it is preserved under base change and composition, and it is local on both X and S. See [**17**, §1.4] for further details. There is also an elegant functorial criterion for a map to be locally of finite presentation [**17**, 8.14.2], whereas no such criterion is know for being locally of finite type, and so the entirety of condition (4) in each of the following two definitions may be described in the language of functors.

DEFINITION 1.1.1.2. A map of schemes $f : S' \to S$ is *étale* if it satisfies one of the following equivalent properties:

(1) For every $s' \in S'$ there is an open neighborhood $U' \subseteq S'$ around s' and an open affine neighborhood $\operatorname{Spec} R \simeq U \subseteq S$ around $f(s')$ with $f(U') \subseteq U$ such that U' is U-isomorphic to an open subscheme of $\operatorname{Spec}(R[T]/g)_{g'}$ for some monic polynomial $g \in R[T]$ (with $g' = \mathrm{d}g/\mathrm{d}T$).
(2) The map f is locally of finite presentation and flat, and each fiber $f^{-1}(s)$ is $\coprod_{i \in I_s} \operatorname{Spec} k_{i,s}$ for finite separable extensions $k_{i,s}$ of $k(s)$.
(3) The map f is locally of finite presentation and flat, and $\Omega^1_{S'/S} = 0$.
(4) The map f is locally of finite presentation and satisfies the functorial criterion for being *formally étale*: for any closed immersion $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ over S with $I = \ker(A \twoheadrightarrow A_0)$ satisfying $I^2 = 0$, the natural map $S'(A) \to S'(A_0)$ is bijective (*i.e.*, solutions to the equations defining S' over S can be uniquely lifted through nilpotent thickenings).

Property (1) is the *structure theorem for étale morphisms*, and this property visibly implies that f is flat. If S is locally noetherian, then in (4) it suffices to use only artin local $\operatorname{Spec} A$ over $s \in S$ with residue field equal to a chosen algebraic closure of $k(s)$. Condition (4) is the one that can be checked in abstract situations with moduli problems, and deducing the structure theorem (and hence flatness) from (4) is the hardest part of the proof that these conditions are equivalent. Étale maps are open (as are locally finitely presented flat maps in general), and any map between étale S-schemes is étale.

EXAMPLE 1.1.1.3. The basic example of an étale scheme over $S = \operatorname{Spec} R$ is the situation described by the equations in the inverse function theorem in analytic geometry: we take S' to be an open subscheme in $\operatorname{Spec}(R[T_1, \ldots, T_n]/(f_1, \ldots, f_n))$ such that the Jacobian matrix $J = (\partial f_i/\partial T_j)$ has determinant that is a unit on S'. The structure theorem is the special case $n = 1$.

It is not obvious by explicit computation that this example satisfies any of the conditions (1), (2), or (3), but the invertibility of J on S' allows us to verify condition (4), as follows. Let $M = A^n$ and let $\vec{f} : M \to M$ be the polynomial map defined by the f_i's. Let $\vec{f_0} : M_0 \to M_0$ be the reduction modulo I. We suppose $v_0 \in M_0$ satisfies $\vec{f_0}(v_0) = 0$ and $\det J(v_0) \in A_0^\times$, and we seek to uniquely lift v_0 to $v \in M$ satisfying $\vec{f}(v) = 0$. Pick $v \in M$ lifting v_0, so $\vec{f}(v) \in IM$ and we seek unique $\varepsilon \in IM$ such that $\vec{f}(v + \varepsilon) = 0$. Since $I^2 = 0$ we have

$$\vec{f}(v + \varepsilon) = \vec{f}(v) + (J(v))(\varepsilon),$$

and $\det(J(v)) \in A$ has reduction $\det(J(v_0)) \in A_0^\times$, so $J(v)$ is invertible. Thus, we may indeed uniquely solve $\varepsilon = -J(v)^{-1}(\vec{f}(v))$.

Definition 1.1.1.4. A map of schemes $f : X \to S$ is *smooth* if it satisfies any of the following equivalent conditions:

(1) For all $x \in X$ there are opens $V \subseteq X$ around x and $U \subseteq S$ around $f(x)$ with $f(V) \subseteq U$ such that V admits an étale U-map to some $\mathbf{A}^n_U$.
(2) The map f is locally of finite presentation and flat, and all fibers $f^{-1}(s)$ are regular and remain so after extension of scalars to some perfect extension of $k(s)$.
(3) The map f is locally of finite presentation and flat, and $\Omega^1_{X/S}$ is locally free with rank near $x \in X$ equal to $\dim_x X_{f(x)}$ (the maximal dimension of an irreducible component of $X_{f(x)}$ through x) for each $x \in X$.
(4) The map f is locally of finite presentation and satisfies the functorial criterion for being *formally smooth*: for any closed immersion $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ over S with $I = \ker(A \twoheadrightarrow A_0)$ satisfying $I^2 = 0$, the natural map $X(A) \to X(A_0)$ is surjective (*i.e.*, solutions to the equations defining X over S can be lifted through nilpotent thickenings).

Property (1) is the *structure theorem for smooth morphisms*. If S is locally noetherian, then in (4) it suffices to use artin local $\operatorname{Spec} A$ with residue field equal to a chosen algebraic closure of the residue field at the image point in S. As with étaleness, (4) is the easy condition to check in abstract situations. The difficult part in the proof of the equivalence is again that (4) implies (1).

Example 1.1.1.5. The basic example of a smooth scheme over $S = \operatorname{Spec} R$ is any open subscheme X in $\operatorname{Spec}(R[T_1, \dots, T_n]/(f_1, \dots, f_r))$ such that the $r \times n$ matrix $J = (\partial f_i / \partial T_j)$ has rank r at all points of X. Conditions (1), (2), and (3) are not easily checked by explicit computation, but condition (4) may be checked by the same method as in Example 1.1.1.3. In the present case, $J(v) = (\partial f_i / \partial T_j)(v)$ is pointwise of rank-r on $\operatorname{Spec} A$, so (by Nakayama's lemma) it is a surjective linear map $A^n \to A^r$.

The following elementary lemma partly justifies the preceding definitions.

Lemma 1.1.1.6. *A map $f : X \to S$ between algebraic $\mathbf{C}$-schemes is étale if and only if f^{an} is a local isomorphism of analytic spaces. Likewise, f is smooth if and only if f^{an} is smooth.*

A *local isomorphism* is a map that is an open immersion locally on the source.

Proof. To pass from f to f^{an} we use condition (1), and to pass from f^{an} to f we use condition (3) and the fact that analytification commutes with formation of complete local rings. Alternatively, with a bit more technique, both the algebraic and analytic cases may be checked by considering (4) with A a finite local $\mathbf{C}$-algebra (analytification does not affect the set of points with values in such rings). $\square$

This lemma shows that étale morphisms in algebraic geometry are a good analogue of the local isomorphisms in complex-analytic geometry. The concept of local analytic isomorphism is topological in the sense that if X is an analytic space and $U \to X$ is a map of topological spaces that is a local homeomorphism, then there exists a unique structure of analytic space on U such that $U \to X$ is a local analytic isomorphism. Hence, the category of analytic X-spaces $U \to X$ with étale structure map is equivalent to the category of topological spaces endowed with a local homeomorphism to X; this equivalence identifies open immersion with open embeddings and analytic fiber products with topological fiber products.

1.1.2. Topological motivation for sites. Let us say that a map of topological spaces is *étale* if it is a local homeomorphism. Étale maps in topology are more general than open embeddings, yet for the purposes of sheaf theory it is not necessary to restrict attention to open subsets and covers by open subsets. As a warm-up to the étale topology of schemes, let us briefly consider the following definition (that will arise naturally in the comparison isomorphisms between topological and étale cohomology).

Definition 1.1.2.1. Let X be a topological space. The *topological étale site* of X consists of the following data:

(1) the category $X_{\text{ét}}$ of étale X-spaces $U \to X$;
(2) the rule τ that assigns to each U in $X_{\text{ét}}$ a distinguished class τ_U of *étale coverings*: τ_U is the collections of maps $\{f_i : U_i \to U\}$ in $X_{\text{ét}}$ such that $\cup_{i \in I} f_i(U_i) = U$.

The étale coverings satisfy the properties that constitute the axioms for a *Grothendieck topology* on the category $X_{\text{ét}}$, but rather than digress into a discussion of general sites and Grothendieck topologies (a *site* is a category endowed with a Grothendieck topology), we shall merely illustrate the key features in the concrete example of the topological étale site. The interested reader is referred to [**6**, Arcata, Ch. I] and [**3**, §6.1–6.2, §8.1] (and the references therein) for basic generalities concerning Grothendieck topologies and the related theory of descent.

A *presheaf* of sets on $X_{\text{ét}}$ is, by definition, a contravariant functor $\mathscr{F} : X_{\text{ét}} \to$ **Set**, and a *morphism* between presheaves of sets on $X_{\text{ét}}$ is (by definition) a natural transformation.

The main point is that the role of overlaps in ordinary sheaf theory is replaced with fiber products: a presheaf $\mathscr{F}$ on $X_{\text{ét}}$ is a *sheaf* if, for all U in $X_{\text{ét}}$ and all coverings $\{U_i \to U\}$ in τ_U, the sheaf axiom holds: the diagram of sets

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,i'} \mathscr{F}(U_i \times_U U_{i'})$$

is *exact* in the sense that the first map is an injection whose image consists of those I-tuples $(c_i) \in \prod \mathscr{F}(U_i)$ such that $\mathscr{F}(\mathrm{pr}_1)(c_i) = \mathscr{F}(\mathrm{pr}_2)(c_{i'})$ in $\mathscr{F}(U_i \times_U U_{i'})$ for all $i, i' \in I$. The category of sheaves of sets on $X_{\text{ét}}$ is denoted $\text{Ét}(X)$, and is called the (étale) *topos* on $X_{\text{ét}}$; strictly speaking, we should say that $\mathscr{F}$ is a sheaf on $(X_{\text{ét}}, \tau)$ rather than on $X_{\text{ét}}$, but no confusion seems likely. The category $\text{Ét}(X)$ admits arbitrary products, has an initial object (the empty sheaf), and has a final object (the sheaf $\underline{X} = \mathrm{Hom}_{X_{\text{ét}}}(\cdot, X)$ that assigns the singleton $\{\emptyset\}$ to all objects in $X_{\text{ét}}$).

Observe that if we define the *ordinary topological site* by replacing étale maps with open embeddings, then the resulting category X_{top} is (equivalent to) the category of open subsets in X, its coverings are the usual coverings of one open by others, and the resulting topos $\mathrm{Top}(X)$ (*i.e.*, category of sheaves of sets on X_{top} with respect to the specified coverings) is the usual category of sheaves of sets on the topological space X. The final and initial objects in $\mathrm{Top}(X)$ are described as in the case of $\text{Ét}(X)$.

Here is an important point: the categories $X_{\text{ét}}$ and X_{top} are not equivalent, since objects in $X_{\text{ét}}$ can have non-trivial automorphisms, yet the associated categories of sheaves are the same:

Lemma 1.1.2.2. *The categories* $\text{Ét}(X)$ *and* $\mathrm{Top}(X)$ *are equivalent.*

Proof. We define functors in both directions and leave it to the reader to check via the sheaf axioms that these are naturally quasi-inverse to each other. For any $\mathscr{F}$ in $\text{Ét}(X)$, we define a sheaf $\iota_*\mathscr{F}$ on the usual topological space X by only evaluating on opens in X. If $\mathscr{F}$ is in $\mathrm{Top}(X)$, then we define $\iota^*\mathscr{F} \in \text{Ét}(X)$ as follows: its value on any $h : U \to X$ is $\Gamma(U, h^*\mathscr{F})$, where $h^*\mathscr{F}$ denotes the usual topological pullback. $\square$

The global-sections functor $\mathscr{F} \rightsquigarrow \mathscr{F}(X)$ on $\mathrm{Top}(X)$ is isomorphic to the functor $\mathrm{Hom}_{\mathrm{Top}(X)}(\underline{X}_0, \cdot)$ of morphisms from the final object $\underline{X}_0 = \mathrm{Hom}_{X_{\mathrm{top}}}(\cdot, X)$, and the subcategory of abelian sheaves on X_{top} is the subcategory of abelian groups in $\mathrm{Top}(X)$, where an *abelian group* in a category C admitting finite products and a final object e is an object G equipped with maps fitting into the diagrams that axiomatize a commutative group (the identity is a map $e \to G$); *i.e.*, the functor $\mathrm{Hom}_C(\cdot, G)$ is endowed with a structure of group-functor. We conclude that sheaf cohomology on the topological space X can be intrinsically described in terms of the category $\mathrm{Top}(X)$: it is the derived functor of the restriction of $\mathrm{Hom}_{\mathrm{Top}(X)}(\underline{X}_0, \cdot)$ to the subcategory of abelian groups in $\mathrm{Top}(X)$, where $\underline{X}_0$ is the final object in $\mathrm{Top}(X)$. Since $\text{Ét}(X)$ is equivalent to $\mathrm{Top}(X)$, we can therefore construct sheaf cohomology in terms of $\text{Ét}(X)$.

To be precise, the *global-sections functor* on $\text{Ét}(X)$ is the functor $\mathscr{F} \rightsquigarrow \mathscr{F}(X)$, and this is clearly the same as the functor of morphisms from the final object $\underline{X}$. The restriction of this functor to the category of abelian groups in $\text{Ét}(X)$ must have sheaf-cohomology as its right derived functor via the equivalence between $\text{Ét}(X)$ and $\mathrm{Top}(X)$. Even though $X_{\text{ét}}$ is not equivalent to X_{top}, both sites give rise to the same theories of abelian sheaf cohomology. Hence, when considering sheaf cohomology, the category of sheaves of sets is more important than the underlying space.

1.1.3. The étale topology and étale topos on a scheme.

Definition 1.1.3.1. Let S be a scheme. The *étale site* of S consists of

(1) the category $S_{\text{ét}}$ of étale S-schemes $U \to S$;
(2) for each U in $S_{\text{ét}}$, the class τ_U of *étale coverings*: collections $\{f_i : U_i \to U\}$ of (necessarily étale) maps in $S_{\text{ét}}$ such that the (necessarily open) subsets $f_i(U_i) \subseteq U$ are a set-theoretic cover of U.

Definition 1.1.3.2. A *presheaf* (of sets) $\mathscr{F}$ on $S_{\text{ét}}$, or an *étale presheaf* on S, is a contravariant functor from $S_{\text{ét}}$ to the category **Set** of sets.

If $f : S'' \to S'$ is a morphism in $S_{\text{ét}}$ and $c \in \mathscr{F}(S')$ is an element, then we usually write $c|_{S''}$ to denote $(\mathscr{F}(f))(c) \in \mathscr{F}(S'')$.

Definition 1.1.3.3. A *sheaf* (of sets) on the étale site of S is a presheaf $\mathscr{F}$ on $S_{\text{ét}}$ that satisfies the *sheaf axiom*: for S' in $S_{\text{ét}}$ and any étale covering $\{S'_i\}$ of S', the diagram

$$\mathscr{F}(S') \to \prod_i \mathscr{F}(S'_i) \rightrightarrows \prod_{(i,i')} \mathscr{F}(S'_{ii'})$$

(with $S'_{ii'} = S'_i \times_{S'} S'_{i'}$) is exact in the sense that the left map is an injection onto the set of those I-tuples $(c_i) \in \prod \mathscr{F}(S'_i)$ such that $c_i|_{S'_{ii'}} = c_{i'}|_{S'_{ii'}}$ in $\mathscr{F}(S'_{ii'})$ for all $i, i' \in I$.

The category of sheaves of sets on $S_{\text{ét}}$ is denoted $\text{Ét}(S)$, and it is called the *étale topos* of S. This category admits arbitrary products, and it has both a final object (with value $\{\emptyset\}$ on all $S' \in S_{\text{ét}}$) and an initial object (the functor $\operatorname{Hom}_{S_{\text{ét}}}(\cdot, S)$). The abelian sheaves on $S_{\text{ét}}$ are the abelian-group objects in the étale topos; these are sheaves with values in the category of abelian groups, and this subcategory of $\text{Ét}(S)$ is denoted $\text{Ab}(S)$. The final object in $\text{Ab}(S)$ is the same as the final object in $\text{Ét}(S)$, but the initial objects are not the same.

Example 1.1.3.4. Let $\mathscr{F}$ be an object in the étale topos $\text{Ét}(S)$. Consider the empty cover of the empty object. Since a product over an empty collection in **Set** is (by universality) the final object $\{\emptyset\}$ in **Set**, we conclude that $\mathscr{F}(\emptyset) = \{\emptyset\}$ and that $\mathscr{F}$ naturally converts disjoint unions into products. The reader who finds this reasoning too bizarre can take the condition $\mathscr{F}(\emptyset) = \{\emptyset\}$ as part of the definition of a sheaf of sets.

If $j : U \to S$ is an étale map, we define the functor $j^* : \text{Ét}(S) \to \text{Ét}(U)$ to send any $\mathscr{F}$ in $\text{Ét}(S)$ to the étale sheaf $j^*\mathscr{F} : U' \to \mathscr{F}(U')$ on U, where an étale U-scheme $U' \to U$ is viewed as an étale S-scheme via composition with j. It is readily checked that $j^*\mathscr{F}$ is a sheaf on $U_{\text{ét}}$, and we usually denote it $\mathscr{F}|_U$.

Example 1.1.3.5. Let $\mathscr{F}$ and $\mathscr{G}$ be objects in $\text{Ét}(S)$. As in ordinary sheaf theory, we can define a presheaf

$$\mathscr{H}om(\mathscr{F}, \mathscr{G}) : U \mapsto \operatorname{Hom}_{\text{Ét}(U)}(\mathscr{F}|_U, \mathscr{G}|_U)$$

on $S_{\text{ét}}$, and the sheaf axioms ensure that this is a sheaf. If $\mathscr{F}$ and $\mathscr{G}$ are abelian sheaves, then we can carry out a similar construction via

$$U \mapsto \operatorname{Hom}_{\text{Ab}(U)}(\mathscr{F}|_U, \mathscr{G}|_U);$$

this is also denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ when the context makes the intended meaning clear (*i.e.*, Hom's of abelian sheaves or of sheaves of sets).

Definition 1.1.3.6. Let $f : S' \to S$ be a map of schemes. The *pushforward* functor $f_* : \text{Ét}(S') \to \text{Ét}(S)$ is

$$(f_*\mathscr{F}')(U) = \mathscr{F}(S' \times_S U)$$

for étale S-schemes U; the presheaf $f_*\mathscr{F}$ is a sheaf.

When $f : S' \to S$ is étale, it is clear that there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathrm{\acute{E}t}(S')}(f^*\mathscr{F}, \mathscr{F}') \simeq \mathrm{Hom}_{\mathrm{\acute{E}t}(S)}(\mathscr{F}, f_*\mathscr{F}'),$$

with f^* defined as in Example 1.1.3.4. There are evident isomorphisms $(f_1 \circ f_2)_* \simeq f_{1*} \circ f_{2*}$ for a composite of scheme maps f_1 and f_2, and pushforward also makes sense for sheaves of abelian groups. Observe that the forgetful functors $\mathrm{\acute{E}t}(S) \to \mathrm{Ab}(S)$ for varying S are compatible with pullback and pushforward functors; from the viewpoint of sheaf theory, the pair of adjoint functors (f_*, f^*) is more important than the geometric map f.

Let us continue with more basic examples.

Example 1.1.3.7. If $S = \mathrm{Spec}\, k$ for a separably closed field k, then étale k-schemes are disjoint unions of copies of S. Thus, $S_{\mathrm{\acute{e}t}}$ is equivalent to the category of sets (the equivalence being defined by the connected-component functor π_0), and for any $\mathscr{F}$ in $\mathrm{\acute{E}t}(S)$ and any object $X = \coprod_{i \in I} S$ we have

$$\mathscr{F}(X) = \prod_{i \in I} \mathscr{F}(S) = \mathrm{Hom}_{\mathbf{Set}}(I, \mathscr{F}(S)) = \mathrm{Hom}_{\mathbf{Set}}(\pi_0(X), \mathscr{F}(S)).$$

It follows that the global-sections functor $\mathscr{F} \rightsquigarrow \mathscr{F}(S)$ is an equivalence of categories from the étale topos on $S = \mathrm{Spec}\, k$ to the category **Set** of sets.

An étale cover $\{S''_j \to S'\}_{j \in J}$ is a *refinement* of an étale cover $\{S'_i \to S'\}_{i \in I}$ if there exists a map $t : J \to I$ and S'-maps $f_j : S''_j \to S'_{t(j)}$ for all $j \in J$; if J is a finite set then this is a *finite refinement*. We call a choice of t and f_j's a *refinement map* between the two covers.

Example 1.1.3.8. An *fpqc* map of schemes is a map $f : T \to S$ that is faithfully flat and quasi-compact (*fidèlement plat et quasi-compact*); such maps are quotient maps for the Zariski topology [**17**, 2.3.12], and an *fpqc cover* of S is a collection of flat quasi-compact maps $f_i : S_i \to U_i$ to Zariski-opens $U_i \subseteq S$ such that any quasi-compact open in S is set-theoretically covered by finitely many $f_i(S_i)$'s. Since étale maps are open, every étale cover $\{S_i\}$ of a quasi-compact scheme S admits a finite refinement $\{S'_j\}$ whose constituents are quasi-compact (and even affine) schemes. It follows that any étale cover of an arbitrary scheme S admits a refinement that is also an fpqc cover.

The reason for interest in fpqc covers is Grothendieck's fundamental discovery that for any S-scheme X, the functor $\underline{X}(T) = \mathrm{Hom}_S(T, X)$ on the category of S-schemes satisfies the sheaf axioms with respect to fpqc covers of any S-scheme S' [**3**, 8.1/1]. That is, if $\{T_i\}$ is an fpqc covering of an S-scheme T, the diagram of sets

$$X(T) \to \prod_i X(T_i) \rightrightarrows \prod_{(i,i')} X(T_i \times_T T_{i'})$$

is exact. It follows from this fact and elementary considerations with the Zariski topology that for any S-scheme X, the functor $\underline{X} : S_{\mathrm{\acute{e}t}} \to \mathbf{Set}$ is a sheaf. Such sheaves play a crucial role in étale sheaf theory.

Example 1.1.3.9. It is not generally true (in the non-noetherian case) that an étale cover $\{S_i\}$ of a quasi-compact scheme S admits a finite refinement $\{S'_j\}$ that is fpqc and such that the overlaps $S'_j \times_S S'_{j'}$ are also quasi-compact. For a counterexample, let A be a ring such that the topological space underlying $\mathrm{Spec}\, A$ is not noetherian (*e.g.*, an infinite product of nonzero rings). As in any non-noetherian topological space, there must exist an open U in $\operatorname{Spec} A$ that is not quasi-compact. Let S be the gluing of $\operatorname{Spec} A$ to itself along U, and let S_1 and S_2 be the two copies of $\operatorname{Spec} A$ that form a Zariski-cover of S.

Suppose that the étale cover $\{S_1, S_2\}$ of S admits a finite refinement $\{S'_j\}$ such that all $S'_j \times_S S'_{j'}$ are quasi-compact. Let V_1 be the disjoint union of the S'_j's that factor through S'_1 and let V_2 be the disjoint union of the S'_j's that factor through S'_2 (some S'_j's may contribute to both V_1 and V_2). Since $V_1 \times_S V_2$ is a finite disjoint union of quasi-compact schemes, it is quasi-compact. The refinement property implies that V_i surjects onto S_i, and so $V_1 \times_S V_2$ surjects onto $S_1 \times_S S_2 = S_1 \cap S_2 = U$. This is inconsistent with the fact that U is not quasi-compact.

By Yoneda's lemma, the functor $S_{\text{ét}} \to \text{Ét}(S)$ defined by $X \rightsquigarrow \underline{X} = \operatorname{Hom}_S(\cdot, X)$ is fully faithful; sheaves arising as $\underline{X}$ for étale S-schemes X are called *representable*. The full-faithfulness implies that the natural map

$$\mathscr{F}(X) \to \operatorname{Hom}_{\text{Ét}(S)}(\underline{X}, \mathscr{F})$$

is bijective for all étale S-schemes X. In particular, the functor $X \rightsquigarrow \underline{X}$ on $S_{\text{ét}}$ carries coproducts (*i.e.*, disjoint unions) to coproducts. The full-faithfulness of $X \rightsquigarrow \underline{X}$ breaks down if we consider X not étale over S (*e.g.*, $S = \operatorname{Spec} \mathbf{C}$, $X = \mathbf{A}^1_{\mathbf{C}}$).

The site $S_{\text{ét}}$ has a final object, namely S, and its associated sheaf $\underline{S}$ evaluates to a singleton $\{\emptyset\}$ on all objects. Thus, we see that $\underline{S}$ is the final object in the étale topos on S. The functor $\operatorname{Hom}_{\text{Ét}(S)}(\underline{S}, \mathscr{F})$ is naturally isomorphic to the functor $\mathscr{F} \rightsquigarrow \mathscr{F}(S)$, and this functor

$$\mathscr{F} \rightsquigarrow \operatorname{Hom}_{\text{Ét}(S)}(\underline{S}, \mathscr{F}) = \mathscr{F}(S)$$

is the *global-sections functor*.

1.1.4. Étale sheaves and Galois modules. Let $S = \operatorname{Spec} k$ for a field k. For a choice of separable algebraic closure k_s of k, we shall explain the functorial equivalence between the étale topos on $\operatorname{Spec} k$ and the category of (left) *discrete* $\operatorname{Gal}(k_s/k)$*-sets*. This will imply that Galois cohomology for k_s/k computes sheaf cohomology on $\operatorname{Ab}(\operatorname{Spec} k)$.

Definition 1.1.4.1. Let G be a profinite group. A (left) *discrete G-set* is a set M equipped with a left action of G that is continuous for the discrete topology on M and for the profinite topology on G; *i.e.*, each $m \in M$ is invariant under an open subgroup of G. If M is an abelian group and the G-action respects this structure, then M is a *discrete G-module*.

When G is presented as a Galois group, the terminology *discrete Galois-set* (or *discrete Galois-module*) will be used.

Remark 1.1.4.2. The abelian category of discrete G-modules admits enough injectives, and so admits a good theory of derived functors for left-exact functors.

To construct injectives, let C_G and C_G^{disc} denote the categories of G-modules and discrete G-modules. The exact inclusion functor $C_G^{\text{disc}} \to C_G$ has a right adjoint given by *discretization*: for any M in C_G, M^{disc} is the G-submodule of elements in M fixed by an open subgroup of G. Since discretization has an exact left adjoint, it carries injectives to injectives; concretely, $\operatorname{Hom}_{C_G^{\text{disc}}}(N, I^{\text{disc}}) = \operatorname{Hom}_{C_G}(N, I)$ is exact in the discrete N for any injective G-module I. Thus, M in C_G^{disc} embeds into the injective object I^{disc}, where I is an injective object in C_G containing M.

Consider the family of functors

$$\widetilde{\mathrm{H}}^\bullet(G, M) = \varinjlim \mathrm{H}^\bullet(G/H, M^H),$$

where H runs over open normal subgroups in G; this is a δ-functor in M, but the terms in the limit are generally not δ-functors in M. For such H, the functor of H-invariants carries injectives in C_G^{disc} to injectives in $C_{G/H}$, and so $\widetilde{\mathrm{H}}^\bullet(G, \cdot)$ is erasable. Thus, this δ-functor is the derived functor of $M \rightsquigarrow \widetilde{\mathrm{H}}^0(G, M) = M^G$.

We are going to prove that for a field k equipped with a choice of separable closure k_s, the category of discrete $\mathrm{Gal}(k_s/k)$-sets is equivalent to the étale topos on $\mathrm{Spec}\, k$, and that this equivalence identifies the functor of $\mathrm{Gal}(k_s/k)$-invariants with the global-sections functor. Thus, once we have set up the general definition of étale sheaf cohomology, it will follow that Galois cohomology for k_s/k computes sheaf cohomology on $\mathrm{Ab}(\mathrm{Spec}\, k)$.

Let k a field, but do not yet make a choice of separable closure. Let $\mathscr{F}$ be an étale sheaf of sets on $\mathrm{Spec}\, k$. Since an étale k-scheme X is a disjoint union $\coprod X_i$, where $X_i = \mathrm{Spec}\, k_i$ for finite separable field extensions k_i/k, the sheaf axioms imply

$$\mathscr{F}(X) = \prod \mathscr{F}(X_i) = \prod \mathscr{F}(\mathrm{Spec}\, k_i).$$

This allows us to focus attention on the restriction of $\mathscr{F}$ to the full subcategory of objects of the form $\mathrm{Spec}\, k'$ for k' finite and separable over k. By considering how $\mathscr{F}(\mathrm{Spec}\, k')$ varies functorially in the extension k'/k, we will be led to a discrete Galois-set that encodes $\mathscr{F}$.

If $g : k' \to k''$ is a morphism between finite separable extensions of k, then we have a morphism $\mathrm{Spec}\, g : \mathrm{Spec}\, k'' \to \mathrm{Spec}\, k'$ in $(\mathrm{Spec}\, k)_{\text{ét}}$, and so there is an induced morphism of sets

$$\mathscr{F}(\mathrm{Spec}\, g) : \mathscr{F}(\mathrm{Spec}\, k') \to \mathscr{F}(\mathrm{Spec}\, k'')$$

with $\mathscr{F}(\mathrm{Spec}\, h) \circ \mathscr{F}(\mathrm{Spec}\, g) = \mathscr{F}(\mathrm{Spec}(h \circ g))$ for $h : k'' \to k'''$ another such map over k; the contravariance of $\mathscr{F}$ and Spec cancel out. We will usually write such functorial maps as $\mathscr{F}(g)$ instead of $\mathscr{F}(\mathrm{Spec}\, g)$, so this notation is covariant in g. Since $\mathrm{Spec}\, g$ is a covering map, it follows from the sheaf axioms that $\mathscr{F}(g)$ is injective.

Consider the special case when $\iota : k' \to k''$ over k is Galois. The group

$$\mathrm{Gal}(k''/k') = \mathrm{Aut}(\mathrm{Spec}\, k''/\mathrm{Spec}\, k')^{\mathrm{opp}}$$

has a natural left action on $\mathscr{F}(\mathrm{Spec}\, k'')$, and the injective map

$$\mathscr{F}(\iota) : \mathscr{F}(\mathrm{Spec}\, k') \hookrightarrow \mathscr{F}(\mathrm{Spec}\, k'')$$

is invariant for this action. To exploit this, consider the diagram

$$\mathrm{Spec}\, k'' \times_{\mathrm{Spec}\, k'} \mathrm{Spec}\, k'' \rightrightarrows \mathrm{Spec}\, k'' \to \mathrm{Spec}\, k'$$

with natural projections on the left. This diagram is identified with the diagram

$$\coprod_{g \in \mathrm{Gal}(k''/k')} \mathrm{Spec}\, k'' \rightrightarrows \mathrm{Spec}\, k'' \to \mathrm{Spec}\, k' \tag{1.1.4.1}$$

where the two maps on the left are the tuple of identity maps on $\mathrm{Spec}\, k''$ and the map that is $\mathrm{Spec}\, g$ on the gth coordinate for all g; explicitly, this identification of diagrams is obtained via the isomorphism

$$k'' \otimes_{k'} k'' \simeq \prod_{g \in \mathrm{Gal}(k''/k')} k''$$
$$x \otimes y \mapsto \big(xg(y)\big)$$

of k''-algebras, where $k'' \otimes_{k'} k''$ is a k''-algebra via the left tensor-factor and $\prod k''$ is a k''-algebra via the diagonal action.

Since we have identified (1.1.4.1) with a covering diagram for an étale cover, the sheaf axioms yield an exact sequence of sets

$$\prod_g \mathscr{F}(\operatorname{Spec} k'') \leftleftarrows \mathscr{F}(\operatorname{Spec} k'') \leftarrow \mathscr{F}(\operatorname{Spec} k') \tag{1.1.4.2}$$

where the top map on the left is the diagonal map $s \mapsto (s, \ldots, s)$ and the bottom map is the action-map $s \mapsto (gs)_{g \in G}$. Thus, exactness of (1.1.4.2) says that $\mathscr{F}(\operatorname{Spec} k') \to \mathscr{F}(\operatorname{Spec} k'')$ is injective and

$$\mathscr{F}(\operatorname{Spec} k') = \mathscr{F}(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')}.$$

Now choose a separable algebraic closure k_s of k, and let Σ be the set of finite Galois extensions of k inside k_s. For each $k' \in \Sigma$, we have a set $\mathscr{F}(\operatorname{Spec} k')$ with a left action of $\mathrm{Gal}(k'/k)$, and for $k'' \in \Sigma$ containing k' inside of k_s we have an injection $\mathscr{F}(\operatorname{Spec} k') \hookrightarrow \mathscr{F}(\operatorname{Spec} k'')$ identifying $\mathscr{F}(\operatorname{Spec} k')$ with $\mathscr{F}(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')}$. Moreover, the map $\mathscr{F}(\operatorname{Spec} k') \to \mathscr{F}(\operatorname{Spec} k'')$ is compatible with Galois actions via the natural surjection $\mathrm{Gal}(k''/k) \twoheadrightarrow \mathrm{Gal}(k'/k)$. Thus,

$$M_{\mathscr{F}} = \varinjlim_{k' \in \Sigma} \mathscr{F}(\operatorname{Spec} k')$$

has a natural structure of discrete left $\mathrm{Gal}(k_s/k)$-set. Furthermore, the natural map $\mathscr{F}(\operatorname{Spec} k') \to M_{\mathscr{F}}$ is an isomorphism onto $M_{\mathscr{F}}^{\mathrm{Gal}(k_s/k')}$ as $\mathrm{Gal}(k'/k)$-sets. This constructs the functor considered in the following theorem.

THEOREM 1.1.4.3. *The functor $\mathscr{F} \rightsquigarrow M_{\mathscr{F}}$ from $\mathrm{\acute{E}t}(\operatorname{Spec} k)$ to the category of discrete left $\mathrm{Gal}(k_s/k)$-sets is an equivalence of categories.*

REMARK 1.1.4.4. This theorem is analogous to the description of local systems via monodromy representations of fundamental groups (for reasonable topological spaces); see §**??**. The choice of k_s corresponds to the choice of a base point.

PROOF. Using the sheaf axioms and the fact that every finite separable extension k'/k is covered by a Galois extension of k, we see that if $\mathscr{F}$ and $\mathscr{G}$ are two étale sheaves on $\operatorname{Spec} k$, then to give a map $\mathscr{F} \to \mathscr{G}$ is the same as to specify a map $\mathscr{F}(\operatorname{Spec} k') \to \mathscr{G}(\operatorname{Spec} k')$ for all finite Galois extensions k' of k with the requirement that these maps be functorial in k'/k. In particular, these maps are required to be compatible with the action of $\mathrm{Gal}(k'/k)$.

We next claim that such functorial data for all finite Galois extensions of k is equivalent to specifying maps $\varphi_{k'} : \mathscr{F}(\operatorname{Spec} k') \to \mathscr{G}(\operatorname{Spec} k')$ of $\mathrm{Gal}(k'/k)$-sets for each $k' \in \Sigma$, functorially with respect to inclusions $k' \subseteq k''$ inside of k_s. To see this equivalence, first note that for any finite Galois extension K'/k we can choose a k-isomorphism $\sigma' : K' \simeq k'$ for a unique $k' \in \Sigma$, with σ' unique up to $\sigma' \rightsquigarrow g \circ \sigma'$ for $g \in \mathrm{Gal}(k'/k)$, and so the composite

$$\varphi_{K'} : \mathscr{F}(\operatorname{Spec} K') \xrightarrow[\simeq]{\mathscr{F}(\sigma')} \mathscr{F}(\operatorname{Spec} k') \xrightarrow{\varphi_{k'}} \mathscr{G}(\operatorname{Spec} k') \xrightarrow[\simeq]{\mathscr{G}(\sigma')^{-1}} \mathscr{G}(\operatorname{Spec} K')$$

is well-defined (*i.e.*, independent of σ') since

$$\mathscr{G}(\sigma')^{-1} \circ \mathscr{G}(g)^{-1} \circ \varphi_{k'} \circ \mathscr{F}(g) \circ \mathscr{F}(\sigma') = \mathscr{G}(\sigma')^{-1} \circ \varphi_{k'} \circ \mathscr{F}(\sigma')$$

by $\mathrm{Gal}(k'/k)$-equivariance of $\varphi_{k'}$. To check that $K' \rightsquigarrow \varphi_{K'}$ is natural on the category of finite Galois extensions K'/k, for an arbitrary k-map $j : K' \to K''$ between finite Galois extensions of k we can find $k', k'' \in \Sigma$ and an inclusion $\iota : k' \subseteq k''$ inside of k_s and k-isomorphisms $\sigma' : K' \simeq k'$, $\sigma'' : K'' \simeq k''$ carrying j over to the inclusion $k' \subseteq k''$. The outside edge of

$$\text{(1.1.4.3)} \qquad \begin{array}{ccccccc} \mathscr{F}(\operatorname{Spec} K') & \xrightarrow{\mathscr{F}(\sigma')} & \mathscr{F}(\operatorname{Spec} k') & \xrightarrow{\varphi_{k'}} & \mathscr{G}(\operatorname{Spec} k') & \xrightarrow{\mathscr{G}(\sigma')^{-1}} & \mathscr{G}(\operatorname{Spec} K') \\ {\scriptstyle \mathscr{F}(j)}\downarrow & & \downarrow{\scriptstyle \mathscr{F}(\iota)} & & \downarrow{\scriptstyle \mathscr{G}(\iota)} & & \downarrow{\scriptstyle \mathscr{G}(j)} \\ \mathscr{F}(\operatorname{Spec} K'') & \xrightarrow{\mathscr{F}(\sigma'')} & \mathscr{F}(\operatorname{Spec} k'') & \xrightarrow{\varphi_{k''}} & \mathscr{G}(\operatorname{Spec} k'') & \xrightarrow{\mathscr{G}(\sigma'')^{-1}} & \mathscr{G}(\operatorname{Spec} K'') \end{array}$$

therefore commutes, since the outer squares commute (by functoriality of $\mathscr{F}$ and $\mathscr{G}$) and the inner square commutes (by compatibility of $\varphi_{k'}$, $\varphi_{k''}$ with respect to the inclusion ι within k_s). Since the composites along the rows in (1.1.4.3) are $\varphi_{K'}$ and $\varphi_{K''}$ respectively, we get the desired naturality of $\varphi : \mathscr{F} \to \mathscr{G}$.

We conclude that to give a map $\mathscr{F} \to \mathscr{G}$ as étale sheaves is the same as to give a map $M_{\mathscr{F}} \to M_{\mathscr{G}}$ as discrete $\mathrm{Gal}(k_s/k)$-sets. That is, the functor $\mathscr{F} \rightsquigarrow M_{\mathscr{F}}$ is fully faithful.

Now let M be a discrete $\mathrm{Gal}(k_s/k)$-set. We will construct a sheaf $\mathscr{F}_M$ on $(\operatorname{Spec} k)_{\text{ét}}$ in a manner that is functorial in M, and this provides a functor in the other direction. For any abstract finite separable extension k' over k, define

$$\mathscr{F}_M(\operatorname{Spec} k') = \left\{ (m_i) \in \prod_{i : k' \hookrightarrow k_s} M \mid m_{g(i)} = g(m_i) \text{ for all } g \in \mathrm{Gal}(k_s/k) \right\}.$$

Here, i runs through the finitely many k-embeddings of k' into k_s. Note that m_i lies in $M^{\mathrm{Gal}(k_s/i(k'))}$. For any $k' \in \Sigma$, projection to the coordinate labeled by the inclusion $k' \hookrightarrow k_s$ arising from membership in Σ defines a bijection

$$\mathscr{F}_M(\operatorname{Spec} k') \simeq M^{\mathrm{Gal}(k_s/k')}.$$

For any k-embedding $j : k' \hookrightarrow k''$ of finite separable extensions of k, the map

$$\text{(1.1.4.4)} \qquad \mathscr{F}_M(j) : \mathscr{F}_M(\operatorname{Spec} k') \to \mathscr{F}_M(\operatorname{Spec} k'')$$

defined by $(\mathscr{F}_M(j)((m_{i'})))_{i''} = m_{i'' \circ j}$ is injective because every $i' : k' \hookrightarrow k_s$ over k does have the form $i'' \circ j$ for some $i' : k' \hookrightarrow k_s$ over k. When $k' = k''$ is a finite Galois extension of k, then the action of $\mathrm{Gal}(k'/k)$ on $\mathscr{F}_M(\operatorname{Spec} k')$ makes the isomorphism of sets $\mathscr{F}_M(\operatorname{Spec} k') \simeq M^{\mathrm{Gal}(k_s/k')}$ into an isomorphism of $\mathrm{Gal}(k'/k)$-sets. Defining

$$\mathscr{F}_M\left(\coprod \operatorname{Spec} k_i\right) = \prod \mathscr{F}_M(\operatorname{Spec} k_i),$$

we have defined a presheaf $\mathscr{F}_M$ on $(\operatorname{Spec} k)_{\text{ét}}$.

The presheaf $\mathscr{F}_M$ is *separated* (*i.e.*, $\mathscr{F}_M(X) \to \prod \mathscr{F}_M(X_i)$ is injective for every cover $\{X_i\}$ of an object X in $(\operatorname{Spec} k)_{\text{ét}}$) because the maps (1.1.4.4) are injective. To check the gluing axiom for the separated $\mathscr{F}_M$, it suffices to consider a cofinal system of covering situations. By connectedness considerations for the objects in our site, we are reduced to checking the gluing axiom in the case of a covering $\operatorname{Spec} k'' \to \operatorname{Spec} k'$ with k''/k' a finite Galois extension inside of k_s. We just need to show that the injection $\mathscr{F}_M(\operatorname{Spec} k') \hookrightarrow \mathscr{F}_M(\operatorname{Spec} k'')$ is an isomorphism onto the $\mathrm{Gal}(k''/k')$-invariants. This is immediate from the way that we defined $\mathscr{F}_M(\operatorname{Spec} k') \to \mathscr{F}_M(\operatorname{Spec} k'')$ and the fact that two k-embeddings $k'' \rightrightarrows k_s$ coincide on k' if and only if they are related by the action of some $g \in \mathrm{Gal}(k''/k')$.

We conclude that $\mathscr{F}_M$ is a sheaf on $(\operatorname{Spec} k)_{\text{ét}}$, and we have isomorphisms

$$\mathscr{F}_M(\operatorname{Spec} k') \simeq M^{\mathrm{Gal}(k_s/k')}$$

for all $k' \in \Sigma$; these isomorphisms are compatible with the action of $\mathrm{Gal}(k'/k)$ and with the inclusions $k' \hookrightarrow k_s$ as subfields of k_s. We therefore have a canonical injection $M_{\mathscr{F}_M} \hookrightarrow M$ compatible with with $\mathrm{Gal}(k_s/k)$-actions. This is an isomorphism because every element of M is fixed by some $\mathrm{Gal}(k_s/k')$ for some finite extension k' of k inside of k_s.

When $M = M_{\mathscr{F}}$, the above construction yields a $\mathrm{Gal}(k'/k)$-compatible bijection

$$\mathscr{F}_{M_{\mathscr{F}}}(\operatorname{Spec} k') \simeq M_{\mathscr{F}}^{\mathrm{Gal}(k_s/k')} = \mathscr{F}(\operatorname{Spec} k') \tag{1.1.4.5}$$

for all $k' \in \Sigma$. The sheaf axioms ensure that these bijections are compatible with inclusions inside k_s, so (1.1.4.5) uniquely extends to an isomorphism of sheaves $\mathscr{F}_{M_{\mathscr{F}}} \simeq \mathscr{F}$. Thus, our constructions are quasi-inverse to each other. □

The equivalence between $\text{Ét}(\operatorname{Spec} k)$ and the category of discrete $\mathrm{Gal}(k_s/k)$-sets must carry final objects to final objects: in $\text{Ét}(\operatorname{Spec} k)$, the final object is the constant sheaf with value $\{\emptyset\}$ on all objects in $(\operatorname{Spec} k)_{\text{ét}}$; in the category of discrete $\mathrm{Gal}(k_s/k)$-sets, the final object is $\{\emptyset\}$ with a trivial Galois-action. The functors co-represented by these objects are the global-sections functor and the functor of Galois-invariants. Restricting to the induced equivalence between the subcategories of abelian-group objects, we thereby get an equivalence between $\mathrm{Ab}(\operatorname{Spec} k)$ and the abelian category of discrete $\mathrm{Gal}(k_s/k)$-modules such that the left-exact global-sections functor goes over to the left-exact functor of Galois-invariants. Since profinite group-cohomology for $\mathrm{Gal}(k_s/k)$ is the derived functor of the Galois-invariants functor on the category of discrete Galois-modules (see Remark 1.1.4.2), we get:

Corollary 1.1.4.5. *Galois cohomology for k_s/k computes the derived functor of the global-sections functor on* $\mathrm{Ab}(\operatorname{Spec} k)$.

Corollary 1.1.4.6. *The functor $X \rightsquigarrow \underline{X}$ is an equivalence of categories from $(\operatorname{Spec} k)_{\text{ét}}$ to $\text{Ét}(\operatorname{Spec} k)$; that is, the category of étale k-schemes is equivalent to the étale topos on $\operatorname{Spec} k$. Moreover, if k_s/k is a separable closure then the discrete Galois-set associated to $\underline{X}$ is $X(k_s)$ equipped with its usual left Galois-action.*

Proof. If X is a finite étale k-scheme and k_s/k is a separable closure, then the sheaf $\underline{X}$ corresponds to a finite discrete $\mathrm{Gal}(k_s/k)$-set; this set is $X(k_s)$ with its usual left Galois action. More generally, since any étale k-scheme is a disjoint union of finite étale k-schemes and the functor $X \rightsquigarrow \underline{X}$ takes coproducts to coproducts, it follows that the discrete $\mathrm{Gal}(k_s/k)$-set associated to $\underline{X}$ for any étale k-scheme X is $X(k_s)$ with its usual left Galois action.

Any discrete $\mathrm{Gal}(k_s/k)$-set is a disjoint union of finite orbits, and an orbit with a chosen element is isomorphic to $\mathrm{Gal}(k_s/k)/\mathrm{Gal}(k'/k)$ for a unique finite extension k'/k contained in k_s. This is the discrete Galois set associated to $X = \operatorname{Spec} k'$, and so the compatibility of $X \rightsquigarrow \underline{X}$ with respect to coproducts implies that every étale sheaf on $\operatorname{Spec} k$ is representable. □

1.1.5. Sheafification and diagram-limits. For a general scheme S, we wish to construct operations in the étale topos $\acute{\mathrm{E}}\mathrm{t}(S)$ as if it were the category of sheaves of sets on an ordinary topological space. For example, we wish to have available operations such as the image of a morphism, the quotient by an equivalence relation, and limits (both direct and inverse). Most of these constructions require the use of sheafification, and so the first issue we need to address is sheafification.

DEFINITION 1.1.5.1. A *sheafification* of a presheaf $\mathscr{F} : S_{\acute{e}t} \to \mathbf{Set}$ is a sheaf $\mathscr{F}^+$ in $\acute{\mathrm{E}}\mathrm{t}(S)$ equipped with a natural transformation $\mathscr{F} \to \mathscr{F}^+$ with the universal property that any natural transformation $\mathscr{F} \to \mathscr{G}$ to a sheaf on $S_{\acute{e}t}$ factors through a unique map $\mathscr{F}^+ \to \mathscr{G}$.

The sheafification of a presheaf $\mathscr{F}$ on $S_{\acute{e}t}$ may be constructed as follows. For any U in $S_{\acute{e}t}$ and any étale cover $\mathfrak{U} = \{U_i\}$ of U, let $\mathrm{H}^0(\mathfrak{U}, \mathscr{F})$ be the set of I-tuples $(s_i) \in \prod \mathscr{F}(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i and j, where $U_{ij} = U_i \times_U U_j$. If $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, any choice of refinement map between these covers defines a map $\mathrm{H}^0(\mathfrak{U}, \mathscr{F}) \to \mathrm{H}^0(\mathfrak{U}', \mathscr{F})$ that is independent of the choice of refinement map, and so by taking $\mathfrak{U}$ to run over a cofinal set of étale covers of U we may form the direct limit $\mathscr{F}_0(U)$ of the $\mathrm{H}^0(\mathfrak{U}, \mathscr{F})$'s. It is clear that $U \mapsto \mathscr{F}_0(U)$ is a separated presheaf and that any map $\mathscr{F} \to \mathscr{G}$ to a sheaf uniquely factors through the evident map $\mathscr{F} \to \mathscr{F}_0$. Thus, to construct $\mathscr{F}^+$ we may suppose $\mathscr{F}$ is separated. In this case, we apply the same process again, and if $\mathscr{F}$ is separated then $\mathscr{F}_0$ is a sheaf.

For our purposes, what matters is not the explicit construction process, but the universal property and the following properties that emerge from the construction:

(1) for any $s \in \mathscr{F}^+(U)$, there exists an étale cover $\{U_i\}$ of U and an element $s_i \in \mathscr{F}(U_i)$ mapping to $s|_{U_i} \in \mathscr{F}^+(U_i)$;
(2) if $s, t \in \mathscr{F}(U)$ have the same image in $\mathscr{F}^+(U)$ then there exists an étale cover $\{U_i\}$ of U such that $s|_{U_i} = t|_{U_i}$ in $\mathscr{F}(U_i)$ for all i.

In particular, if a presheaf map $\mathscr{F} \to \mathscr{G}$ is a monomorphism in the sense that $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all $U \in S_{\acute{e}t}$, then $\mathscr{F}^+(U) \to \mathscr{G}^+(U)$ is injective for all U. More generally, an exact diagram of presheaves of sets

$$\mathscr{F} \rightrightarrows \mathscr{F}' \to \mathscr{F}''$$

sheafifies to an exact diagram of sheaves of sets.

In $\acute{\mathrm{E}}\mathrm{t}(S)$, we may use sheafification to construct the image of a map and the quotient by an equivalence relation: these are defined by sheafifying the evident set-theoretic presheaf construction, exactly as in ordinary sheaf theory. The expected universal properties carry over, and a similar technique works in $\mathrm{Ab}(S)$ to construct images and cokernels.

Here is a more subtle construction. Let $j : U \to S$ be an étale morphism; we seek to construct a functor $j_!^{\mathbf{Set}} : \acute{\mathrm{E}}\mathrm{t}(U) \to \acute{\mathrm{E}}\mathrm{t}(S)$ that is left-adjoint to the restriction functor $j^* : \acute{\mathrm{E}}\mathrm{t}(S) \to \acute{\mathrm{E}}\mathrm{t}(S')$; *i.e.*,

$$\mathrm{Hom}_{\acute{\mathrm{E}}\mathrm{t}(S)}(j_!^{\mathbf{Set}} \mathscr{F}', \mathscr{F}) \simeq \mathrm{Hom}_{\acute{\mathrm{E}}\mathrm{t}(U)}(\mathscr{F}', j^* \mathscr{F}),$$

and we also want such an adjoint $j_!^{\mathbf{Ab}}$ between the categories $\mathrm{Ab}(U)$ and $\mathrm{Ab}(S)$.

Let us first recall the analogues in ordinary sheaf theory for an open embedding $j : U \hookrightarrow S$. The functor $j_!^{\mathbf{Ab}}$ is the *extension-by-zero*, and the functor $j_!^{\mathbf{Set}}$ is the *extension-by-$\emptyset$*. These are constructed by sheafifying the presheaf that sends V to $\mathscr{F}(V)$ when $V \subseteq U$ and otherwise sends V to the initial object 0 (resp. $\emptyset$) in the target category (abelian groups or sets). A uniform description that covers both cases is to say that we sheafify the presheaf

$$V \rightsquigarrow \coprod_{f \in \mathrm{Hom}_S(V,U)} \mathscr{F}(V \xrightarrow{f} U),$$

since the coproduct is taken over the singleton when $V \subseteq U$ and is taken over the empty set (and so assigns the initial object to V) when $V \not\subseteq U$.

The analogous constructions in étale sheaf theory requires slightly more care than in the classical case; the extra complications are due to the fact that an étale S-scheme V may factor through $j : U \to S$ in more than one way, whereas such a factorization in ordinary topology for an open embedding j is unique if it exists.

Definition 1.1.5.2. With notation as above, $j_!^{\mathbf{Set}} : \text{Ét}(U) \to \text{Ét}(S)$ sends $\mathscr{F}$ to the sheafification of the presheaf

$$S' \rightsquigarrow \coprod_{f \in \mathrm{Hom}_S(S',U)} \mathscr{F}(S' \xrightarrow{f} U).$$

The functor $j_!^{\mathbf{Ab}} : \mathrm{Ab}(U) \to \mathrm{Ab}(S)$ is defined by the same categorical operation, where coproducts are taken in the category of abelian groups (*i.e.*, direct sums).

It is straightforward to check that the functors $j_!^{\mathbf{Set}}$ and $j_!^{\mathbf{Ab}}$ are left-adjoint to j^*, though note that they are not compatible with the forgetful functors $\mathrm{Ab}(U) \to \text{Ét}(U)$ and $\mathrm{Ab}(S) \to \text{Ét}(S)$ because coproducts and initial objects in the category of abelian groups are not the same as in the category of sets. When the context makes the intended meaning clear, we will simply write $j_!$ to denote either functor.

As a consequence of these concrete constructions, arguments as in ordinary sheaf theory prove several properties in both the étale topos $\text{Ét}(S)$ and its subcategory of abelian groups $\mathrm{Ab}(S)$:

- the monic maps are precisely the subsheaf inclusions (and so $j_!^{\mathbf{Set}}$ and $j_!^{\mathbf{Ab}}$ carry monics to monics for any étale j);
- the epic maps are precisely the étale-local surjections;
- the epic monomorphisms are precisely the isomorphisms.

In particular, $\mathrm{Ab}(S)$ is an abelian category.

For later purposes, it is convenient to have available various kinds of limits, of a type more general than limits over directed sets. Such general limits will be used to formulate important exactness properties for pullback-functors on categories of sheaves of sets. Here are the limits we wish to consider:

Definition 1.1.5.3. Let C be a category, and let I be a diagram in **Set** with objects indexed by a set; consider I as a category. We define an *I-indexed diagram* D in C to be a covariant functor $D : I \to C$ (this is just a diagram in C with objects indexed by $\mathrm{ob}(I)$ and morphisms arising functorially from those in I). An *inverse limit* of D is a final object in the category of pairs $(L, \{\phi_i : L \to D(i)\}_{i \in \mathrm{ob}(I)})$ consisting of an object L in C and maps ϕ_i such that

$$\begin{tikzcd} L \arrow[r, "\phi_i"] \arrow[dr, "\phi_{i'}"'] & D(i) \arrow[d, "D(f)"] \\ & D(i') \end{tikzcd}$$

commutes for all maps $f : i \to i'$ in I.

A *direct limit* is an initial object among pairs $(L, \{\phi_i : D(i) \to L\}_{i \in \mathrm{ob}(I)})$ satisfying the commutativity of diagrams

$$\begin{tikzcd} D(i) \arrow[r, "\phi_i"] \arrow[d, "D(f)"'] & L \\ D(i') \arrow[ur, "\phi_{i'}"'] & \end{tikzcd}$$

for all maps $f : i \to i'$ in I.

If the set of objects and arrows in I is finite, the corresponding inverse limit (resp. direct limit) is a *finite inverse limit* (resp. *finite direct limit*).

Example 1.1.5.4. If the category I is a set with only identity arrows, then an inverse limit over an I-indexed diagram is a product indexed by $\mathrm{ob}(I)$. The corresponding direct-limit concept is that of coproduct indexed by $\mathrm{ob}(I)$.

Suppose I is the category on a partially ordered set (with a single arrow from i to i' when $i \le i'$), and require that I be *cofiltering* (resp. *filtering*) in the sense that finite subsets have a common lower bound (resp. upper bound). In this case, the corresponding concept of inverse limit (resp. direct limit) for I-indexed diagrams on C is called a *filtered inverse limit* (resp. *filtered direct limit*). These are the limits encountered in basic algebra.

Since any (small) diagram in **Set** is a rising union of finite diagrams, any inverse limit (resp. direct limit) can be expressed as a filtered inverse limit of finite inverse limits (resp. filtered direct limit of finite direct limits), provided finite direct and inverse limits exist in C. Consequently, considerations with limits generally reduce to two kinds: finite limits and (co)filtered limits. In general, by inducting on the size of a diagram, we see that a category admitting finite products and finite fiber products admits all finite inverse limits, and a category admitting finite coproducts and finite pushouts (the opposite of a finite fiber product) admits all finite direct limits. In an abelian category, a finite inverse limit can always be expressed as the kernel of a morphism, while a finite direct limit can always be expressed as the cokernel of a morphism.

Example 1.1.5.5. Let S be a scheme. The category $\text{Ét}(S)$ admits all direct limits and all inverse limits. Indeed, the constructions on the set-theoretic level in ordinary sheaf theory carry over *verbatim*, using sheafification for direct limits.

Definition 1.1.5.6. Let C be a category admitting finite products and finite fiber products. Let $\phi_1, \phi_2 : \mathscr{F}' \to \mathscr{F}''$ be maps in C. A diagram $\mathscr{F} \xrightarrow{\iota} \mathscr{F}' \rightrightarrows \mathscr{F}''$ resting on the ϕ_j's such that $\phi_1 \circ \iota = \phi_2 \circ \iota$ is an *equalizer diagram* (or an *equalizer kernel*) if the commutative square

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota \times \iota"] \arrow[d, "\phi"'] & \mathscr{F}' \times \mathscr{F}' \arrow[d, "\phi_1 \times \phi_2"] \\ \mathscr{F}'' \arrow[r, "\Delta"'] & \mathscr{F}'' \times \mathscr{F}'' \end{tikzcd}$$

is cartesian, where $\phi = \phi_1 \circ \iota = \phi_2 \circ \iota$.

Yoneda's lemma translates this definition as follows: for every object T of C, the induced diagram of sets

$$\mathrm{Hom}(T, \mathscr{F}) \to \mathrm{Hom}(T, \mathscr{F}') \rightrightarrows \mathrm{Hom}(T, \mathscr{F}'')$$

is exact. Equalizer diagrams in general categories play the role of left-exact sequences in abelian categories.

The purpose of the preceding general nonsense is to enable us to make the following definition.

Definition 1.1.5.7. Let $F : C \to C'$ be a covariant functor between two categories that admit finite direct limits and finite inverse limits. The functor F is *left exact* if it commutes with formation of finite inverse limits, and it is *right exact* if it commutes with formation of finite direct limits. It is *exact* if it is both left and right exact.

It is straightforward to check that a left exact functor commutes with formation of equalizer kernels, and if C and C' are abelian categories then a covariant additive functor $F : C \to C'$ is left (resp. right) exact in the sense of the preceding definition if and only if it is exact as usually defined in the theory of abelian categories.

Example 1.1.5.8. The functor $f_* : \acute{\mathrm{E}}\mathrm{t}(S') \to \acute{\mathrm{E}}\mathrm{t}(S)$ defined by a map $f : S' \to S$ commutes with the formation of arbitrary inverse limits, since presheaf products and presheaf fiber-products carry sheaves to sheaves. In particular, f_* is left exact. When f is étale, we defined a left adjoint $f^* : \acute{\mathrm{E}}\mathrm{t}(S) \to \acute{\mathrm{E}}\mathrm{t}(S')$. It follows from adjointness that f^* must commute with arbitrary direct limits, and so is right exact. The construction of f^* shows that f^* also commutes with the formation of finite products and finite fiber products. Thus, f^* is exact.

1.1.6. Pullback of étale sheaves. Let $f : S' \to S$ be a map of schemes. Our aim is to prove that f_* always has a left adjoint f^*, and that this adjoint is exact. As in our construction of $j_!$ for an étale map j, the construction of f^* is more delicate than in ordinary sheaf theory because there can be more than one map between objects in $S_{\acute{e}t}$. To construct f^* and to see that it yields an exact left adjoint, we proceed in two steps.

Step 1. Consider commutative diagrams of the form

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "f"'] & S \end{tikzcd} \tag{1.1.6.1}$$

with étale right side and fixed étale left side (and of course fixed bottom side). These diagrams form a category in an evident manner, and this category is co-filtered in the sense that if X' admits maps ϕ_1 and ϕ_2 to étale S-schemes X_1 and X_2, then the ϕ_j's factor through the S-map $\phi_1 \times \phi_2 : X' \to X_1 \times_S X_2$. If κ is a cardinal that bounds the number of open affines in X', then the map $X' \to X$ factors through an open subscheme of X that is κ-*small* in the sense that it can be covered (in the Zariski topology) by $\leq \kappa$ open affines.

Thus, we may form the filtered direct limit of sets

$$(f^{-1}\mathscr{F})(X' \to S') = \varinjlim \mathscr{F}(X \to S)$$

by restricting attention to κ-small X, though the choice of κ does not affect this construction. Note that two diagrams (1.1.6.1) with the same right, bottom, and left sides (but different top sides) may be compatible via a non-trivial S-automorphism of X, and so there may be a non-trivial self-map on $\mathscr{F}(X \to S)$ in the formation of the above direct limit.

Step 2. Observe that $f^{-1}\mathscr{F}$ is defined as a filtered direct limit, and so it is easy to check that the functor $\mathscr{F} \rightsquigarrow f^{-1}\mathscr{F}$ from $\mathrm{\acute{E}t}(S)$ to presheaves on $S_{\mathrm{\acute{e}t}}$ is compatible with the formation of finite products and fiber products. The sheafification of the presheaf $f^{-1}\mathscr{F}$ is denoted $f^*\mathscr{F}$, and it follows that $f^* : \mathrm{\acute{E}t}(S) \to \mathrm{\acute{E}t}(S')$ commutes with formation of finite products and finite fiber products, and hence commutes with all finite inverse limits. Thus, f^* is a left exact functor; it is a left adjoint to f_* because in any diagram (1.1.6.1), the map $X' \to X$ factors uniquely through $X \times_S S' \to X$ over $S' \to S$. In particular, f^* is exact. There are obvious concrete maps $f^*f_*\mathscr{F} \to \mathscr{F}$ and $\mathscr{F} \to f_*f^*\mathscr{F}$ yielding the bijection

$$\mathrm{Hom}_{\mathrm{\acute{E}t}(S')}(f^*\mathscr{F}, \mathscr{F}') \simeq \mathrm{Hom}_{\mathrm{\acute{E}t}(S)}(\mathscr{F}, f_*\mathscr{F}').$$

EXAMPLE 1.1.6.1. The functor $S_{\mathrm{\acute{e}t}} \to \mathrm{\acute{E}t}(S)$ converts base change into pullback. That is, if $X \to S$ is étale then we have an equality $f^*(\underline{X}) = \underline{X \times_S S'}$ that is natural in f and X. This follows from the equality

$$\begin{aligned} \mathrm{Hom}_{S'}(f^*\underline{X}, \mathscr{F}) &= \mathrm{Hom}_S(\underline{X}, f_*\mathscr{F}) \\ &= (f_*\mathscr{F})(X) \\ &= \mathscr{F}(X \times_S S') \\ &= \mathrm{Hom}_{S'}(\underline{X \times_S S'}, \mathscr{F}). \end{aligned}$$

If X is not étale over S then this calculation does not make sense and typically in such situations $f^*\underline{X}$ is not easily described in terms of $X \times_S S'$.

EXAMPLE 1.1.6.2. Let $k \to k'$ be an extension of fields with compatible choices of separable closures. Let $f : \mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ be the natural map. Under the equivalence between Galois-sets and étale sheaves of sets, the operation f^* sends a discrete $\mathrm{Gal}(k_s/k)$-set to a discrete $\mathrm{Gal}(k'_s/k')$-set. via composition of the action-map with the continuous map $\mathrm{Gal}(k'_s/k') \to \mathrm{Gal}(k_s/k)$.

As in ordinary sheaf theory, we may directly construct natural transformations $f^* \circ g^* \to (g \circ f)^*$ that are compatible with triple composites and compatible (via adjointness) with the evident identification $(f \circ g)_* \simeq f_* \circ g_*$. In particular, the natural transformation $f^* \circ g^* \to (g \circ f)^*$ is an isomorphism.

It is an important fact (immediate from the construction) that if $\mathscr{F}$ is an abelian sheaf then $f^*\mathscr{F}$ is naturally an abelian sheaf, and that this defines a functor $f^* : \mathrm{Ab}(S) \to \mathrm{Ab}(S')$ that is left adjoint to the left exact $f_* : \mathrm{Ab}(S') \to \mathrm{Ab}(S)$. Moreover, because finite inverse limits of abelian groups coincide with finite inverse limits on the underlying sets, f^* retains its left exactness when restricted to a functor between categories of abelian sheaves. The left exactness of the right-adjoint f_* between abelian-sheaf categories therefore implies that f^* is an exact functor between abelian-sheaf categories. The exactness of f^* implies that its left-adjoint f_* carries injective abelian sheaves to injective abelian sheaves; this will underlie the construction of the Leray spectral sequence in étale cohomology.

Example 1.1.6.3 (Compatibility of extension-by-zero and pullback). Consider a cartesian diagram of schemes

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "j'"'] & U \arrow[d, "j"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

where j and j' are étale. In both the categories of sheaves of sets and sheaves of abelian groups, there is a natural adjunction map $\mathrm{id} \to j^*j_!$, and so we have a natural transformation

$$f'^* \to f'^*j^*j_! \simeq j'^*f^*j_!;$$

by adjointness again, this gives a map $\xi : j'_!f'^* \to f^*j_!$. We claim that ξ is an isomorphism. The functors $j'_!f'^*$ and $f^*j_!$ are respectively left-adjoint to $f'_*j'^*$ and j^*f_*, and so the canonical isomorphism between these latter two functors sets up an isomorphism of their left adjoints. This isomorphism is ξ.

Remark 1.1.6.4 (Topological invariance of the étale site). The étale site (and hence the étale topos) on a scheme are topological invariants in the following sense. If $f : S' \to S$ is radicial, integral and surjective (*i.e.*, if it is a universal homeomorphism [**17**, 18.12.11]), then the natural transformations $\mathrm{id} \to f_*f^*$ and $f^*f_* \to \mathrm{id}$ are isomorphisms, and so f_* and f^* are inverse equivalences between $\mathrm{\acute{E}t}(S')$ and $\mathrm{\acute{E}t}(S)$; more generally, the functor $X \rightsquigarrow X \times_S S'$ is an equivalence between $S_{\mathrm{\acute{e}t}}$ and $S'_{\mathrm{\acute{e}t}}$. To establish such an equivalence of étale sites, we may work locally on S, so we can assume S and S' are affine; since any integral ring extension is a direct limit of finite subextensions, consideration with direct limits reduces us to the case of finite radicial surjections, and this case is treated in [**13**, Exp. IX, 4.10].

The important special case when S' is a closed subscheme in S defined by an ideal sheaf of nilpotent functions is settled in [**17**, 18.1.2]; this allows us to harmlessly pass to underlying reduced schemes in many proofs in the étale topology. For another example, suppose k' is a purely inseparable field extension of a field k (*e.g.*, a perfect closure), and let X is a k-scheme. The categories $\mathrm{\acute{E}t}(X)$ and $\mathrm{\acute{E}t}(X_{/k'})$ are identified via the functors π_* and π^*, with $\pi : X_{/k'} \to X$ the projection. In particular, étale sheaf theory on a scheme X over a field k is identified with étale sheaf theory on $X_{/k_p}$, where k_p is a perfect closure of k.

1.1.7. Locally-constant and constructible sheaves. For a set Σ, let $\underline{\Sigma}_S$ in $\mathrm{\acute{E}t}(S)$ denote the sheafification of the presheaf $U \mapsto \Sigma$. Equivalently, as in ordinary sheaf theory, this is the sheaf represented by the disjoint union of copies of S indexed by Σ. In particular, if $f : S' \to S$ is a map of schemes then we naturally have $f^*\underline{\Sigma}_S \simeq \underline{\Sigma}_{S'}$. For this reason, we usually write $\underline{\Sigma}$ rather than $\underline{\Sigma}_S$. An object in $\mathrm{\acute{E}t}(S)$ that is isomorphic to $\underline{\Sigma}$ for a set Σ is called a *constant sheaf* (or *a constant sheaf on the set* Σ).

Definition 1.1.7.1. An object $\mathscr{F}$ in the étale topos $\mathrm{\acute{E}t}(S)$ is *locally constant* if there exists an étale cover $\{S_i \to S\}_{i \in I}$ such that each $\mathscr{F}|_{S_i}$ is constant. If in addition the associated set over each S_i is finite, then $\mathscr{F}$ is *locally constant constructible* (abbreviation: *lcc*).

The importance of lcc sheaves is due to the fact that the lcc condition is defined in a manner that is local for the étale topology, yet such sheaves have a simple global classification:

THEOREM 1.1.7.2 (Classification of lcc sheaves). *The functor* $X \rightsquigarrow \underline{X}$ *from* $S_{\text{ét}}$ *to* $\text{Ét}(S)$ *restricts to an equivalence of categories between finite étale* S*-schemes and lcc sheaves on* $S_{\text{ét}}$.

PROOF. We first check that $\underline{X}$ is lcc if X is finite and étale over S. We may work over the disjoint open covering $\{U_n\}$ of S such that $X|_{U_n} \to U_n$ has constant degree $n \geq 0$, so we may assume X has some constant degree $n \geq 0$ over S. We will induct on n by splitting off pieces of X, the case $n = 0$ being clear (so we may assume $n > 0$). We will imitate the technique in field theory that makes a splitting field for a separable irreducible polynomial $f \in k[T]$.

Note that for any separated (*e.g.*, finite) étale map $Y \to Z$ and any section $s : Z \to Y$, the map s is both étale and a closed immersion, and so it must be an open and closed immersion. That is, $Y = Y' \coprod s(Z)$. Hence, if a finite étale map has constant degree $n \geq 1$ and admits a section, then the section splits off and its complement is finite étale over the base with constant degree $n - 1$.

To apply this in our situation, we observe that $U = X \to S$ is an étale cover, and the base change of $p : X \to S$ by this cover is the degree-n map $\text{pr}_2 : X \times_S X \to X$. This map has a section, namely the diagonal, and so induction on n completes the proof that there exists an étale cover $\{S_i \to S\}_{i \in I}$ such that $X \times_S S_i$ is isomorphic to a finite product of copies of S_i. That is, $\underline{X}$ is lcc.

Conversely, suppose that $\mathscr{F}$ is an étale sheaf on S and there is an étale covering $\{f_i : S_i \to S\}_{i \in I}$ such that there are isomorphisms

$$\mathscr{F}|_{(S_i)_{\text{ét}}} \simeq \underline{\Sigma_i} = \Sigma_i \times \underline{S_i}$$

for some finite sets Σ_i of size $n_i \geq 0$. We want to prove that $\mathscr{F}$ is represented by a finite étale S-schene X.

We may work Zariski-locally over S, due to full faithfulness of the embedding of $S_{\text{ét}}$ into the étale topos $\text{Ét}(S)$, and so openness of the f_i's allows us to work separately over the pairwise-disjoint open unions U_n consisting of those $f_i(S_i)$'s such that $n_i = n$. Thus, we may assume $n_i = n$ for all i, and so $S' = \coprod S_i$ is an étale cover of S such that there are isomorphisms $\xi' : \mathscr{F}|_{S'} \simeq \Sigma \times \underline{S'}$ with Σ a finite set of size $n \geq 0$.

Since $S' \to S$ is open, we may shrink S and replace S' with a finite disjoint union of opens so as to assume that S and S' are both affine. Thus, $S' \to S$ is faithfully flat and quasi-compact. Let $p_1, p_2 : S' \times_S S' \rightrightarrows S'$ be the projections, so if the base-change functors $S_{\text{ét}} \to S'_{\text{ét}}$ are denoted p_j^* then the étale $S' \times_S S'$-scheme $p_j^*(\Sigma \times S')$ represents the sheaf $p_j^*(\underline{\Sigma})$. There is an isomorphism

$$p_1^*(\underline{\Sigma}) \overset{p_1^*(\xi')^{-1}}{\simeq} p_1^*(\mathscr{F}|_{S'_{\text{ét}}}) = \mathscr{F}|_{(S' \times_S S')_{\text{ét}}} = p_2^*(\mathscr{F}|_{S'_{\text{ét}}}) \overset{p_2^*(\xi')}{\simeq} p_2^*(\underline{\Sigma}).$$

This gives rise to an isomorphism of $S' \times_S S'$-schemes

$$\varphi : p_1^*(\Sigma \times S') \simeq p_2^*(\Sigma \times S')$$

satisfying the cocycle condition $p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi)$ over the triple fiber product of $S' \to S$; *i.e.*, φ constitutes *descent data* on $\Sigma \times S'$ with respect to the fpqc map $S' \to S$.

Since $\Sigma \times S' \to S'$ is an affine (and even finite) morphism, the effectivity of fpqc-descent for affine (and even just finite) morphisms [**3**, 6.1] implies that there exists an S-scheme X equipped with an S'-isomorphism

$$\Sigma \times S' \simeq X \times_S S'$$

carrying the descent data φ on the left over to the canonical descent data on the right. By [**17**, 2.7.1], the map $X \to S$ must be finite étale with constant degree n because these properties are acquired after applying the fpqc base change to S'. Thus, X is an object in $S_{\text{ét}}$, and so ξ' is an isomorphism

$$\xi'_X : \mathscr{F}|_{S'} \simeq \underline{X}|_{S'}$$

in $\text{Ét}(S')$ such that $p_1^*(\xi'_X) = p_2^*(\xi'_X)$ in $\text{Ét}(S' \times_S S')$. Since ξ'_X is a section of $\mathscr{H}om(\mathscr{F}, \underline{X})$ over S' and its restrictions under the pullbacks to $S' \times_S S'$ coincide, it uniquely descends to $\xi_X \in \text{Hom}_{\text{Ét}(S)}(\mathscr{F}, \underline{X})$. The same applies to its inverse, so ξ_X is an isomorphism. □

As one application of the representability of lcc sheaves, the unramifiedness property of a Galois representation can be expressed in terms of the associated étale sheaf:

Corollary 1.1.7.3. *Let R be a Dedekind domain and let $i : \operatorname{Spec} K \to \operatorname{Spec} R$ be the inclusion of the generic point. The functor $\mathscr{F} \rightsquigarrow i^*\mathscr{F}$ is a fully faithful functor from the category of lcc sheaves on $(\operatorname{Spec} R)_{\text{ét}}$ to the category of lcc sheaves on $(\operatorname{Spec} K)_{\text{ét}}$, with essential image equal to the category of those finite discrete $\text{Gal}(K_s/K)$-sets that are unramified at all closed points of $\operatorname{Spec} R$.*

In particular, if $\mathscr{G}$ is an lcc sheaf on $(\operatorname{Spec} K)_{\text{ét}}$ then its associated continuous representation of $\text{Gal}(K_s/K)$ on a finite set is unramified at all places of R if and only if $\mathscr{G}$ extends to an lcc sheaf over $(\operatorname{Spec} R)_{\text{ét}}$.

Proof. By the classification of lcc sheaves, we have to prove that if $X_K \to \operatorname{Spec} K$ is finite étale and $X(K_s)$ is unramified at all closed points of $\operatorname{Spec} R$, then there is a finite étale R-scheme X with $X \times_{\operatorname{Spec} R} \operatorname{Spec} K \simeq X_K$, and that such an X is functorial in X_K. Since R is normal, it follows that every étale R-scheme is normal, and so the R-finite and flat X must be the R-finite and flat normalization $\widetilde{X}_K$ of $\operatorname{Spec} R$ in X_K. Thus, the problem is to prove that $\widetilde{X}_K$ is étale over $\operatorname{Spec} R$ when $X_K(K_s)$ is unramified over all closed points of $\operatorname{Spec} R$.

We may assume that R is a discrete valuation ring and that X_K is connected; thus, it remains to prove that for any finite separable extension L/K with $\text{Hom}_K(L, K_s)$ unramified as a $\text{Gal}(K_s/K)$-set, the finite R-flat integral closure R_L of R in L is étale over R. The unramifiedness hypothesis on the Galois-set $\text{Hom}_K(L, K_s)$ says that all inertia groups for all maximal ideals of the semi-local R_L act trivially. Since the splitting-field of our Galois-set is a Galois closure of L over K, this says exactly that R_L is everywhere unramified over R in the sense of valuation theory, or in other words the closed fiber of the finite flat map $\operatorname{Spec} R_L \to \operatorname{Spec} R$ is a finite union of Spec's of finite separable extensions of the residue field of R. This says that the finite flat map $\operatorname{Spec} R_L \to \operatorname{Spec} R$ is étale, as desired. □

Definition 1.1.7.4. Let S be a noetherian topological space. A *stratification* of S is a finite set $\{S_i\}$ of pairwise-disjoint non-empty subsets S_i that are locally closed in S and satisfy $\cup S_i = S$, with the closure of S_i equal to a union of S_j's.

The S_i's are called the *strata* of the stratficiation, and the empty space admits only the *empty stratification*.

If S is a noetherian scheme, then a sheaf $\mathscr{F}$ on $S_{\text{ét}}$ is *constructible* if there exist a stratification $\{S_i\}$ of the underlying Zariski topological space of S such that the restriction of $\mathscr{F}$ to each stratum is lcc.

The way stratifications arise in practice is this: we find a dense open $U_0 \subseteq S$ where some property of interest holds, and then find a dense open U_1 in $Z_1 = S - U_0$ where the same property holds, and then find a dense open U_2 in $Z_2 = Z_1 - U_1$ where this property holds, and so on until (by noetherian induction) we reach the situation $U_j = \emptyset$. Each U_i is open in its closure Z_i, and so U_i is locally closed. Note that $Z_1, \dots, Z_{j-1}$ is a decreasing chain of closed sets in S, with U_i having closure equal to $Z_i = U_i \cup U_{i+1} \cup \cdots$. The collection $\{U_0, U_1, \dots, U_{j-1}\}$ is a stratification of S.

When S is a noetherian scheme and a stratification $\{S_i\}$ is given, we may consider each stratum as a subscheme of S by viewing it as an open subscheme of a choice of closed-subscheme structure on its closure in S. The choice of such scheme structure on the strata does not affect the definition of constructibility, due to the topological invariance of the étale site. Note also that if $\{S_i\}$ and $\{S'_j\}$ are two stratifications of S, then $\{S_i \cap S'_j\}$ is a stratification (upon removing any empty overlaps). This fact is implicitly used when carrying out noetherian-induction arguments with stratifications.

The definition of the lcc condition is étale-local, but the definition of constructibility is not. Thus, we must prove that locality holds:

Theorem 1.1.7.5 (Local nature of constructibility). *Let S be a noetherian scheme, and $\{U_i\}$ an étale cover. If $\mathscr{F}$ is an object in* $\text{Ét}(S)$ *and* $\mathscr{F}|_{U_i} \in \text{Ét}(U_i)$ *is constructible for all i, then $\mathscr{F}$ is constructible.*

Proof. By noetherian induction, it suffices to work in a Zariski-neighborhood of each generic point of S. Thus, we may assume that S is irreducible and that there is a finite étale cover $S' \to S$ such that $\mathscr{F}|_{S'}$ is constructible (a *finite étale cover* of a scheme S is an S-scheme whose structure map is finite, étale, and surjective). A cofinal system of Zariski-opens in S' containing the generic points is given by preimages of Zariski-opens in S around the generic point. Thus, since $\mathscr{F}|_{S'}$ is lcc on some Zariski-dense open in S', shrinking some more around the generic point of S allows us assume $\mathscr{F}|_{S'}$ is lcc. The lcc property is local for the étale topology, so $\mathscr{F}$ is lcc. $\square$

We now give some useful examples of constructible sheaves.

Example 1.1.7.6. Let $X \to S$ be a quasi-compact étale map to a noetherian scheme S. We claim that $\underline{X}$ is constructible. Since the formation of $\underline{X}$ commutes with base change on S, noetherian induction reduces us to finding a Zariski-dense open $U \subseteq S$ such that $X_U \to U$ is finite. The map $X \to S$ is quasi-finite and locally of finite presentation, and so it is finite over some Zariski-dense open U in S.

Example 1.1.7.7. The technique of noetherian induction enables us to prove that constructibility is preserved under many functors. For example, consider the following constructions: pullback, image under a map, and finite limits (such as equalizer-kernels and quotients by equivalence relations). We claim that the output of these operations is constructible when the input-sheaves are constructible.

Noetherian induction reduces us to the case when the input-sheaves are lcc, and since constructibility is local for the étale topology we may assume that these lcc sheaves are constant. The passage from a finite set to its associated constant-sheaf commutes with all of the constructions under consideration.

In the subcategory $\mathrm{Ab}(S) \subseteq \mathrm{\acute{E}t}(S)$, some of the basic operations are not the same as in $\mathrm{\acute{E}t}(S)$: finite direct limits are not the same (*e.g.*, a finite coproduct of abelian groups has underlying set equal to the product, not the disjoint union). However, the same method as above may be applied get the preservation of constructibility in $\mathrm{Ab}(S)$ under finite direct limits of abelian sheaves.

Example 1.1.7.8. Let $j : U \to S$ be a quasi-compact étale map to a noetherian scheme S, and let $\mathscr{F}$ be a constructible sheaf on U in either $\mathrm{\acute{E}t}(U)$ or $\mathrm{Ab}(U)$. We claim that $j_!\mathscr{F}$ is constructible on S. Since $j_!$ is compatible with pullback, noetherian induction allows us to work near the generic points on S. Thus, we may assume j is finite étale.

Working étale-locally on S, we may split the finite étale map $j : U \to S$. That is, we can assume $U = \Sigma \times S$ for a finite set Σ. Let $\mathscr{F}_\sigma$ be the constructible restriction of $\mathscr{F}$ to the factor $S = \{\sigma\} \times S$ in U. By adjointness of $j_!$ and j^*,

$$\mathrm{Hom}_S(j_!\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_U(\mathscr{F}, j^*\mathscr{G}) = \prod_{\sigma\in\Sigma} \mathrm{Hom}_S(\mathscr{F}_\sigma, \mathscr{G}) = \mathrm{Hom}_S(\coprod \mathscr{F}_\sigma, \mathscr{G}),$$

where the coproduct is categorical (in the category of abelian groups or the category of sets). By Yoneda's lemma, it follows that $j_!\mathscr{F}$ is the coproduct of the $\mathscr{F}_\sigma$'s, and so (treat $\mathrm{\acute{E}t}(S)$ and $\mathrm{Ab}(S)$ separately) it is constructible.

The importance of constructible sheaves in the general theory is due to:

Theorem 1.1.7.9. *Let S be a noetherian scheme.*

(1) *Every $\mathscr{F}$ in $\mathrm{\acute{E}t}(S)$ is the filtered direct limit of its constructible subsheaves, and subsheaves of constructible sheaves are constructible.*
(2) *If $\mathscr{F}$ is in $\mathrm{Ab}(S)$ and each section of $\mathscr{F}$ is locally killed by a nonzero integer, then $\mathscr{F}$ is the filtered direct limit of its constructible abelian subsheaves.*
(3) *An object in $\mathrm{Ab}(S)$ is noetherian (i.e., its subobjects satisfy the asecending chain condition) if and only if it is constructible.*

Proof. See [**9**, Ch. I, §4, pp. 42-3]. □

The noetherian property of constructible abelian sheaves is due to the noetherian property of noetherian topological spaces. The category of abelian étale sheaves on a noetherian scheme S has very few artinian objects (*i.e.*, abelian sheaves whose subobjects satisfy the descending chain condition); however, all objects in the abelian category of lcc abelian sheaves on $S_{\mathrm{\acute{e}t}}$ are artinian.

1.1.8. Stalks of étale sheaves. For a map $x : \mathrm{Spec}\, k \to S$ and an étale sheaf of sets $\mathscr{F}$ on $S_{\mathrm{\acute{e}t}}$, we write $\mathscr{F}_x$ to denote $x^*\mathscr{F}$ in $\mathrm{\acute{E}t}(\mathrm{Spec}\, k(x))$. When k is separably closed, we may identify $\mathscr{F}_x$ with a set. The functor $\mathscr{F} \rightsquigarrow \mathscr{F}_x$ is exact, since pullbacks are exact in étale sheaf theory.

Definition 1.1.8.1. A *geometric point* of $S_{\mathrm{\acute{e}t}}$ (or of S) is a map $\overline{s} : \mathrm{Spec}\, k \to S$ with k a separably closed field. If $f : S \to S'$ is a map of schemes, then $f(\overline{s})$ denotes the geometric point $f \circ \overline{s}$ of S'. The functor $\mathscr{F} \rightsquigarrow \mathscr{F}_{\overline{s}}$ from $\mathrm{\acute{E}t}(S)$ to **Set** is the *fiber* *functor* (or *stalk functor*) at $\overline{s}$. Two geometric points $\overline{s}$ and $\overline{s}'$ of S are *equivalent* when their physical image points in S are the same.

Observe that the fiber functors associated to equivalent geometric points are (non-canonically) isomorphic to each other, and that applying this definition in ordinary topology does yield the usual notion of the stalk of a sheaf at a point on a topological space.

To put the preceding definition in perspective, we note that the concept of a geometric point can be defined more generally. Let us briefly describe how this goes. For an arbitrary site (category with a Grothendieck topology), with T its associated category of sheaves of sets (the *topos* of the site), a *geometric point* of T is an exact functor $\xi : T \to \mathbf{Set}$ such that ξ commutes with arbitrary (not just finite) direct limits. Two geometric points in this sense are (by definition) equivalent when they are naturally isomorphic as functors. In two important cases, Grothendieck classified the geometric points in the sense of this abstract definition (see [**15**, Exp. VIII, §7.8*ff.*] for details):

- Let S be a topological space such that irreducible closed sets have unique generic points (*e.g.*, a normal Hausdorff space, or the underlying space of a scheme). Let T be the category of sheaves of sets on this space. Every geometric point of T is equivalent to the ordinary stalk functor at a unique point of S.
- Let S be a scheme. Every abstract geometric point of $\mathrm{\acute{E}t}(S)$ is equivalent to the stalk functor at a point $\overline{s} : \mathrm{Spec}(k) \to S$, where k is a separable algebraic closure of the residue field at the image of $\overline{s}$ in S. Stalk-functors based at distinct ordinary points of S are not equivalent.

For a geometric point $\overline{s} : \mathrm{Spec}\, k \to S$, an *étale neighborhood* of $\overline{s}$ is an étale map $U \to S$ equipped with a map $u : \mathrm{Spec}\, k \to U$ whose composite with $U \to S$ is $\overline{s}$:

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "u"] \arrow[dr, "\overline{s}"'] & U \arrow[d] \\ & S \end{tikzcd}$$

In ordinary topology, stalks may be described as direct limits of sets of sections over open neighborhoods, and we have an analogue of this in the étale topology: for $s \in S$ and a geometric point $\overline{s} : \mathrm{Spec}\, \overline{k} \to S$ over s with $\overline{k}$ a separable closure of $k(s)$, the natural map

$$\varinjlim_{(U,u)\to(S,\overline{s})} \mathscr{F}(U) \to \mathscr{F}_{\overline{s}} = (\overline{s}^*\mathscr{F})(\mathrm{Spec}\, \overline{k})$$

is a bijection; the direct limit is taken over the category of étale neighborhoods of $\overline{s}$. This description is an immediate consequence of the construction of pullback functors, together with the observation that the sheafification of a presheaf $\mathscr{G}$ on $\mathrm{Spec}\, \overline{k}$ has value $\mathscr{G}(\overline{k})$ on $\mathrm{Spec}\, \overline{k}$. In particular, two elements a and b in $\mathscr{F}(S)$ are equal in some étale neighborhood of $\overline{s}$ if and only if $a_{\overline{s}} = b_{\overline{s}}$ in $\mathscr{F}_{\overline{s}}$. It follows that if Σ is a set of geometric points $s : \mathrm{Spec}\, k \to S$ whose images cover S set-theoretically, then a map $\varphi : \mathscr{F} \to \mathscr{G}$ in $\mathrm{\acute{E}t}(S)$ is epic (resp. monic) if and only if φ_s is epic (resp. monic) for all $s \in \Sigma$; the same holds for the property of being an isomorphism. Likewise, maps $\varphi_1, \varphi_2 : \mathscr{F} \rightrightarrows \mathscr{G}$ are equal if and only if $\varphi_{1,s} = \varphi_{2,s}$ for all $s \in \Sigma$.

1.2. Cohomology basics

1.2.1. The étale fundamental group. For any topological space X, the degree-1 topological cohomology $\mathrm{H}^1(X, M)$ with coefficients in an abelian group M is isomorphic to $\mathrm{Hom}(\mathrm{H}_1(X, \mathbf{Z}), M)$, with $\mathrm{H}_1(X, \mathbf{Z})$ isomorphic to the abelianization of the fundamental group (at any base point). In the development of étale cohomology, it is important to have an analogue of this fact right at the start, where the étale fundamental group plays the role of the topological π_1. Thus, we shall now discuss the theory of the étale fundamental group of a connected (pointed) scheme, and we will establish that it can be used to compute $\mathrm{H}^1_{\text{ét}}$ with coefficients in a finite abelian group. The reader may wish to compare some of our arguments with the topological arguments in Appendix **??**.

The avoidance of noetherian conditions forces us to keep in mind at the outset that the connected components of a general scheme need not be open (example: $\mathrm{Spec}\prod_{n=1}^{\infty}\mathbf{F}_2$). However, this is not a problem for us, because a finite étale cover of a connected scheme has only finitely many connected components (and so all such components are open and closed). This finiteness is readily seen by inducting on the degree of the cover.

Let S be an arbitrary connected scheme, and let $\overline{s}$ be a geometric point of S. Let $S' \to S$ be a finite étale map. Connectivity of S forces this map to have constant degree, say n. We assume that S' is connected, and we choose a geometric point $\overline{s}'$ of S' over $\overline{s}$. To streamline the exposition, we impose the (harmless) condition that $\overline{s}$ and $\overline{s}'$ have the same residue field k.

The uniqueness principle for lifting through covering spaces in topology has the following analogue in the étale topology:

Lemma 1.2.1.1 (Rigidity of pointed étale covers). *If $f, g : S' \rightrightarrows S''$ are S-maps to a separated étale S-scheme S'', and $(S', \overline{s})$ is a pointed connected scheme such that $f(\overline{s}') = g(\overline{s}')$ in $S''(k)$, then $f = g$.*

Proof. The closed immersion

$$\Delta : S'' \to S'' \times_S S''$$

is étale, hence open, so $S'' \times_S S''$ can be written as $S'' \coprod Y$ with S'' equal to the diagonal. Since S' is connected and (by assumption) the image of the map $f \times g : S' \to S'' \times_S S''$ meets Δ, the image lies in Δ. □

Example 1.2.1.2. As an important application of the rigidity lemma, we may construct the (strict) henselization of a local ring $(R, \mathfrak{m})$; this generalizes the maximal unramified extension of a complete discrete valuation ring, and is constructed as follows.

Let us say that an R-algebra R' is *essentially étale* if it is a localization of an étale R-algebra. Fix a separable closure $i_0 : R/\mathfrak{m} \to k$, and consider pairs (R', i) where $R \to R'$ is an essentially étale local map and i is an $R/\mathfrak{m}$-embedding of $R'/\mathfrak{m}'$ into k over i_0. By the rigidity lemma and denominator-chasing, the category of such pairs is rigid in the sense that there exists at most one map between any two objects. This category is also filtered, since for any two objects we may find a map to a common third object by localizing a tensor product over R.

The *strict henselization* $R^{\mathrm{sh}}_{i_0}$ of R is the direct limit of all pairs (R', i). Any connected finite étale cover $X \to \mathrm{Spec}\, R^{\mathrm{sh}}_{i_0}$ must arise as the base-change of a connected finite étale cover X' of some $\mathrm{Spec}\, R'$ as above, but we can pick a point

x' in the closed fiber of X' and embed it into k over $R'/\mathfrak{m}'$, so $\mathscr{O}_{X',x'}$ occurs in the direct limit defining $R^{\mathrm{sh}}_{i_0}$. This provides a section to the connected finite étale cover $X \to \operatorname{Spec} R^{\mathrm{sh}}_{i_0}$, and so this cover must have degree 1. That is, $\operatorname{Spec} R^{\mathrm{sh}}_{i_0}$ has no non-trivial finite étale covers. The *henselization* R^{h} is constructed similarly, except that we require the residue-field extension at the maximal ideals to be trivial. An argument as above shows that $\operatorname{Spec} R^{\mathrm{h}}$ has no non-trivial finite étale covers with trivial residue field extensions. Both R^{h} and $R^{\mathrm{sh}}_{i_0}$ are faithfully flat and local over R.

When $R = R^{\mathrm{h}}$, then we say R is a *henselian ring*, and when R is henselian with separably closed residue field then it is a *strictly henselian ring*; for example, considerations with direct limits imply that R^{h} is henselian and $R^{\mathrm{sh}}_{i_0}$ is strictly henselian. These constructions are characterized by simple universal properties (so, for example, $R^{\mathrm{sh}}_{i_0}$ is functorial in (R, i_0)), and the theory of local henselian rings is developed in [**23**] and [**17**, §18.5–18.9]; we note the non-obvious fact that the noetherian property is preserved under both constructions [**17**, 18.6.6, 18.8.8].

Passage to the (strict) henselization is less drastic than the operation of completion but has many of the good properties of completions. For example, finite algebras over a henselian local ring decompose into a product of finite local algebras. Just as ordinary localization $A_{\mathfrak{p}}$ at a prime ideal is constructed as a direct limit of coordinate rings of Zariski-open neighborhoods of $\mathfrak{p}$ in $\operatorname{Spec} A$, strict henselization plays the role of a local ring (at a geometric point) for the étale topology; see Example 1.2.6.1 for a precise version of this idea.

Returning to the general setting, let S be a connected scheme and let S' be a connected finite étale S-scheme with $\deg(S'/S) = n$. The rigidity lemma implies $\#\operatorname{Aut}(S'/S) \leq n$ since there are exactly n geometric points in the fiber over $\overline{s}$. A generalization of the method of construction of Galois closures [**13**, Exp. V, §2–4] shows that we can always find a finite étale map $S'' \to S'$ with S'' connected and $\#\operatorname{Aut}(S''/S) = \deg(S''/S)$.

Definition 1.2.1.3. A finite étale map $S' \to S$ between connected non-empty schemes is *Galois* (or a (finite) *Galois covering*) if $\#\operatorname{Aut}(S'/S) = \deg(S'/S)$, and then the *Galois group* of S' over S is the opposite group $\operatorname{Gal}(S'/S) \stackrel{\mathrm{def}}{=} \operatorname{Aut}(S'/S)^{\mathrm{op}}$.

Example 1.2.1.4. Let R be a Dedekind domain with fraction field K, and let K'/K be a finite separable extension, with R' the integral closure of R in K'. In general, $\operatorname{Spec} R' \to \operatorname{Spec} R$ is Galois in the sense of Definition 1.2.1.3 if and only if K'/K is Galois and R' is everywhere unramified over R. In particular, the condition that K'/K be Galois is not enough to make R' Galois over R.

For a more global example, let $S' \to S$ be a finite map between two smooth proper connected curves over an algebraically closed field k. The scheme S' is Galois over S if and only if the map of curves is generically Galois (*i.e.*, $k(S')$ is Galois over $k(S)$) and has no ramification in the sense of valuation theory.

If $S' \to S$ is a finite Galois covering with Galois group G, then G acts on S' on the right, and the action map

$$S' \times G \to S' \times_S S'$$

defined by $(s', g) \mapsto (s', s'g)$ is a map of finite étale S'-schemes if we use pr_1 as the structure map for $S' \times_S S'$. In particular, the action map is finite étale; a calculation on fibers over geometric points of S' shows that this map must have degree 1, so it is an isomorphism. This generalizes the isomorphism in field theory

$$K \otimes_k K \simeq \prod_{g \in G} K$$

defined by $x \otimes y \mapsto (g(y)x)_g$ for a finite Galois extension of fields K/k with Galois group G.

The generalization of the absolute Galois group of a field (equipped with a choice of separable closure) is the *étale fundamental group* of a pointed scheme $(S, \overline{s})$. To construct this, consider two (connected) finite Galois covers $(S', \overline{s}') \to (S, \overline{s})$ and $(S'', \overline{s}'') \to (S, \overline{s})$ of a connected scheme S. Rigidity implies that there is at most one S-map $(S'', \overline{s}'') \to (S', \overline{s}')$ taking $\overline{s}''$ to $\overline{s}'$. When such a map exists, for any $f'' \in \mathrm{Aut}(S''/S)$ there exists a unique $f' \in \mathrm{Aut}(S'/S)$ fitting into a commutative diagram

$$\begin{tikzcd} S'' \arrow[r, "f''"] \arrow[d, "\pi"'] & S'' \arrow[d, "\pi"] \\ S' \arrow[r, "f'"'] & S' \end{tikzcd}$$

Indeed, $\pi \circ f''(\overline{s}'') \in S'_{\overline{s}'}$ has the form $f'(\overline{s}')$ for a unique $f' \in \mathrm{Aut}(S'/S)$, due to the simple transitivity of Galois-group actions on geometric fibers, and so $(f \circ \pi)(\overline{s}'') = (\pi \circ f'')(\overline{s}'')$. Thus, $f \circ \pi = \pi \circ f''$. It is clear that this map

$$\mathrm{Aut}(S''/S) \to \mathrm{Aut}(S'/S) \tag{1.2.1.1}$$

is a group homomorphism that moreover is surjective. This leads to:

Definition 1.2.1.5. The *étale fundamental group* of a connected scheme S (with respect to a geometric point $\overline{s}$) is the profinite group

$$\pi_1(S, \overline{s}) = \varprojlim_{(S', \overline{s}')} \mathrm{Aut}(S'/S)^{\mathrm{op}} = \varprojlim_{(S', \overline{s}')} \mathrm{Gal}(S'/S),$$

where the inverse limit is taken over connected finite Galois covers $S' \to S$ endowed with a geometric point $\overline{s}' : \mathrm{Spec}\, k(\overline{s}) \to S'$ over $\overline{s}$.

The surjectivity of (1.2.1.1) ensures that $\pi_1(S, \overline{s}) \to \mathrm{Gal}(S'/S)$ is surjective for all pointed connected finite Galois covers $(S', \overline{s}') \to (S, \overline{s})$. Let us now explain how $\pi_1(S, \overline{s})$ is covariant in the pair $(S, \overline{s})$.

Let $f : (T, \overline{t}) \to (S, \overline{s})$ be a morphism of connected pointed schemes. Let $(S', \overline{s}') \to (S, \overline{s})$ be a pointed connected Galois cover of some degree n with $k(\overline{s}') = k(\overline{s})$, so $S'_{/T}$ is a degree-n finite étale T-scheme endowed with a canonical point $\overline{t}' = \overline{t} \times_{\overline{s}} \overline{s}'$ with $k(\overline{t}') = k(\overline{t})$. Since $S'_{/T}$ is finite étale over the connected scheme T, it has only finitely connected components, and so each is therefore open and closed in T. Let T' be the unique such component containing $\overline{t}'$. The group $G = \mathrm{Gal}(S'/S)$ acts simply transitively on geometric fibers of $S' \to S$, and so it also acts simply transitively on geometric fibers of $S'_{/T} \to T$. If $H \subseteq G$ is the stabilizer of T' in T, then H must have order equal to $\deg(T'/T)$, and so T' is a finite Galois cover of T with Galois group H. The continuous composite

$$\pi_1(T, \overline{t}) \twoheadrightarrow \mathrm{Gal}(T'/T) = H \hookrightarrow G = \mathrm{Gal}(S'/S).$$

is compatible with (1.2.1.1), and so passing to the inverse limit over $(S', \overline{s}')$'s defines a continuous map $\pi_1(f) = \pi_1(T, \bar{t}) \to \pi_1(S, \overline{s})$ that respects composition in f.

This covariance recovers the functoriality of absolute Galois groups of fields relative to choices of separable closures:

EXAMPLE 1.2.1.6. Let $S = \operatorname{Spec} k$ for a field k, and choose a separable closure $\overline{s} : \operatorname{Spec} k_s \to \operatorname{Spec} k$, so we get a canonical isomorphism of profinite groups $\pi_1(S, \overline{s}) \simeq \operatorname{Gal}(k_s/k)$. If

$$\begin{tikzcd} k_s \arrow[r] & k_s' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

is a commutative diagram, then the diagram

$$\begin{tikzcd} \pi_1(S, \overline{s}) \arrow[d, equal] & \pi_1(S', \overline{s}') \arrow[l] \arrow[d, equal] \\ \operatorname{Gal}(k_s/k) & \operatorname{Gal}(k_s'/k') \arrow[l] \end{tikzcd}$$

commutes, where the top row is induced by covariant functoriality and the bottom row is the classical map from algebra.

1.2.2. First applications of π_1. In Galois theory, forming a tensor product of a field extension K/k against arbitrary finite separable extensions k'/k always yields a field if and only if K and k_s are linearly disjoint over k. In topology (for reasonable spaces), pullback preserves connectivity of arbitrary connected covering spaces if and only if the induced map on topological π_1's is surjective. These viewpoints are synthesized in:

THEOREM 1.2.2.1 (Connectivity criterion via π_1's). *Let $f : X \to Y$ be a map of connected schemes. Pick a geometric point $\overline{x}$ of X and define $\overline{y} = f(\overline{x})$. The map $\pi_1(f) : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$ is surjective if and only if $X \times_Y Y'$ is connected for all connected finite étale covers $Y' \to Y$.*

PROOF. The map $\pi_1(f)$ has image that is compact and hence closed, so $\pi_1(f)$ is surjective if and only if it has dense image' *i.e.*, if and only if the composite

$$\pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y}) \twoheadrightarrow \operatorname{Gal}(Y'/Y) \tag{1.2.2.1}$$

is surjective for every connected pointed finite étale cover $(Y', \overline{y}')$ of $(Y, \overline{y})$. Since $X \times_Y Y'$ is finite étale over the connected X, it has finitely many connected components (not more than the degree of Y' over Y). The geometric point $\overline{x}' = (\overline{x}, \overline{y}')$ lies in a unique connected component X', and (1.2.2.1) factors through a canonical surjection $\pi_1(X, \overline{x}) \twoheadrightarrow \operatorname{Gal}(X'/X)$. Thus, the surjectivity of $\pi_1(f)$ is equivalent to that of

$$\operatorname{Gal}(X'/X) \to \operatorname{Gal}(Y'/Y)$$

for all connected finite étale Y' over Y.

In the construction of covariant functoriality of π_1, we saw that this map of finite Galois groups is injective, so it is surjective if and only both groups have the same size. The order of the Galois group is the degree of the covering, and by construction $[X' : X] \leq [Y' : Y]$ with equality if and only if $X' = X \times_Y Y'$. Such equality says exactly that $X \times_Y Y'$ is connected. $\square$

Example 1.2.2.2. Building on Example 1.2.1.6, let X be an irreducible normal scheme (*i.e.*, all local rings are normal domains), and let K be its function field. The most important case for us is $X = \operatorname{Spec} R$ with R an integrally closed domain; we do not assume X is locally noetherian. Fix a choice of separable closure K_s of K, and view this as a geometric point $\overline{x} : \operatorname{Spec} K_s \to X$. Consider the natural continuous map

$$\mathrm{Gal}(K_s/K) = \pi_1(\operatorname{Spec} K, \overline{x}) \to \pi_1(X, \overline{x}).$$

We claim that this map is surjective, and that the resulting quotient $\mathrm{Gal}(K'/K) = \pi_1(X, \overline{x})$ of $\mathrm{Gal}(K_s/K)$ cuts out the subextension $L \subseteq K_s$ that is the maximal extension of K inside of K_s such that the normalization of X in each finite subextension of L is finite étale over X (such a property for two finite subextensions is clearly preserved under formation of composites, so the compositum of all such extensions is Galois over K). For example, if X is Dedekind then L/K is the maximal extension that is everywhere unramified over X in the sense of valuation theory.

To prove surjectivity of the π_1-map, the connectivity criterion via π_1's reduces us to checking that for a connected finite étale cover $X' \to X$, the finite étale cover $X'_{/K}$ of $\operatorname{Spec} K$ is connected. Since X' is étale over the normal X, and hence is normal, its local rings are domains and so distinct irreducible components of X' cannot intersect. By X-flatness, any generic point of X' lies over the unique generic point of X. Thus, there are only finitely many irreducible components of X'. The pairwise disjointness of such components, together with the connectedness of $X' \neq \emptyset$, forces X' to be irreducible (and hence integral). Thus, $X'_{/K}$ is clearly connected. It is likewise clear from this argument that X' is the normalization of X in the function field of X', and moreover that the function fields K' of such X' are exactly those finite separable extensions of K such that the normalization of X in K' is finite étale over X. This establishes the desired description of the subextension of K_s/K cut out by the quotient $\pi_1(X, \overline{x})$ of $\mathrm{Gal}(K_s/K)$.

One consequence is that if X is irreducible and normal then $\pi_1(U, \overline{x}) \to \pi_1(X, \overline{x})$ is surjective for any Zariski-open U in X containing $\overline{x}$. This is false if we remove normality. For example, the projective plane curve $X : y^2 z = x^3$ over an algebraically closed field has normalization $\widetilde{X}$ that is isomorphic to $\mathbf{P}^1$, and $\widetilde{X} \to X$ a universal homeomorphism, so the topological invariance of the étale site implies that $\pi_1(X)$ is trivial. However, the complement U of the cusp $[0, 0, 1]$ is identified with the affine line, and Artin–Schreier theory provides many degree-p connected finite étale covers of the affine line in characteristic $p > 0$.

Theorem 1.2.2.3 (Grothendieck). *There is a canonical equivalence of categories between the category of finite étale morphisms $S' \to S$ and the category of finite discrete left $\pi_1(S, \overline{s})$-sets, given by $S' \rightsquigarrow S'(\overline{s})$, where the connected covers correspond to the finite sets with transitive $\pi_1(S, \overline{s})$-action.*

This equivalence is functorial in $(S, \overline{s})$ in the sense that if $f : (T, \overline{t}) \to (S, \overline{s})$ is a map from another pointed connected scheme, then the natural equality of sets $(T \times_S S')(\overline{t}) = S'(\overline{s})$ respects $\pi_1(T, \overline{t})$-actions, where $\pi_1(T, \overline{t})$ acts on $S'(\overline{s})$ through $\pi_1(f)$.

Proof. Since S' has only finitely many connected components, we can reduce to the case of a connected S'. Consider a connected Galois cover $S'' \to S$ admitting a factorization $S'' \xrightarrow{\pi} S' \to S$, so $S'' \to S'$ is also Galois. Choose a geometric point $\overline{s}'' \in S''$ over $\overline{s}$ in order to uniquely define a continuous surjection $\pi_1(S, \overline{s}) \twoheadrightarrow$

$\mathrm{Aut}(S''/S)^{\mathrm{op}}$. Define a map of sets $\mathrm{Aut}(S''/S)^{\mathrm{op}} \to S'_{\overline{s}}$ by sending g to $\pi \circ g(\overline{s}'')$. If $\pi \circ h \circ g(\overline{s}'') = \pi \circ g(\overline{s}'')$, then $\pi, \pi \circ h : S'' \rightrightarrows S'$ are S-maps that coincide on $g(\overline{s}'')$, and so rigidity forces them to agree. That is, we have a bijection of sets

$$\mathrm{Aut}(S''/S)^{\mathrm{op}}/\mathrm{Aut}(S''/S')^{\mathrm{op}} \simeq S'(\overline{s}).$$

The induced left $\pi_1(S, \overline{s})$-set structure on $S'(\overline{s})$ is independent of the choice of $(S'', \overline{s}'')$, and any discrete finite left $\pi_1(S, \overline{s})$-set with a transitive action arises in this way. $\square$

Using the classification of lcc sheaves, we get:

Corollary 1.2.2.4. *Let S be a connected scheme and $\overline{s}$ a geometric point. The fiber-functor $\mathscr{F} \rightsquigarrow \mathscr{F}_{\overline{s}}$ sets up an equivalence of categories between lcc sheaves of sets on $S_{\text{ét}}$ and finite discrete left $\pi_1(S, \overline{s})$-sets; this is functorial in $(S, \overline{s})$. In particular, the abelian category of lcc abelian sheaves on $S_{\text{ét}}$ is equivalent to the category of finite discrete left $\pi_1(S, \overline{s})$-modules.*

Example 1.2.2.5. Let X be a connected scheme, and choose a point $x \in X$ and a geometric point $\overline{x} : \mathrm{Spec}\, k(x)_s \to X$ over x. For any étale sheaf $\mathscr{F}$ on X, the stalk $\mathscr{F}_x$ at x is an étale sheaf on $(\mathrm{Spec}\, k(x))_{\text{ét}}$, and thus is a $\mathrm{Gal}(k(\overline{x})/k(x))$-set via the choice of $\overline{x}$. If $\mathscr{F}$ is an lcc sheaf on X, then the pointed map $(x, \overline{x}) \to (X, \overline{x})$ induces a continuous map of profinite groups $\mathrm{Gal}\big(k(\overline{x})/k(x)\big) = \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x})$. Thus, the finite discrete $\pi_1(x, \overline{x})$-set $(\mathscr{F}_x)_{\overline{x}}$ is canonically constructed from the finite discrete $\pi_1(X, \overline{x})$-set $\mathscr{F}_{\overline{x}}$ via $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x})$. This is an example of the functoriality in Theorem 1.2.2.3.

Example 1.2.2.6. In number theory, the following procedure is frequently used: we consider a discrete module M for the Galois group of a global field, and if this module is unramified at some non-archimedean place v then we may choose a decomposition group at v and use this choice to view M as a Galois module for the residue field at v. We would like to describe this procedure more generally in the language of fundamental groups, since the π_1-language is used when working with Galois representations via étale cohomology.

Suppose that R is a Dedekind domain with fraction field K, and let $x \in X = \mathrm{Spec}\, R$ be a closed point. Choose a geometric point $\overline{x}$ over x. We would like to describe how to identify the $\overline{x}$-fiber and the geometric generic fiber of certain étale sheaves on $\mathrm{Spec}\, R$. Let R_x^{h} denote the henselization of the local ring R_x of R at x. By [**17**, 18.5.15], every finite étale cover of $\mathrm{Spec}\, k(x)$ uniquely lifts to a finite étale cover of $\mathrm{Spec}\, R_x^{\mathrm{h}}$. Equivalently, the map $\mathrm{Spec}\, k(x) \to \mathrm{Spec}\, R_x^{\mathrm{h}}$ induces an isomorphism $\pi_1(x, \overline{x}) \simeq \pi_1(\mathrm{Spec}\, R_x^{\mathrm{h}}, \overline{x})$.

The map

$$\pi_1(x, \overline{x}) \simeq \pi_1(\mathrm{Spec}\, R_x^{\mathrm{h}}, \overline{x}) \to \pi_1(\mathrm{Spec}\, R_x, \overline{x})$$

is a description of the procedure of choosing a decomposition group at x in the maximal quotient $\pi_1(\mathrm{Spec}\, R_x, \overline{x})$ of $\mathrm{Gal}(K_s/K)$ that is everywhere unramified over x; here we are using Example 1.2.2.2.

We now apply the preceding generalities. Let M be a finite discrete $\mathrm{Gal}(K_s/K)$-set and let $\mathscr{F}_K$ be the associated sheaf on $(\mathrm{Spec}\, K)_{\text{ét}}$. Assume that the Galois-action is everywhere unramified, and let $\mathscr{F}$ be the unique associated lcc sheaf over $(\mathrm{Spec}\, R)_{\text{ét}}$ via Corollary 1.1.7.3. For any closed point $x \in X = \mathrm{Spec}\, R$, once we choose a strict henselization R_x^{sh} inside of K_s we may canonically identify the underlying set of the $\pi_1(x, \overline{x})$-module $\mathscr{F}_x$ with the underlying set M of the

$\mathrm{Gal}(K_s/K)$-set $\mathscr{F}_K$ that is everywhere unramified at x. This procedure gives us a compatibility of geometric points on R and K, as in the commutative diagram

$$\begin{tikzcd} & \operatorname{Spec} K_s \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\ \operatorname{Spec} k(\overline{x}) \arrow[r] & \operatorname{Spec} R_x^{\mathrm{sh}} \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

This comes down to the following. Let $\mathscr{G}$ be an lcc sheaf on $S = \operatorname{Spec} R$. By the classification of lcc sheaves, $\mathscr{G}$ is represented by a finite étale S-scheme X. If $S' = \operatorname{Spec} R_{\overline{x}}^{\mathrm{sh}}$, then $X \times_S S'$ is a finite étale cover of the strictly henselian local scheme S', and so it is split. Thus, $X \times_S S'$ is a finite disjoint union of copies of S'. In particular, there is a canonical bijection between the underlyings sets of the closed fiber and generic fiber in this split covering over the connected base S'. However, $X(\overline{x}) = \mathscr{G}_{\overline{x}}$ and the choice of injection of $R_{\overline{x}}^{\mathrm{sh}}$ into K_s identifies the points in the generic fiber of $X \times_S S' \to S'$ with $\mathscr{G}_{K_s}$. This yields the desired identification of stalks at closed and generic geometric points, depending on a choice of embedding of a strict henselization into a geometric generic-fiber field.

1.2.3. Specialization. Example 1.2.2.6 uses a special case of the technique of specialization. Let us discuss this technique in general, as it leads to a useful criterion for a constructible sheaf to be lcc. Let S be a scheme, $\mathscr{F}$ a sheaf of sets on $S_{\text{ét}}$, and $\overline{s}$ and $\overline{\eta}$ two geometric points of S. The point $\overline{s}$ is a *specialization* of the point $\overline{\eta}$ (and $\overline{\eta}$ is a *generization* of $\overline{s}$) if $\overline{\eta} \to S$ admits a factorization through $\operatorname{Spec} \mathscr{O}_{S,\overline{s}}^{\mathrm{sh}}$; strictly speaking, such a factorization should be specified when we say that $\overline{s}$ specializes $\overline{\eta}$. The openness of étale maps implies that $\overline{s}$ is a specialization of $\overline{\eta}$ if and only if the underlying physical points $s, \eta \in S$ satisfy $s \in \overline{\{\eta\}}$.

Given a factorization $i : \overline{\eta} \to \operatorname{Spec} \mathscr{O}_{S,\overline{s}}^{\mathrm{sh}}$, the *specialization mapping* $\mathscr{F}_{\overline{s}} \to \mathscr{F}_{\overline{\eta}}$ is the composite

$$\mathscr{F}_{\overline{s}} \simeq \varinjlim_{(U,u)\to(S,\overline{s})} \mathscr{F}(U) \xrightarrow{i^*} \mathscr{F}_{\overline{\eta}};$$

this is functorial in $\mathscr{F}$. This generalizes the Zariski-topology fact that sections of a sheaf over a neighborhood of s admit canonical specializations into stalks any point η that lies in all such neighborhoods.

Example 1.2.3.1. The specialization map takes on a concrete form for $S = \operatorname{Spec} R$ when R is strictly henselian, with residue field $\overline{k}$. In this case, [**17**, 18.5.11(c)] and [**17**, 18.5.15] ensure that for any separated étale neighborhood U of $\overline{s}$, there is a unique decomposition $U = X' \coprod S$ with $\overline{s}$ factoring through the component S. (This non-trivial input is the analogue of the elementary topological fact that a section to a separated local homeomorphism of topological spaces splits off as an open and closed component of the source.) It follows that the identity map $S \to S$ is a cofinal system of étale neighborhoods of $\overline{s}$, and so $\mathscr{F}_{\overline{s}} = \mathscr{F}(S)$. With this identification, the specialization map is the evident map $\mathscr{F}(S) \to \mathscr{F}_{\overline{\eta}}$ that is pullback on sections.

Our interest in specialization maps is due to:

Theorem 1.2.3.2 (Specialization criterion for local constancy). *Assume S is noetherian, and let $\mathscr{F}$ be a constructible sheaf of sets on S. The sheaf $\mathscr{F}$ is lcc if and only if all specialization maps $\mathscr{F}_{\overline{s}} \to \mathscr{F}_{\overline{\eta}}$ are bijective.*

PROOF. If $\mathscr{F}$ is lcc, then the classification of lcc sheaves implies $\mathscr{F} = \underline{X}$ for a finite étale S-scheme X. We may assume S is strictly henselian with closed point $\overline{s}$, and so X must be a split cover of S. Hence, $\mathscr{F}$ is constant in $\acute{\mathrm{E}}\mathrm{t}(S)$, and so its specialization maps are clearly isomorphisms.

Conversely, assume all specialization maps are isomorphisms. To prove that $\mathscr{F}$ is lcc, it is enough to verify this in an étale neighborhood of each geometric point. Pick a geometric point $\overline{x}$ of X, and let $\Sigma = \mathscr{F}_{\overline{x}}$. By the universal property of constant sheaves, restricting to an étale neighborhood of $\overline{x}$ allows us to assume there is a map $\varphi : \underline{\Sigma} \to \mathscr{F}$ inducing the given equality on $\overline{x}$-fibers. Since the specialization maps for $\underline{\Sigma}$ and $\mathscr{F}$ are isomorphisms, it follows that $\varphi_{\overline{x}'}$ is bijective for any geometric point $\overline{x}'$ of X that may be linked to $\overline{x}$ by finitely many generizations and specializations. By linking through geometric generic points of irreducible components, this exhausts all points in the Zariski-open connected component of X containing $\overline{x}$. □

1.2.4. Étale cohomology groups. Let S be a scheme. The category $\mathrm{Ab}(S)$ of abelian-group objects in the étale topos $\acute{\mathrm{E}}\mathrm{t}(S)$ is an abelian category, and that the global-sections functor $\acute{\mathrm{E}}\mathrm{t}(S) \to \mathbf{Set}$ is left-exact. Thus, this functor defines a left-exact functor $\mathrm{Ab}(S) \to \mathbf{Ab}$ to the category of abelian groups.

LEMMA 1.2.4.1. *The abelian category* $\mathrm{Ab}(S)$ *has enough injectives.*

PROOF. A *generating object* $\mathbf{U}$ in an abelian category A is an object with the property that $\mathrm{Hom}(\mathbf{U}, C') \to \mathrm{Hom}(\mathbf{U}, C)$ is not bijective whenever $C' \to C$ is a monic map that is not an isomorphism; *i.e.*, maps from $\mathbf{U}$ can detect proper subobjects. By [**11**, Thm. 1.10.1], an abelian category has enough injectives if the following three axioms holds: it has a generating object, admits arbitrary direct sums, and satisfies the condition

$$B \cap \left(\sum A_i\right) = \sum (B \cap A_i)$$

for any monomorphism $B \hookrightarrow B'$ and any increasing filtered family $\{A_i\}$ of subobjects in B'. The final two axioms are easily verified in $\mathrm{Ab}(S)$, and so it suffices to construct a generating object. Since $j_!$ is a left adjoint to j^* for étale maps $j : U \to S$, a generating object is given by the sheaf $\mathscr{F} = \oplus j_! \underline{\mathbf{Z}}$, where the direct sum is taken over a cofinal set of étale maps $j : U \to S$ (*i.e.*, every étale S-scheme admits an S-map from one of the U's in our chosen collection). □

DEFINITION 1.2.4.2. *Étale cohomology* $\mathrm{H}^{\bullet}_{\acute{\mathrm{e}}\mathrm{t}}(S, \cdot)$ on $\mathrm{Ab}(S)$ is the right derived functor of the left-exact global-sections functor $\mathscr{F} \rightsquigarrow \mathscr{F}(S)$. If $f : X \to S$ is a map of schemes, the δ-functor $\mathrm{R}^{\bullet} f_*$ of *higher direct images* is the right derived functor of the left-exact functor $f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(S)$.

For any map of schemes $f : X \to S$, recall that $f^* : \mathrm{Ab}(S) \to \mathrm{Ab}(X)$ is an exact functor between abelian categories. This exactness implies that there is a unique δ-functorial map $\mathrm{H}^{\bullet}_{\acute{\mathrm{e}}\mathrm{t}}(S, \mathscr{G}) \to \mathrm{H}^{\bullet}_{\acute{\mathrm{e}}\mathrm{t}}(X, f^*\mathscr{G})$ extending the canonical map in degree-0; this *cohomological pullback* is transitive with respect to composites in f (as may be checked in degree 0). Since f_* has exact left-adjoint f^*, pushforward carries injectives to injectives. Thus, we obtain *Leray spectral sequences*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(S, \mathrm{R}^q f_*(\cdot)) \Rightarrow \mathrm{H}^{p+q}(X, \cdot), \ \ \mathrm{E}_2^{p,q} = \mathrm{R}^p h_* \circ \mathrm{R}^q f_* \Rightarrow \mathrm{R}^{p+q}(h \circ f)_*$$

(where $h : S \to S'$ is any map of schemes).

THEOREM 1.2.4.3. *Let $j : U \to S$ be an étale map. If $\mathscr{I} \in \mathrm{Ab}(S)$ is injective, then so is $j^*\mathscr{I} \in \mathrm{Ab}(U)$.*

PROOF. We want the functor

$$\mathrm{Hom}_{\mathrm{Ab}(U)}(\mathscr{G}, j^*\mathscr{I}) = \mathrm{Hom}_{\mathrm{Ab}(S)}(j_!\mathscr{G}, \mathscr{I})$$

to be exact in $\mathscr{G}$, and so it suffices to prove exactness of the left adjoint $j_! : \mathrm{Ab}(U) \to \mathrm{Ab}(S)$. The right exactness is a tautological consequence of the fact that $j_!$ is a left adjoint that is exact (by construction, $j_!$ carries monics to monics). $\square$

COROLLARY 1.2.4.4. *Let $j : U \to S$ be étale. The δ-functor $\mathrm{H}^\bullet_{\text{ét}}(U, j^*(\cdot))$ is erasable, and so it is the derived functor of its degree-0 term $\mathscr{F} \rightsquigarrow \mathscr{F}(U)$ on $\mathrm{Ab}(S)$.*

As in the topological case, we will write $\mathrm{H}^\bullet_{\text{ét}}(U, \mathscr{F})$ to denote either of the two (*a posteriori* isomorphic) δ-functors considered in the corollary. This corollary ensures that for any map $f : X \to S$ and any $\mathscr{F}$ in $\mathrm{Ab}(X)$, $\mathrm{R}^\bullet f_*\mathscr{F}$ is naturally identified with the sheafification of $V \rightsquigarrow \mathrm{H}^\bullet(f^{-1}(V), \mathscr{F})$ on $S_{\text{ét}}$. As an application of this sheafification process, or by using a universal δ-functor argument (or adjointness), we can relativize cohomological pullback: for any commutative square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

we have a *base change morphism*

$$g^* \circ \mathrm{R}^\bullet f_* \to (\mathrm{R}^\bullet f'_*) \circ g'^*$$

as δ-functors $\mathrm{Ab}(X) \to \mathrm{Ab}(S')$, and this enjoys transitivity properties exactly as in topology.

1.2.5. Čech theory and applications. An important theoretical tool in the investigation of étale cohomology is the Čech to derived-functor spectral sequence. In the topological case, this formalism is summarized in §**??**, and we now describe and apply its étale-topology variant. Let $\mathfrak{U}$ be an étale cover of a scheme X. Using the same definitions as in the topological case, for any abelian presheaf $\mathscr{F}$ on $X_{\text{ét}}$ we may construct the Čech complex $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ of presheaves on $X_{\text{ét}}$. Thus, we get functors $\mathrm{H}^q_{\text{ét}}(\mathfrak{U}, \cdot)$ on abelian presheaves on $X_{\text{ét}}$. For any $q > 0$ and any injective $\mathscr{I}$ in $\mathrm{Ab}(X)$, we have $\mathrm{H}^q_{\text{ét}}(U, j^*\mathscr{I}) = 0$ for any étale $j : U \to X$. Thus, we may carry over the universal δ-functor arguments in §**??** to construct a Čech to derived-functor cohomology spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(\mathfrak{U}, \underline{\mathrm{H}}^q_{\text{ét}}(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathscr{F})$$

for $\mathscr{F}$ in $\mathrm{Ab}(X)$.

This spectral sequence is compatible with refinement in $\mathfrak{U}$, and this compatibility is independent of the choice of refinement maps. Hence, we may pass to the direct limit over a cofinal system of étale covers of X, obtaining

$$\mathrm{E}_2^{p,q} = \check{\mathrm{H}}^p_{\text{ét}}(X, \underline{\mathrm{H}}^q_{\text{ét}}(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathscr{F}).$$

The term $\mathrm{E}_2^{0,q} = \check{\mathrm{H}}^0_{\text{ét}}(X, \underline{\mathrm{H}}^q_{\text{ét}}(\mathscr{F}))$ is the separated presheaf associated to the presheaf $U \rightsquigarrow \mathrm{H}^q_{\text{ét}}(U, \mathscr{F})$ for $U \in X_{\text{ét}}$. Since $\mathrm{H}^q_{\text{ét}}(U, \mathscr{F})$ may be computed via injective resolutions of $\mathscr{F}$ in $\mathrm{Ab}(X)$, for $q > 0$ it follows that any element of $\mathrm{H}^q_{\text{ét}}(U, \mathscr{F})$ restricts to 0 when we localize on U. Hence, $\mathrm{E}_2^{0,q} = 0$ for $q > 0$. In degree 1, this yields:

THEOREM 1.2.5.1. *The edge map* $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathscr{F}) \to \mathrm{H}^1_{\text{ét}}(X, \mathscr{F})$ *is an isomorphism for all* $\mathscr{F}$ *in* $\mathrm{Ab}(X)$.

The theorem has immediate consequences for the classification of torsors; this requires a definition:

DEFINITION 1.2.5.2. Let X a scheme and G a group-object in $\text{Ét}(X)$. An *étale left G-torsor* is an object $\mathscr{F}$ in $\text{Ét}(X)$ that has non-empty stalks and is equipped with a left G-action such that the canonical map $G \times \mathscr{F} \to \mathscr{F} \times \mathscr{F}$ defined by $(g, s) \mapsto (gs, s)$ is an isomorphism. The notion of *right G-torsor* is defined similarly.

The object G is a left G-torsor via left multiplication on itself. This is the *trivial* left G-torsor.

EXAMPLE 1.2.5.3. Let X be connected and $X' \to X$ a connected Galois cover with Galois group G (in the sense of Definition 1.2.1.3). The X-scheme X' (or the étale sheaf it represents) is a right torsor for the X-group $X \times G$; we shall usually just say that X' is a right G-torsor over X.

Let $\mathscr{F}$ be a left G-torsor, with G a group in $\text{Ét}(X)$. For any $U \in X_{\text{ét}}$ such that $\mathscr{F}(U)$ is nonempty, a choice of element in $\mathscr{F}(U)$ determines an isomorphism $G|_U \simeq \mathscr{F}|_U$ in $\text{Ét}(U)$ as left G-torsors. Since $\mathscr{F}$ has sections étale-locally on X, it follows that $\mathscr{F}$ and G are étale-locally isomorphic. Thus, if G is lcc then $\mathscr{F}$ is lcc, and in such cases we may use the classification of lcc sheaves to conclude that G is represented by a finite étale X-group scheme (denoted G) and $\mathscr{F}$ is represented by a finite étale X-scheme X', with the torsor-structure given by a left G-action on X' over X such that the action-map

$$G \times_X X' \to X' \times_X X'$$

defined by $(g, x') \mapsto (gx', x')$ is an isomorphism. The lcc sheaf G is classified by the left $\pi_1(X, \overline{x})$-group $G_{\overline{x}}$, and (as in [**24**, Ch. X, §2]) the set of isomorphism classes of left $G_{\overline{x}}$-torsors in the category of finite discrete left $\pi_1(X, \overline{x})$-modules is in natural bijection with the pointed set $\mathrm{H}^1(\pi_1(X, \overline{x}), G_{\overline{x}})$, where this H^1 is taken in the sense of non-abelian profinite-group cohomology; *cf.* [**24**, Ch VII, Appendix].

When G is in $\mathrm{Ab}(X)$, we can say more. In this case $\mathrm{H}^1(\pi_1(X, \overline{x}), G_{\overline{x}})$ has a commutative group structure, and the set of isomorphism classes of left G-torsors has a commutative group structure as follows. For two torsors $\mathscr{F}$ and $\mathscr{F}'$, the product $\mathscr{F} \times \mathscr{F}'$ is a torsor for the group $G \times G$ and the addition map $G \times G \to G$ is a map of groups, so we may pass to the quotient $\mathscr{F}''$ of $\mathscr{F} \times \mathscr{F}'$ by the action of the anti-diagonal subgroup of $(g, -g)$'s; *i.e.*, we form the unique G-torsor $\mathscr{F}''$ equipped with a map $\mathscr{F} \times \mathscr{F}' \to \mathscr{F}''$ that is equivariant for the addition map $G \times G \to G$. The trivial G-torsor serves as an identity for this commutative law of composition, and the inverse $-\mathscr{F}$ of a G-torsor $\mathscr{F}$ is given by having G act on $\mathscr{F}$ through the composition of negation on G and the usual action on F (this is an inverse because a G-torsor with a section is a trivial G-torsor). This group law on the set of isomorphism classes of left G-torsors clearly goes over to the group law on $\mathrm{H}^1(\pi_1(X, \overline{x}), G_{\overline{x}})$.

THEOREM 1.2.5.4. *Let G be an abelian sheaf on $X_{\text{ét}}$. The group $\mathrm{H}^1_{\text{ét}}(X, G)$ is isomorphic to the group of isomorphism classes of left G-torsors in* $\mathrm{Ab}(X)$*, and this identification is bifunctorial in the pair (X, G).*

In particular, if X is connected and $\overline{x}$ is a geometric point, and if G in $\mathrm{Ab}(X)$ *is lcc, then there is a bifunctorial isomorphism of groups*

$$\mathrm{H}^1_{\text{ét}}(X, G) \simeq \mathrm{H}^1(\pi_1(X, \overline{x}), G_{\overline{x}}).$$

PROOF. The Čech to derived-functor spectral sequence identifies $\mathrm{H}^1_{\text{ét}}(X, G)$ with $\check{\mathrm{H}}^1_{\text{ét}}(X, G)$ in a bifunctorial manner, and so it remains to identify $\check{\mathrm{H}}^1_{\text{ét}}(X, G)$ with the group of isomorphism classes of left G-torsors in $\mathrm{Ab}(X)$. The arguments in §**??** carry over *verbatim* to the étale topology, and in the notation in that discussion we take $D = X$ and $U = \emptyset$. The étale version of Theorem **??** in this special case provides the desired result. $\square$

The preceding theorem says that, under an lcc hypothesis, we may compute degree-1 sheaf cohomology as profinite-group cohomology. The finiteness aspect of the lcc condition is crucial. For example, if we consider degree-1 cohomology for the constant sheaf attached to an infinite abelian group, then the connection with group cohomology of π_1 is not as straightforward as in the topological case:

EXAMPLE 1.2.5.5. Consider the nodal plane cubic $C_1 : y^2w = x^3 - x^2w$ over a separably closed field. Let C denote a chain of projective lines indexed by $\mathbf{Z}$, with 0 on each line joined to ∞ on the next line; there is a natural étale surjection $C \to C_1$ that is equivariant for the evident translation-action of $\mathbf{Z}$ on C. For $n \geq 1$, the quotient of this map by $n\mathbf{Z}$ is a degree-n connected Galois cover $C_n \to C_1$ with covering group $\mathbf{Z}/n\mathbf{Z}$. Using the fact that C_1 has normalization $\mathbf{P}^1$, and that finite étale covers of the projective line over a separably closed field are split, it follows that these C_n's exhaust the connected finite étale covers of C_1 up to isomorphism. The étale sheaf $\mathscr{F}$ in $\text{Ét}(C_1)$ represented by C is an étale $\mathbf{Z}$-torsor.

For all $n \geq 1$,

$$\mathscr{F}(C_n) = \mathrm{Hom}_{C_1}(C_n, C) = \emptyset.$$

Thus, the étale torsor $\mathscr{F}$ is not split by any finite étale cover of C_1. We conclude that $\mathrm{H}^1_{\text{ét}}(C_1, \mathbf{Z})$ is non-trivial, whereas $\mathrm{Hom}_{\text{cont}}(\pi_1(C_1, \overline{x}), \mathbf{Z}) = 0$.

For normal connected noetherian schemes X, this defect in π_1 goes away: every locally constant étale sheaf on such an X is split by a finite étale covering. This follows from Example 1.2.2.2.

1.2.6. Comparison of Zariski and étale cohomology. As a further important application of Čech methods, we shall now compute the étale cohomology of a quasi-coherent sheaf. We first need to review the passage between Zariski-sheaves and étale-sheaves. This is a prototype for the *comparison isomorphism* to be studied later, and is modelled on the mechanism of comparison of cohomology for different topologies as in Serre's GAGA theorems. Let S be a scheme. Since every Zariski-open in S is étale over S, there is a natural functor

$$\iota_* : \text{Ét}(S) \to \mathrm{Zar}(S)$$

to the category of sheaves of sets for the Zariski topology, namely $(\iota_*(\mathscr{F}))(U) = \mathscr{F}(U)$ for Zariski-open U in S. This functor has a right adjoint $\iota^* : \mathrm{Zar}(S) \to \text{Ét}(S)$;

explicitly, $\iota^*(\mathscr{F})$ is the sheafification of the presheaf

$$(h : U \to S) \rightsquigarrow \varinjlim_{V \supseteq h(U)} \mathscr{F}(V)$$

on $S_{\text{ét}}$. It is clear that ι_* is left-exact, so ι^* is right-exact, and in fact ι^* is exact. The same holds for the restricted functors between categories of abelian sheaves, and the exactness of ι^* implies (by a universal δ-functor argument) that there exists a unique δ-functorial *Zariski-étale comparison morphism*

$$\mathrm{H}^\bullet(S, \mathscr{F}) \to \mathrm{H}^\bullet_{\text{ét}}(S, \iota^*\mathscr{F})$$

that extends the evident map in degree 0. This is rarely an isomorphism.

When $\mathscr{F}$ is an $\mathscr{O}_S$-module, we can make a variant as follows. The additive group scheme $\mathbf{G}_a$ defines an abelian sheaf in $\mathrm{Ab}(S)$, also denoted $\mathbf{G}_a$. This may be viewed as a sheaf of rings, and we write $\mathscr{O}_{S_{\text{ét}}}$ to denote this sheaf of rings; explicitly, $\mathscr{O}_{S_{\text{ét}}}(U) = \mathscr{O}_U(U)$. There is a natural map $\mathscr{O}_S \to \iota_*\mathscr{O}_{S_{\text{ét}}}$ as sheaves of rings. Adjointness defines a map $\iota^*\mathscr{O}_S \to \mathscr{O}_{S_{\text{ét}}}$; this is generally not an isomorphism, because on stalks at $\overline{s}$ it is the natural map $\mathscr{O}_{S,s} \to \mathscr{O}^{\text{sh}}_{S,\overline{s}}$ with $s \in S$ the physical point underlying $\overline{s}$. This map of sheaves of rings enables us to define an $\mathscr{O}_{S_{\text{ét}}}$-module

$$\mathscr{F}_{\text{ét}} = \mathscr{O}_{S_{\text{ét}}} \otimes_{\iota^*\mathscr{O}_S} \iota^*\mathscr{F}$$

for any $\mathscr{O}_S$-module $\mathscr{F}$ on the Zariski topology of S, and the functor $\mathscr{F} \rightsquigarrow \mathscr{F}_{\text{ét}}$ from $\mathscr{O}_S$-modules to $\mathscr{O}_{S_{\text{ét}}}$-modules is exact because strict henselization is flat. The sheaf $\iota^*\mathscr{F}$ serves as a pullback for $\mathscr{F}$ from the Zariski topology to the étale topology, and $\mathscr{F}_{\text{ét}}$ is the analogous pullback when we work with the subcategories of module objects over $\mathscr{O}_S$ and $\mathscr{O}_{S_{\text{ét}}}$.

EXAMPLE 1.2.6.1. If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_S$-module and $h : U \to S$ is étale, then the restriction of $\mathscr{F}_{\text{ét}}$ to the Zariski topology on U is the usual Zariski-pullback $h^*\mathscr{F}$. This is not a tautology, so let us prove it. Consider the presheaf

$$\widetilde{\mathscr{F}} : (h' : U' \to S) \mapsto \Gamma(U', h'^*\mathscr{F})$$

on $S_{\text{ét}}$. It follows from fpqc-descent for quasi-coherent sheaves [**3**, 6.1] that this presheaf is a sheaf. Its stalk at $\overline{s}$ is $\mathscr{O}^{\text{sh}}_{S,\overline{s}} \otimes_{\mathscr{O}_{S,s}} \mathscr{F}_s$.

By construction, $\widetilde{\mathscr{F}}$ is an $\mathscr{O}_{S_{\text{ét}}}$-module; if $\mathscr{F} = \mathscr{O}_S$ then $\widetilde{\mathscr{F}} = \mathscr{O}_{S_{\text{ét}}}$. For example, the stalks of $\mathscr{O}_{S_{\text{ét}}}$ are the strict henselizations of the local rings on the scheme S. There is an evident map $\mathscr{F} \to \iota_*\widetilde{\mathscr{F}}$ that is linear over the natural map $\mathscr{O}_S \to \iota_*\mathscr{O}_{S_{\text{ét}}}$, and so adjointness gives a map

$$\mathscr{F}_{\text{ét}} = \mathscr{O}_{S_{\text{ét}}} \otimes_{\iota^*\mathscr{O}_S} \iota^*\mathscr{F} \to \widetilde{\mathscr{F}}$$

as $\mathscr{O}_{S_{\text{ét}}}$-modules. By computing on stalks, we see that this is an isomorphism. This gives a concrete description of $\mathscr{F}_{\text{ét}}$, and returning to our original étale $h : U \to S$ we conclude that the restriction of $\mathscr{F}_{\text{ét}}$ to the Zariski-topology on U is $h^*\mathscr{F}$.

Since $\mathscr{F}_{\text{ét}}$ is exact as a functor of the $\mathscr{O}_S$-module $\mathscr{F}$, we may compose the Zariski-étale comparison morphism with the natural map $\iota^*\mathscr{F} \to \mathscr{F}_{\text{ét}}$ to define a modified δ-functorial comparison morphism

$$\mathrm{H}^\bullet(S, \mathscr{F}) \to \mathrm{H}^\bullet_{\text{ét}}(S, \mathscr{F}_{\text{ét}}). \qquad (1.2.6.1)$$

for $\mathscr{O}_S$-modules $\mathscr{F}$ on S.

THEOREM 1.2.6.2. *The modified comparison morphism* (1.2.6.1) *is an isomorphism when* $\mathscr{F}$ *is quasi-coherent.*

PROOF. We first use two applications of the technique of the *spectral sequence for an affine cover* to reduce to the case of an affine scheme. This goes as follows. Fix an open affine covering $\mathfrak{U}$ of S, and so (by exactness of $\mathscr{F} \rightsquigarrow \mathscr{F}_{\text{ét}}$) the comparison morphism induces a morphism between the Čech to derived-functor spectral sequences for $\mathscr{F}$ on S and $\mathscr{F}_{\text{ét}}$ on $S_{\text{ét}}$ (with respect to the Zariski and étale covers induced by $\mathfrak{U}$). Our aim is to prove that the map on derived-functor abutments is an isomorphism, so it is enough to establish that the induced map between $\mathrm{E}_2^{p,q}$-terms is an isomorphism. This reduces the isomorphism problem to the case of a scheme S that is a finite overlap of affines. In particular, we can assume S is separated. Running through the spectral sequence again for separated S, the finite overlaps of open affines are affine, and so we are reduced to the case of affine S.

When S is affine, the situation is clear in degree-0 and we have to prove a vanishing claim in étale cohomology in positive degrees. Any étale cover of an affine scheme admits a refinement $\mathfrak{U}$ by finitely many affines, and the higher étale overlaps (*i.e.*, fiber products) of these are again affine. Example 1.2.6.1 and fpqc-descent for quasi-coherent sheaves ensure that the Čech complex $\mathrm{C}^\bullet(\mathfrak{U}, \mathscr{F}_{\text{ét}})$ built on a finite affine étale cover of an affine scheme is exact in positive degrees. Thus, $\mathscr{F}_{\text{ét}}$ has the property that its Čech complexes with respect to a cofinal system of covers are exact in positive degrees. In ordinary topology, Cartan's lemma [**11**, §3.8, Cor. 4] says that a sheaf with such a property must have vanishing higher derived-functor cohomology. The proof of Cartan's lemma is a spectral-sequence argument with the Čech to derived-functor cohomology spectral sequence. We have this spectral sequence for the étale topology, and so the proof of Cartan's lemma carries over *verbatim*. This proves that $\mathscr{F}_{\text{ét}}$ has vanishing higher étale cohomology on an affine scheme. $\square$

1.2.7. Kummer and Artin–Schreier sequences. We have seen that $\mathrm{H}^1_{\text{ét}}$ with finite constant coefficients may be understood through the theory of the étale fundamental group. To understand higher cohomology, let alone to prove vanishing results in large degrees, a new idea is needed. This problem is solved via an étale-topology version of Kummer theory, and in positive characteristic there is an analogue of Artin–Schreier theory that is used to study p-torsion cohomology in characteristic p. We now explain both theories.

For any scheme S, define $\mathbf{G}_m \in \mathrm{Ab}(S)$ to be the abelian sheaf of points of the smooth group scheme $\mathbf{G}_m$: its value on U is $\mathrm{H}^0(U, \mathscr{O}_U^\times)$. Likewise, for any nonzero integer n we define μ_n to be kernel of $x \mapsto x^n$ on $\mathbf{G}_m$; this is the sheaf of points of the finite flat commutative group scheme of nth roots of unity. If the integer n is a unit on S, then for any affine S-scheme $\operatorname{Spec} A$ and any $u \in A^\times = \mathbf{G}_m(A)$, the A-algebra $A[X]/(X^n - u)$ is etale over A. Thus, for such n, the (n-torsion) *Kummer sequence*

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{x^n} \mathbf{G}_m \to 0$$

of abelian sheaves on $S_{\text{ét}}$ is not only left exact (as is obvious on the level of presheaves, since μ_n is a scheme-theoretic kernel), but it is exact for the étale topology. Indeed, with notation as above, any $u \in \mathbf{G}_m(A)$ is an nth power in $\mathbf{G}_m(A')$ on the étale cover $\operatorname{Spec} A' = \operatorname{Spec} A[T]/(T^n - u) \twoheadrightarrow \operatorname{Spec} A$.

EXAMPLE 1.2.7.1. By Theorem 1.2.5.4, $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ classifies étale torsors for $\mathbf{G}_m$, or equivalently it classifies étale line bundles (*i.e.*, locally free $\mathscr{O}_{X_{\text{ét}}}$-sheaves of rank 1). The set of isomorphism classes of étale line bundles is denoted $\mathrm{Pic}(X_{\text{ét}})$, and it has a natural group structure via tensor products; the identification between $\mathrm{Pic}(X_{\text{ét}})$ and $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ is an isomorphism of groups.

Example 1.2.6.1 and fpqc-descent for quasi-coherent sheaves ensure that $\mathscr{F} \rightsquigarrow \mathscr{F}_{\text{ét}}$ is an equivalence of categories between quasi-coherent sheaves on X and $\mathscr{O}_{X_{\text{ét}}}$-modules that are étale-locally isomorphic to the étale sheaf associated to a quasi-coherent sheaf (on an étale X-scheme). Thus, the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\text{ét}})$ induced by $\mathscr{L} \rightsquigarrow \mathscr{L}_{\text{ét}}$ is an isomorphism of groups.

By following the sign convention in (**??**) for both X and $X_{\text{ét}}$, we conclude that the natural composite map

$$\mathrm{H}^1(X, \mathscr{O}_X^\times) \to \mathrm{H}^1_{\text{ét}}(X, \iota^*(\mathscr{O}_X^\times)) \to \mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$$

is an isomorphism, where the first step is the Zariski-étale comparison morphism and the second step is induced by the unit-group map of the map of sheaves of rings $\iota^*\mathscr{O}_X \to \mathscr{O}_{X_{\text{ét}}}$.

The importance of the Kummer sequence is this: if X is a $\mathbf{Z}[1/n]$-scheme such that μ_n is constant over X (*e.g.*, X is a scheme over a separably closed field of characteristic not dividing n), then we may identify $\mathbf{Z}/n$ with μ_n and so the Kummer sequence allows us to analyze cohomology with $\mathbf{Z}/n$-coefficients by understanding the higher cohomology of $\mathbf{G}_m$. For example, suppose X is a connected proper smooth genus-g curve over a separably closed field k whose characteristic does not divide n. We have $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m) = \mathrm{Pic}(X)$, and the nth-power map is surjective on $\mathbf{G}_m(X) = k^\times$, so

$$\mathrm{H}^1_{\text{ét}}(X, \mu_n) = \mathrm{Pic}(X)[n] = \mathrm{Pic}^0_{X/k}[n];$$

by the algebraic theory of the Jacobian, this gives the right answer: $(\mathbf{Z}/n\mathbf{Z})^{2g}$.

From the viewpoint of their applications to cohomology, Kummer theory corresponds to the analytic exponential sequence. The study of torsion that is divisible by the characteristic rests on Artin–Schreier theory, an analogue of de Rham theory. Suppose p is a prime and $p = 0$ on S. Since $t \mapsto t^p$ is additive in characteristic p, we can define the *Artin-Schreier sequence*

$$0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{G}_a \xrightarrow{t^p - t} \mathbf{G}_a \to 0$$

in $\mathrm{Ab}(S)$. We claim that this is an exact sequence. The left-exactness is clear since $t^p - t = \prod_{j \in \mathbf{F}_p}(t - j)$. To prove the exactness on the right, we just have to observe that $t^p - t$ is an étale surjection of group schemes. Explicitly, for any ring A with characteristic p and any $a \in A$, the equation $t^p - t = a$ can be solved in the finite étale extension $A[T]/(T^p - T - a)$.

It follows from the long-exact cohomology sequence associated to the Artin-Schreier sequence that the cohomology with $\mathbf{Z}/p\mathbf{Z}$-coefficients in characteristic p is controlled by the étale cohomology of $\mathscr{O}_{X_{\text{ét}}}$, and this is just Zariski-cohomology for the quasi-coherent sheaf $\mathscr{O}_X$. For example, if X is a proper curve then the vanishing $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $i > 1$ implies $\mathrm{H}^i(X, \mathbf{Z}/p\mathbf{Z}) = 0$ for $i > 2$. Since $\mathrm{H}^1(X, \mathscr{O}_X)$ is finite-dimensional in the proper case, the additive self-map induced by $t^p - t$ may be viewed as the map on k-points induced by an étale self-map of the connected additive algebraic k-group underlying the vector space $\mathrm{H}^1(X, \mathscr{O}_X)$. Any étale self-map of a connected algebraic k-group is surjective on k-points (since k is separably closed), and so $\mathrm{H}^2(X, \mathbf{Z}/p\mathbf{Z}) = 0$ for such k.

The Artin–Schreier resolution of $\mathbf{Z}/p\mathbf{Z}$ by coherent sheaves is a characteristic-p analogue of the holomorphic Poincaré lemma on a complex manifold (a resolution of $\underline{\mathbf{C}}$ by coherent sheaves $\Omega^\bullet$). Likewise, the long exact cohomology sequence that fits the étale cohomology of $\mathbf{Z}/p\mathbf{Z}$ between coherent cohomologies is an analogue of the Hodge to de Rham spectral sequence that computes the topological cohomology of $\underline{\mathbf{C}}$ on a complex manifold via the Hodge filtration and the spectral sequence whose E_1-terms are cohomology of coherent sheaves.

Definition 1.2.7.2. Let S be a scheme. An abelian étale sheaf $\mathscr{F}$ on $S_{\text{ét}}$ is a *torsion sheaf* if every section of $\mathscr{F}$ is locally killed by a non-zero integer; equivalently, all stalks are torsion abelian groups. If all sections are locally killed by powers of a prime p, then $\mathscr{F}$ is a *p-power torsion sheaf.*

The following theorem is proved by using the Kummer and Artin–Schreier sequences on curves:

Theorem 1.2.7.3. *Let X be a separated finite-type scheme of dimension ≤ 1 over a separably closed field k. For any torsion sheaf $\mathscr{F}$ on X, the groups $\mathrm{H}^i_{\text{ét}}(X, \mathscr{F})$ vanish for $i > 2$. If $\mathscr{F}$ is a constructible sheaf, then $\mathrm{H}^i_{\text{ét}}(X, \mathscr{F})$ is finite for $i \leq 2$.*

When X is affine, the vanishing of $\mathrm{H}^i_{\text{ét}}(X, \mathscr{F})$ also holds for $i = 2$ and any torsion sheaf $\mathscr{F}$ with torsion-orders not divisible by the characteristic. When X is proper then such vanishing holds for $i = 2$ when k has positive characteristic p and $\mathscr{F}$ is a p-power torsion sheaf.

Proof. See [**9**, Ch. I, §5] or [**15**, Exp. IX, §4] for a proof when torsion-orders are not divisible by the characteristic of k, and see [**19**, Exp. XXII, §1–§2] for the extra arguments needed to handle the p-part in characteristic p. □

1.3. Advanced torsion cohomology

We now discuss some deeper foundational theorems in étale cohomology at the level of torsion sheaves. The ℓ-adic enhancement is given in §1.4.

1.3.1. Quasi-separatedness and limits of schemes. Many proofs of general results in étale cohomology involve reduction to the case of cohomology of constructible sheaves on noetherian schemes. For example, the proof of the general cohomological vanishing theorem for torsion sheaves on curves (Theorem 1.2.7.3) uses limit arguments to reduce to the study of the cohomology of constructible sheaves. The starting point for all such arguments is the fact that any abelian torsion sheaf on a noetherian scheme is the direct limit of its constructible abelian subsheaves (Theorem 1.1.7.9). To exploit this, we need to know that direct limits can be moved through cohomology functors.

There is another kind of limit whose interaction with cohomology must be understood: we occasionally need to compute cohomology over a direct-limit ring. For example, we want a *Lefschetz principle* to affirm that the theory of étale cohomology for finite-type schemes over an algebraically closed field k of characteristic zero should be essentially the same as the theory in the special case $k = \mathbf{C}$. The algebraic mechanism that will make such a principle work is the fact that an extension of algebraically closed fields can always be expressed as a direct limit of smooth ring extensions, and so it is necessary to understand the behavior of cohomology with respect to a direct-limit process in coordinate rings. Geometrically, we must understand how cohomology behaves as we move in a projective system of schemes.

The basic property we wish to investigate is the behavior of étale cohomology with respect to limits of both schemes and abelian étale sheaves. On compact Hausdorff spaces, topological sheaf cohomology commutes with direct limits of abelian sheaves (Lemma **??**). This suggests that quasi-compact schemes are the ones to consider for the problem of moving limits through cohomology. However, we will see that an additional condition (always satisfied in the noetherian case) is necessary:

Definition 1.3.1.1. A scheme morphism $f : X \to S$ is *quasi-separated* if the diagonal $X \to X \times_S X$ is quasi-compact, and is *finitely presented* if it is quasi-separated and locally of finite presentation. A scheme X is *quasi-separated* if its structure morphism to $\operatorname{Spec} \mathbf{Z}$ is quasi-separated.

Concretely, a scheme is quasi-separated if and only if any two open affines in X lying over a common open affine in S have quasi-compact overlap in X; equivalently, any two quasi-compact opens in X have quasi-compact overlap. The finite presentation condition says that, locally over the base, X is described with a finite amount of information: finitely many equations in finitely many variables, and a finite amount of affine gluing data. Two good properties are that any map between quasi-separated S-schemes is quasi-separated and, more importantly, that any map between quasi-separated and quasi-compact S-schemes is also quasi-compact. See [**17**, §1.2] for a detailed treatment of these conditions.

Example 1.3.1.2. When S and X are locally noetherian, any map $f : X \to S$ is quasi-separated. Any map between noetherian schemes is therefore quasi-compact and quasi-separated. In general, a quasi-compact and quasi-separated scheme is a scheme that admits a finite open affine covering with the overlaps also admitting finite open affine covers. For example, a noetherian scheme is quasi-compact and quasi-separated, even when it is not separated.

In Example 1.1.3.9, we constructed a quasi-compact scheme S and an étale cover of S admitting no finite refinement with quasi-compact double overlaps. That example was constructed by gluing two affines along a non-quasi-compact open, and so S is not quasi-separated. Schemes that are not quasi-separated are generally regarded as pathological.

Although quasi-compactness and quasi-separatedness were defined in terms of the Zariski topology, we can characterize the combined conditions by using the étale topology:

Lemma 1.3.1.3. *A scheme X is quasi-compact and quasi-separated if and only if every étale cover $\{U_i\}$ of X admits a finite refinement $\{V_j\}$ such that all double-overlaps $V_j \times_X V_{j'}$ are quasi-compact.*

Proof. Assume that X is quasi-compact and quasi-separated. Openness of étale maps ensures that every étale cover of X admits a finite refinement consisting of affines whose image in X is contained in an affine open. Maps between affines are quasi-compact, and quasi-separatedness implies that $U \hookrightarrow X$ is a quasi-compact map for any open affine $U \subseteq X$. Thus, a cofinal system of étale covers is given by covers $\{V_j\}_{j \in J}$ with J finite, each $V_j \to X$ both quasi-compact and quasi-separated (even separated). Thus, $V_j \times_X V_{j'}$ is quasi-compact over X, and so it quasi-compact, for all j and j'.

Conversely, assume that all étale covers of X admit finite refinements $\{V_j\}$ with quasi-compact double-overlaps. Consider an open affine covering $\{U_i\}$ of X for the Zariski topology. Since this admits a finite refinment for the étale topology, it certainly admits a finite subcover for the Zariski topology. Thus, X is quasi-compact. To prove quasi-separatedness, let $\{U_i\}$ be a finite Zariski-cover of X by open affines; we want each $U_{ii'} = U_i \times_X U_{i'}$ to be quasi-compact. Since $\{U_i\}$ is an étale cover of X, it admits a finite refinement $\{V_j\}$ with each $V_j \times_X V_{j'}$ quasi-compact. Since each V_j factors through at least one U_i, if we define $V^{(i)}$ to be the disjoint union of the V_j's whose open image in X lies in U_i (so some V_j's may contribute to more than one $V^{(i)}$) then $V^{(i)}$ surjects onto U_i. Thus, $V^{(i)} \times_X V^{(i')}$ surjects onto $U_i \times_X U_{i'} = U_i \cap U_{i'}$. However, $V^{(i)} \times_X V^{(i')}$ is a finite disjoint union of quasi-compact products $V_j \times_X V_{j'}$. $\square$

Quasi-separatedness and quasi-compactness will turn out to be a sufficient criterion for passing direct limits through étale cohomology on a scheme. However, we also will need to consider the behavior of cohomology with respect to limits of geometric objects (such as $\mathbf{C}$ considered as a limit of finitely generated extensions of $\overline{\mathbf{Q}}$). For this purpose we require Grothendieck's theory of inverse limits for schemes, so let us summarize how Grothendieck's theory works. The basic example is $T = \operatorname{Spec} A$ with $A = \varinjlim A_\lambda$ a filtered direct limit of rings A_λ. We define $T_\lambda = \operatorname{Spec} A_\lambda$. There are natural compatible maps $T \to T_\lambda$, and so there is a map $T \to \varprojlim T_\lambda$ as topological spaces. Every ideal in A is the limit of its contractions to the A_λ's, and this procedure expresses prime ideals as limits of prime ideals and respects containments. Thus, in view of the combinatorial definition of the Zariski topology, we see that the continuous map $T \to \varprojlim T_\lambda$ is a homeomorphism. More importantly, for any scheme X the natural map of sets

$$T(X) \to \varprojlim T_\lambda(X)$$

is a bijection. Indeed, for affine $X = \operatorname{Spec} B$ this expresses the universal property of A as a direct limit of the rings A_λ, and we can globalize to any X by covering X with open affines U_i (and covering the overlaps $U_i \cap U_j$ with open affines U_{ijk}) to reduce to the result in the affine case. By universality, this construction commutes with restriction over an open affine in any T_{λ_0} (keeping only those $\lambda \geq \lambda_0$).

In the more general situation of a (cofiltered) inverse system of schemes $\{T_\lambda\}$ such that the transition maps $T_{\lambda'} \to T_\lambda$ are affine morphisms, the functor $X \rightsquigarrow \varprojlim T_\lambda(X)$ is represented by a scheme T such that the maps $T \to T_\lambda$ are affine and the topological-space map $T \to \varprojlim T_\lambda$ is a homeomorphism. Explicitly, we just choose some λ_0 and over any open affine U in T_{λ_0} the system of $T_\lambda|_U$'s for $\lambda \geq \lambda_0$ fits into the affine situation already treated; this gives a solution over U and it is straightforward to glue these over all U's in T_{λ_0} to make a global solution T with the desired properties. The scheme T is denoted $\varprojlim T_\lambda$. The theory of such inverse limits of schemes is exhaustively developed in [**17**, §8*ff*].

Let $\{T_\lambda\}$ be an inverse system of quasi-compact and quasi-separated schemes with affine transition maps, and let T be the inverse limit. Since T is affine over any T_{λ_0}, it is quasi-compact and quasi-separated. The general principle in the theory of inverse limits of schemes is that any finitely presented construction over T should descend over some T_λ, with any two descents becoming isomorphic over $T_{\lambda'}$ for some $\lambda' \geq \lambda$ (with any two such isomorphisms becoming equal over $T_{\lambda''}$ for some $\lambda'' \geq \lambda'$), and that any finitely presented situation over T_λ that acquires a reasonable property (flatness, surjectivity, properness, *etc.*) over T should acquire that property over $T_{\lambda'}$ for some $\lambda' \geq \lambda$.

1.3.2. The direct limit formalism.

THEOREM 1.3.2.1 (Compatibility of cohomology and filtered limits I). *Let X be a quasi-compact and quasi-separated scheme, and $\{\mathscr{F}_\lambda\}$ a directed system of abelian sheaves on $X_{\text{ét}}$ with direct limit $\mathscr{F}$. The map*

$$\varinjlim_{\lambda} \mathrm{H}^i_{\text{ét}}(X, \mathscr{F}_\lambda) \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{F})$$

is an isomorphism for all i.

PROOF. We first treat degree-0 by a direct argument. Any étale cover of a quasi-compact and quasi-separated scheme U admits a finite refinement $\{U_i\}$ with each étale map $U_i \to U$ quasi-compact and quasi-separated, and so all finite fiber products among the U_i's are quasi-compact over U. This is the key point.

Let $\mathscr{G}$ be the presheaf $\mathscr{G}(U) = \varinjlim \mathscr{F}_\lambda(U)$ whose sheafification $\mathscr{G}^+$ is $\varinjlim \mathscr{F}_\lambda$. Since $\{\mathscr{F}_\lambda\}$ is filtered, we see that if an étale map $U \to X$ is quasi-compact and quasi-separated, and if $\mathfrak{U} = \{U_i\}$ is an étale cover of U by finitely many étale U-schemes U_i that are quasi-compact and quasi-separated over U, then $\mathscr{G}(U) = \mathrm{H}^0(\mathfrak{U}, \mathscr{G})$. Such covers $\mathfrak{U}$ are cofinal among étale covers of U, so the first step of the sheafification process for $\mathscr{G}$ does not change the values on quasi-compact and quasi-separated X-schemes U. Hence, the second step of the sheafification process does not change the values on such objects U. By taking $U = X$, we get the desired result in degree 0.

Since filtered direct limits commute with finite products and any étale cover of X admits a finite refinement $\mathfrak{U} = \{U_i\}$ such that all finite products among the U_i's are quasi-compact and quasi-separated, it follows from the degree-0 case that for a cofinal system of covers $\mathfrak{U}$, $\mathrm{H}^\bullet_{\text{ét}}(\mathfrak{U}, \cdot)$ commutes with filtered direct limits. By the theorem on exchange of iterated filtered limits, we conclude that $\check{\mathrm{H}}^\bullet_{\text{ét}}(X, \cdot)$ also commutes with filtered direct limits. It is not true in general that Čech cohomology agrees with étale cohomology, but we have a spectral sequence

$$\mathrm{E}_2^{p,q} = \check{\mathrm{H}}^p_{\text{ét}}(X, \underline{\mathrm{H}}^q_{\text{ét}}(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathscr{F}).$$

Thus, the problem of moving filtered direct limits through $\mathrm{H}^n_{\text{ét}}(X, \mathscr{F})$ is reduced to that of the $\mathrm{E}_2^{p,q}$-terms for $p+q = n$. Since $\mathrm{E}_2^{0,n} = 0$, we may suppose $q < n$. We also have $\mathrm{E}_2^{n,0} = \check{\mathrm{H}}^n_{\text{ét}}(X, \mathscr{F})$, so we may suppose $p > 0$. This settles $n = 1$, and so we may induct (with an inductive hypothesis that is quantified over all quasi-compact and quasi-separated étale X-schemes). □

If P is a set of primes, let us define a *P-sheaf* to be a torsion sheaf whose torsion-orders are not divisible by primes in P.

THEOREM 1.3.2.2 (Noetherian descent). *Let S be a quasi-separated and quasi-compact scheme, and let P be a set of primes.*

1. *There exists an inverse system of noetherian schemes $\{S_\lambda\}$ with affine transition maps such that $S \simeq \varprojlim S_\lambda$, and the S_λ's may be taken to be finite type over $\mathbf{Z}$. For any such system $\{S_\lambda\}$ and any P-sheaf $\mathscr{F}$ on $S_{\text{ét}}$, $\mathscr{F}$ is a direct limit of P-sheaves $\mathscr{F}_i$ such that each $\mathscr{F}_i$ a pullback of a constructible P-sheaf on some $(S_\lambda)_{\text{ét}}$.*

(2) *For any quasi-separated finite-type S-scheme X there exists a closed immersion $X \hookrightarrow \overline{X}$ into a finitely presented S-scheme, and if X is separated over S then $\overline{X}$ can be chosen to be separated over S.*
(3) *In (2), there exists λ_0 and a finite type S_{λ_0}-scheme $\overline{X}_{\lambda_0}$ whose base change to S is $\overline{X}$. Defining $\overline{X}_\lambda = S_\lambda \times_{S_{\lambda_0}} \overline{X}_{\lambda_0}$ for $\lambda \geq \lambda_0$, the map $\overline{X}_\lambda \to S_\lambda$ is separated for some large λ if and only if $\overline{X} \to S$ is separated; the same holds for properness.*

PROOF. The first assertion in (1) is [**25**, Thm. C9], and it proceeds via a nontrivial induction on the size of an affine open cover. The existence of $\overline{X}_{\lambda_0}$ in (3) is [**17**, 8.8.2], and the descent of the properties of separatedness and properness is a special case of [**17**, 8.10.5]. We shall use descent of finitely presented schemes to settle the second claim in (1), and then we will sketch the proof of (2).

Let $\mathscr{F}$ be a P-sheaf on $S_{\text{ét}}$. Consider triples $(\lambda, \mathscr{G}, \phi)$ where $\mathscr{G}$ is a constructible P-sheaf on S_λ and $\phi : \pi_\lambda^* \mathscr{G} \to \mathscr{F}$ is a map (with $\pi_\lambda : S \to S_\lambda$ the canonical map). These triples form a (small) category $C_{\mathscr{F}}$, where a *morphism*

$$(\lambda, \mathscr{G}, \phi) \to (\lambda', \mathscr{G}', \phi')$$

for $\lambda' \geq \lambda$ is a map $f : \pi_{\lambda',\lambda}^* \mathscr{G} \to \mathscr{G}'$ that satisfies $\phi' \circ \pi_{\lambda'}^*(f) = \phi$. There is an evident diagram $D_{\mathscr{F}}$ of P-sheaves $\pi_\lambda^* \mathscr{G}$ on $S_{\text{ét}}$ indexed by the category C, and the direct limit $\mathscr{L}(\mathscr{F})$ of this diagram is a P-sheaf on $S_{\text{ét}}$ equipped with a natural map $\mathscr{L}(\mathscr{F}) \to \mathscr{F}$. Since $\mathscr{L}(\mathscr{F})$ is a filtered direct limit of the finite limits over finite subdiagrams in $D_{\mathscr{F}}$, to prove (1) it is enough to show that $\mathscr{L}(\mathscr{F}) \to \mathscr{F}$ is an isomorphism.

For surjectivity, we note that any section s of $\mathscr{F}$ is locally the image of a map $\phi : i_!(\mathbf{Z}/n\mathbf{Z}) \to \mathscr{F}$ for some étale map $i : U \to S$ with U étale over an open affine in S. Since $i : U \to S$ is finitely presented (it is quasi-separated because of the hypotheses on S), it descends to an finite-type map $i_\lambda : U_\lambda \to S_\lambda$ for some λ, and by [**17**, 17.7.8] we may increase λ to make this map étale. The pullback of $i_{\lambda !}(\mathbf{Z}/n\mathbf{Z})$ to S is $i_!(\mathbf{Z}/n\mathbf{Z})$. Thus, $(\lambda, i_{\lambda !}(\mathbf{Z}/n\mathbf{Z}), \phi)$ is an object in $C_{\mathscr{F}}$, and so we see that $\mathscr{F}' \to \mathscr{F}$ is surjective. For injectivity, it suffices to show that for any object $(\lambda, \mathscr{G}, \phi)$ in $C_{\mathscr{F}}$, the kernel of $\phi : \pi_\lambda^* \mathscr{G} \to \mathscr{F}$ is killed by the natural map $\pi_\lambda^* \mathscr{G} \to \mathscr{L}(\mathscr{F})$. A section in the kernel is locally the image of a map $i'_!(\mathbf{Z}/n\mathbf{Z}) \to \pi_\lambda^* \mathscr{G}$ with $i' : U' \to S$ an étale map factoring through an open affine in S. We may increase λ so that (U', i') descends to $i'_\lambda : U'_\lambda \to S_\lambda$. Thus, we obtain another object $(\lambda, i'_{\lambda !}(\mathbf{Z}/n\mathbf{Z}), 0)$ in $C_{\mathscr{F}}$ equipped with a map to $(\lambda, \mathscr{G}, \phi)$, and we want $i'_!(\mathbf{Z}/n\mathbf{Z}) \to \mathscr{L}(\mathscr{F})$ to vanish. There is an evident morphism

$$(\lambda, i'_{\lambda !}(\mathbf{Z}/n\mathbf{Z}), 0) \to (\lambda, 0, 0)$$

in $C_{\mathscr{F}}$, and this completes the proof of injectivity.

It remains to prove (2). Suppose we can construct a closed immersion of X into some finitely presented S-scheme $\overline{X}$. Since $\overline{X}$ is quasi-compact and quasi-separated we may express the quasi-coherent ideal $\mathscr{I}$ of X on $\overline{X}$ as the direct limit of its finitely generated quasi-coherent subideals $\mathscr{I}_\alpha$ [**17**, 1.7.7]. Thus, the zero-schemes $\overline{X}_\alpha$ of the $\mathscr{I}_\alpha$ are an inverse system of finitely presented S-schemes with affine transition maps and inverse limit X. When the limit object X is separated over S, then we claim that some $\overline{X}_\alpha$ must be separated over S, and so may be used in the role of $\overline{X}$. This assertion is local on S, so we may assume S is affine, and then it is [**25**, Prop. C7]. The construction of $\overline{X}$ is a tedious exercise with the techniques in [**25**, App. C], so we only describe the basic principle that generalizes the observation that any finite-type algebra is a direct limit of finitely presented algebras (express an ideal of relations as a limit of finitely generated subideals):

If X is affine and lies over an open affine U in S, we can take $\overline{X} = \mathbf{A}^n_{U'}$ for some n. In general we may cover X by r such open affines, and we induct on r. This immediately reduces to the following general situation: X is covered by quasi-compact (and hence finite type) opens U and V such that there exist closed immersions $U \hookrightarrow \overline{U}$ and $V \hookrightarrow \overline{V}$ into finitely presented S-schemes. Since $U \cap V$ is quasi-compact, there exist quasi-compact opens U' and V' in $\overline{U}$ and $\overline{V}$ such that $U' \cap U = U \cap V$ and $V' \cap V = U \cap V$. We express the ideals of U and V as limits of finitely generated quasi-coherent subideals, and study these ideals to find finitely presented closed subschemes in $\overline{U}$ and $\overline{V}$ containing U and V such that these finitely presented closed subschemes can be glued along suitable opens that thicken the quasi-compact $U \cap V$, with this gluing providing the desired $\overline{X}$ (after appropriate further shrinking around U and V). $\square$

Theorem 1.3.2.3 (Compatibility of cohomology and filtered limits II). *Let $\{T_\lambda\}$ be an inverse system of quasi-compact and quasi-separated schemes, with affine transition maps $\pi_{\lambda,\lambda'} : T_{\lambda'} \to T_\lambda$ whenever $\lambda' \geq \lambda$. Let $T = \varprojlim T_\lambda$. Let $\{\mathscr{F}_\lambda\}$ be a compatible family of abelian étale sheaves the T_λ's in the sense that we are given transitive maps $\pi^*_{\lambda',\lambda}\mathscr{F}_\lambda \to \mathscr{F}_{\lambda'}$ for all $\lambda' \geq \lambda$. Define $\mathscr{F}$ to be the direct limit of the pullbacks of these sheaves to $T_{\text{ét}}$. The natural map $\varinjlim_\lambda \mathrm{H}^i_{\text{ét}}(T_\lambda, \mathscr{F}_\lambda) \to \mathrm{H}^i_{\text{ét}}(T, \mathscr{F})$ is an isomorphism for all i.*

The most important instance of this theorem is when the transition maps on sheaves are isomorphisms, so $\mathscr{F}$ is the pullback of each $\mathscr{F}_\lambda$.

Proof. Since T is quasi-compact and quasi-separated (it is even affine over any T_λ), considerations with finite presentations imply that any étale cover of T admits a finite refinement that descends to an étale cover of some T_λ. This fact enables us to carry over the spectral-sequence argument via Čech cohomology, as in the proof of Theorem 1.3.2.1, provided the case of degree 0 is settled.

For degree 0, we apply the limit methods in [**17**, §8*ff*] as follows. Consider an étale map $U \to T$ that is quasi-compact and quasi-separated. This descends to an étale map $U_{\lambda_0} \to T_{\lambda_0}$ that is quasi-compact and quasi-separated; moreover, any two such descents become isomorphic under base change to some $T_{\lambda'}$ with $\lambda' \geq \lambda_0$, and two such isomorphisms over $T_{\lambda'}$ become equal over $T_{\lambda''}$ for some $\lambda'' \geq \lambda'$. Thus, if we define $U_\lambda = U_{\lambda_0} \times_{T_{\lambda_0}} T_\lambda$ for $\lambda \geq \lambda_0$ then the group

$$\mathscr{G}(U) \stackrel{\text{def}}{=} \varinjlim \mathscr{F}_\lambda(U_\lambda)$$

is functorially independent of all choices. The strategy is to prove that $\mathscr{G}$ uniquely extends to a sheaf on $T_{\text{ét}}$. This will be done by building up to $T_{\text{ét}}$ through two full subcategories that admit their own étale topologies.

We may view $\mathscr{G}$ as a presheaf on the full subcategory $T^{\text{qcqs}}_{\text{ét}}$ in $T_{\text{ét}}$ whose objects are étale T-schemes U with quasi-compact and quasi-separated structure map to T. All morphisms in $T^{\text{qcqs}}_{\text{ét}}$ are quasi-compact and quasi-separated (as well as étale), so $T^{\text{qcqs}}_{\text{ét}}$ is stable under finite fiber products in $T_{\text{ét}}$. Thus, $T^{\text{qcqs}}_{\text{ét}}$ may be endowed with an étale topology in the evident manner, and left-exactness considerations show that $\mathscr{G}$ is a sheaf for this topology. Let $T^{\text{qs}}_{\text{ét}}$ be the full subcategory in $T_{\text{ét}}$

consisting of étale T-schemes with quasi-separated structure map. All morphisms in this category are quasi-separated, so $T_{\text{ét}}^{\text{qs}}$ is stable under finite fiber products in $T_{\text{ét}}$. Thus, $T_{\text{ét}}^{\text{qs}}$ admits a natural étale topology. Every T' in $T_{\text{ét}}^{\text{qs}}$ admits an étale cover $\{U_i\}$ by objects in $T_{\text{ét}}^{\text{qcqs}}$, and the T-maps $U_i \to T'$ are necessarily quasi-compact and quasi-separated [**17**, 1.2.2, 1.2.4], so fiber products of the U_i's over T' lie in $T_{\text{ét}}^{\text{qcqs}}$; roughly speaking, the étale topology on $T_{\text{ét}}^{\text{qs}}$ is generated by $T_{\text{ét}}^{\text{qcqs}}$.

Since the $T_{\text{ét}}^{\text{qcqs}}$-covers of any $U \in T_{\text{ét}}^{\text{qcqs}}$ are cofinal among all étale covers of U in $T_{\text{ét}}$, we conclude that there is a unique étale sheaf $\mathscr{G}'$ on $T_{\text{ét}}^{\text{qs}}$ restricting to $\mathscr{G}$ on $T_{\text{ét}}^{\text{qcqs}}$. We now wish to repeat the process to uniquely extend $\mathscr{G}'$ to a sheaf $\mathscr{G}''$ on $T_{\text{ét}}$. This can be done because every T' in $T_{\text{ét}}$ has a cofinal system of covers $\{U_i\}$ with each U_i a separated étale scheme over an open affine in T' that lies over an open affine in T (so all fiber products among U_i's over T' are T-separated and hence lie in $T_{\text{ét}}^{\text{qs}}$). In brief, we have proved that the sites $T_{\text{ét}}^{\text{qcqs}}$ and $T_{\text{ét}}$ give rise to the same topos.

There is a unique map $\mathscr{G}'' \to \mathscr{F}$ extending the evident maps on U-points for U in $T_{\text{ét}}^{\text{qcqs}}$, and local section-chasing shows that this map of sheaves is an isomorphism. Thus, $\mathscr{F}(T) = \mathscr{G}''(T) = \mathscr{G}(T) = \varinjlim \mathscr{F}_\lambda(T_\lambda)$. $\square$

1.3.3. Cohomology as a Galois module. As an application of the compatibility of cohomology and limits, we can analyze some interesting Galois-modules that arise from étale cohomology on schemes over a field K. Let $f : X \to \operatorname{Spec} K$ be a map, let K_s/K be a separable closure, and let $\mathscr{F}$ be an abelian étale sheaf on X. Since higher direct images may be identified with sheafified cohomology, the compatibility of cohomology and direct limits implies that the geometric stalk of the étale sheaf $(\mathrm{R}^i f_* \mathscr{F})_{K_s}$ on $(\operatorname{Spec} K_s)_{\text{ét}}$ is

$$\varinjlim_{K'} \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K', \mathscr{F}_{K'}\big) \simeq \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathscr{F}_{K_s}\big), \tag{1.3.3.1}$$

where the limit is taken over finite extensions of K inside of K_s. Note also that we have a canonical left action of $\operatorname{Gal}(K_s/K)$ on (1.3.3.1) via
(1.3.3.2)

$$\begin{array}{ccc} \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathscr{F}_{K_s}\big) & \xrightarrow{(1\times g)^*} & \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, (1\times g)^* \mathscr{F}_{K_s}\big) \\ & & \downarrow \simeq \\ & & \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathscr{F}_{K_s}\big) \end{array}$$

for $g \in \operatorname{Gal}(K_s/K)$, where the second step uses the commutative diagram

$$\begin{tikzcd} X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s \arrow[rr, "1\otimes g"] \arrow[dr] & & X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s \arrow[dl] \\ & X & \end{tikzcd}$$

Theorem 1.3.3.1. *The* $\operatorname{Gal}(K_s/K)$*-action in* (1.3.3.2) *is the same as that coming from the structure of* $\mathrm{R}^i f_* \mathscr{F}$ *as an étale sheaf on* $\operatorname{Spec} K$.

Remark 1.3.3.2. This theorem includes the implicit assertion (that can be checked directly) that the description (1.3.3.2) is continuous with respect to the discrete topology. It is useful that we can encode this $\mathrm{Gal}(K_s/K)$-action intrinsically in terms of the étale sheaf $\mathrm{R}^i f_* \mathscr{F}$, without having to mention K_s or work with K_s-schemes.

PROOF. To verify the compatibility between (1.3.3.2) and the $\mathrm{Gal}(K_s/K)$-action on $(\mathrm{R}^i f_* \mathscr{F})_{K_s}$, we consider the canonical group isomorphism

$$\varinjlim_{K'} \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K', \mathscr{F}_{K'}\big) \to \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathscr{F}_{K_s}\big) = (\mathrm{R}^i f_* \mathscr{F})_{K_s}$$

where K'/K runs over finite Galois extensions inside of K_s. Using functoriality in K_s, this identifies the abstract $\mathrm{Gal}(K_s/K)$-action on $\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathscr{F}_{K_s})$ with that induced by the (compatible) actions on $\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec} K} \operatorname{Spec} K', \mathscr{F}_{K'})$ by $\mathrm{Gal}(K'/K)$ for all K'. This concrete action on the direct limit is exactly the action arising on the Galois module associated to the stalk of the étale sheaf $\mathrm{R}^i f_* \mathscr{F}$, since this higher direct image sheaf on $(\operatorname{Spec} K)_{\text{ét}}$ is naturally identified with the sheafification of the presheaf

$$U \mapsto \mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec} K} U, \mathscr{F}_U)$$

on $(\operatorname{Spec} K)_{\text{ét}}$, where $\mathscr{F}_U$ denotes the pullback of $\mathscr{F}$ to $X \times_{\operatorname{Spec} K} U$. $\square$

1.3.4. Proper base change theorem. One of the most important theorems in the theory of étale cohomology is the analogue of topological proper base change:

THEOREM 1.3.4.1 (Proper base change). *Let S be a scheme, let $f : X \to S$ be a proper map, and let $\mathscr{F}$ be a torsion abelian étale sheaf on X. For any cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

the natural base-change map $g^* \mathrm{R}^i f_* \mathscr{F} \to \mathrm{R}^i f'_* (g'^* \mathscr{F})$ is an isomorphism.

The first step in the proof uses limit methods and noetherian descent to reduce to the case when $S = \operatorname{Spec} R$ for a strictly henselian local noetherian ring R, the map $g : S' \to S$ is the inclusion of the closed point $\{s\}$, and $\mathscr{F}$ is a constructible $\mathbf{Z}/n\mathbf{Z}$-module. The natural map $\mathrm{H}^i_{\text{ét}}(X, \mathscr{F}) \to \mathrm{H}^i_{\text{ét}}(X_s, \mathscr{F}_s)$ must be shown to be an isomorphism in this case. As is explained in [**9**, Ch. I, 6.1], a combination of limit techniques and arguments with spectral sequences and fibrations reduces this general problem to the special case $\mathscr{F} = \mathbf{Z}/n\mathbf{Z}$ and $i \le 1$. This special case is the heart of the argument, and so we shall now explain how it is proved.

The basic principle is that H^0 and H^1 with constant coefficients have concrete meaning in terms of π_0 and π_1: these low-degree cohomologies measure connectivity and torsors. It is these explicit interpretations that will be studied in the proof.

THEOREM 1.3.4.2. *Let $X \to \operatorname{Spec} R$ be a proper, with R a henselian noetherian local ring. Let X_0 denote the closed fiber. The pullback map*

$$\mathrm{H}^i_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{H}^i_{\text{ét}}(X_0, \mathbf{Z}/n\mathbf{Z})$$

is an isomorphism for all $n > 0$ and for $i = 0, 1$.

PROOF. For noetherian S, sections in $\mathrm{H}^0_{\text{ét}}(S, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}_S(S, \coprod_{j \in \mathbf{Z}/n\mathbf{Z}} S)$ assign elements $j_\alpha \in \mathbf{Z}/n\mathbf{Z}$ to each (closed and open) connected component S_α of S. Thus, for $i = 0$ the theorem says that X and its closed fiber X_0 have the same number of connected components. To establish this connectivity claim, we may assume the proper X is non-empty, so $X_0 \neq \emptyset$, and we must show that the primitive idempotents of $\mathrm{H}^0(X, \mathscr{O}_X) \simeq \mathrm{H}^0(\mathrm{Spec}\, R, f_*\mathscr{O}_X)$ and $\mathrm{H}^0(X_0, \mathscr{O}_{X_0})$ correspond under reduction. Since $\mathrm{H}^0(\mathrm{Spec}\, R, f_*\mathscr{O}_X)$ is finite over the henselian local R, it is a finite product of local R-algebras. Such a product decomposition gives rise to a disjoint open decomposition of X, so upon reducing to the case of connected X we conclude that $A = \mathrm{H}^0(X, \mathscr{O}_X)$ must be local, and we need to prove that X_0 is connected.

If $\widehat{R}$ denotes the completion of R, then the finite local R-algebra A has the property that $\widehat{R} \otimes_R A = \widehat{A}$ is the completion of A and hence is local. By faithful flatness of $\widehat{R}$ over R we know that $\widehat{A} = \mathrm{H}^0(X', \mathscr{O}_{X'})$, where $X' = X \otimes_R \widehat{R}$. Thus, the ring of global functions $\widehat{A}$ on X' is local and hence has no non-trivial idempotents, and so X' is connected. Since X' has the same closed fiber as X, we have reduced our connectedness problem to the case when R is complete. In this case, a separation of X_0 would uniquely lift to compatible separations of each infinitesimal closed fiber $X_m \stackrel{\text{def}}{=} X \bmod \mathfrak{m}_R^{m+1}$, and so would give rise to compatible non-trivial idempotents in $\mathrm{H}^0(X_m, \mathscr{O}_{X_m})'s$. By the theorem on formal functions, these define a non-trivial idempotent in $\mathrm{H}^0(X, \mathscr{O}_X)$, and hence a separation of X, contrary to hypothesis. Thus, X_0 must be connected. This settles the case $i = 0$. In particular, to check the case $i = 1$ we may assume that X and X_0 are both connected. The use of infinitesimal neighborhoods of X_0 in the preceding argument is analogous to the use of tubular neighborhoods of fibers in the proof of the topological proper base change theorem.

For $i = 1$, the computation of constant-coefficient H^1 in terms of π_1 translates our problem into showing that every étale $\mathbf{Z}/n\mathbf{Z}$-torsor Y_0 over X_0 is induced by a unique étale $\mathbf{Z}/n\mathbf{Z}$-torsor over X. Such a torsor Y_0 is a degree-n finite étale cover of X_0. As a first step we change the base to $\widehat{R}$ and we will construct such a unique torsor over $X \otimes_R \widehat{R}$. By [**17**, 18.3.4], if Z is a proper scheme over a complete local noetherian ring A then the functor $(\cdot) \times_A A/\mathfrak{m}$ is an equivalence of categories between finite étale covers of Z and finite étale covers of its closed fiber. Since an étale $\mathbf{Z}/n\mathbf{Z}$-torsor over a scheme Z is a finite étale Z-scheme Z' equipped with an action of $\mathbf{Z}/n\mathbf{Z}$ such that the action map $(\mathbf{Z}/n\mathbf{Z}) \times Z' \to Z' \times_Z Z'$ of finite étale Z-schemes is an isomorphism, we conclude that $Y_0 \to X_0$ uniquely lifts to an étale $\mathbf{Z}/n\mathbf{Z}$-torsor over $X \otimes_R \widehat{R}$.

Thus, there is bijectivity on $\mathrm{H}^1_{\text{ét}}$'s when we replace X with $X \otimes_R \widehat{R}$, and so for injectivity on $\mathrm{H}^1_{\text{ét}}$'s in general it suffices to show that any two étale $\mathbf{Z}/n\mathbf{Z}$-torsors over on X that are isomorphic over $X \otimes_R \widehat{R}$ must be isomorphic. Equivalently,

$$\mathrm{Hom}_{\text{cont}}(\pi_1(X), \mathbf{Z}/n\mathbf{Z}) \to \mathrm{Hom}_{\text{cont}}(\pi_1(X \otimes_R \widehat{R}), \mathbf{Z}/n\mathbf{Z}) \tag{1.3.4.1}$$

is injective (for compatible choices of base points). Here we are implicitly using the connectivity of $X \otimes_R \widehat{R}$, and this connectivity follows from the case $i = 0$ applied over the henselian ring $\widehat{R}$ (and the fact that $X \otimes_R \widehat{R}$ has closed fiber X_0 that is connected by assumption).

To see the injectivity of (1.3.4.1), it suffices to show that $\pi_1(X \otimes_R \widehat{R}) \to \pi_1(X)$ is surjective. The connectivity criterion via π_1's translates this into checking that if

$Y \to X$ is a connected finite étale cover, then the finite étale cover $Y \times_X (X \otimes_R \widehat{R}) = Y \otimes_R \widehat{R}$ of $X \otimes_R \widehat{R}$, is still connected. For this we may apply the $i = 0$ case to $Y \to \operatorname{Spec} R$ to deduce that Y_0 is connected, and then applying the $i = 0$ case to $Y \otimes_R \widehat{R}$ (whose closed fiber is Y_0) we conclude that $Y \otimes_R \widehat{R}$ is connected.

Now that we have uniqueness, we need to check the existence of liftings of torsors; this is a problem if R is not complete. Fix an étale $\mathbf{Z}/n\mathbf{Z}$-torsor Y_0 over X_0. Write $R = \varinjlim R_i$ as a direct limit of finite type $\mathbf{Z}$-algebras. Since R is henselian local, if R_i' is the henselization of the localization of R_i at the contraction of the maximal ideal of R, then R is also the direct limit of the R_i''s. The R-scheme X is proper and finitely presented (since R is noetherian), so we can apply the extensive direct limit techniques from [**17**, §8–§11] to ensure that for some i_0 there exists a proper R'_{i_0}-scheme X_{i_0} that gives rise to X via base change to R. Let $X_i = X_{i_0} \times_{R'_{i_0}} R'_i$ for $i \geq i_0$. By viewing X_0 as a limit of closed fibers of X_i's for $i \geq i_0$, the same limit techniques applied to $Y_0 \to X_0$ show that we may bring down the chosen étale $\mathbf{Z}/n\mathbf{Z}$-torsor Y_0 to one over the closed fiber of some X_{i_1} with $i_1 \geq i_0$. Thus, our initial setup descends to R'_{i_1}, and so to prove existence over R it suffices to work over R'_{i_1} (since we can apply a base change).

In other words, it suffices to assume that R is the henselization of a $\mathbf{Z}$-algebra essentially of finite type. To summarize, we have a proper map $X \to \operatorname{Spec} R$, and we have an étale $\mathbf{Z}/n\mathbf{Z}$-torsor $Y_0 \to X_0$; we want to lift this to an étale $\mathbf{Z}/n\mathbf{Z}$-torsor on X. We have seen how to make a unique étale $\mathbf{Z}/n\mathbf{Z}$-torsor $\widehat{Y}$ over $X \otimes_R \widehat{R}$ that lifts $Y_0 \to X_0$. Thus, given such a cover $\widehat{Y}$ we wish to find an étale $\mathbf{Z}/n\mathbf{Z}$-torsor $Y \to X$ with the same closed fiber over R. To find such a Y, express $\widehat{R}$ as the direct limit of its finitely generated R-subalgebras; the direct limit techniques from [**17**, §8–§11] ensure that there exists a finite type R-subalgebra $A \subseteq \widehat{R}$ and an étale $\mathbf{Z}/n\mathbf{Z}$-torsor $Y_A \to X \otimes_R A$ whose base change by $A \to \widehat{R}$ is $\widehat{Y}$. Write $A = R[T_1, \ldots, T_r]/(f_1, \ldots, f_m)$, so the (inclusion) map $g : A \to \widehat{R}$ corresponds to a set of solutions of the polynomial equations $f_1 = \cdots = f_m = 0$ in $\widehat{R}$. If we can find a solution to this system in R that induces the same solution in the residue field, then we would get a map $s : A \to R$ over R with $A \to R \to \widehat{R}$ and $g : A \to \widehat{R}$ inducing the same composite to the residue field. Clearly $Y_A \times_{\operatorname{Spec} A, s} \operatorname{Spec} R$ would be the desired $\mathbf{Z}/n\mathbf{Z}$-cover of X, realizing the closed-fiber cover Y_0.

Thus, we want to show that for a ring R that is the henselization of an local ring essentially of finite type over $\mathbf{Z}$, if we are given a solution in $\widehat{R}$ to a finite system of polynomials over R then there exists a solution in R with the same image in $R/\mathfrak{m} \simeq \widehat{R}/\mathfrak{m}$. This is a special case of Artin approximation [**3**, 3.6/16]. $\square$

1.3.5. Smooth base change and vanishing cycles. We now discuss the étale analogue of the homotopy-invariance of the cohomology of local systems in topology. Let us first revisit the topological situation to motivate what to prove in the étale topology.

Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

of topological spaces, and assume that g is *smooth* in the sense that each $s' \in S'$ has an neighborhood that is homeomorphic to $U \times B$ with $U \subseteq S$ a neighborhood of $g(s')$ and $B = [0,1]^n$.

THEOREM 1.3.5.1 (Topological smooth base change). *For any abelian sheaf $\mathscr{F}$ on X, the natural base-change morphisms*

$$g^*\mathrm{R}^i f_*\mathscr{F} \to \mathrm{R}^i f'_*({g'}^*\mathscr{F})$$

are isomorphisms when $\mathscr{F}$ is a local system on X.

PROOF. Working locally on S' allows us to suppose $S' = S \times B$, and since higher direct images are sheafifications of cohomology it is enough to prove that the pullback $\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X \times B, {g'}^*\mathscr{F})$ is an isomorphism. This pullback map is a section to any pullback map

$$\mathrm{H}^i(X \times B, {g'}^*\mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F})$$

to a fiber $X \times \{b\}$ for any $b \in B$, and these fibral-pullbacks are isomorphisms because $X \times \{b\}$ is a deformation retract of $X \times B$ and the cohomology of local systems is invariant under deformation retract (Corollary **??**). Thus, the topological smooth base change theorem is a rephrasing of the homotopy-invariance of topological cohomology. □

Another viewpoint on this homotopy calculation arises in the theory of vanishing cycles. Here is the basic setup. Let $p : Y \to I = [0,1]$ be a continuous map, and let $p_\eta : Y_\eta \to I_\eta$ be the restriction over $I_\eta = I - \{0\}$. Let us determine whether the maps $\mathrm{H}^\bullet(Y, \underline{\mathbf{Z}}) \to \mathrm{H}^\bullet(Y_\eta, \underline{\mathbf{Z}})$ are isomorphisms. To analyze these maps, we will realize them as edge maps in a spectral sequence. Let $\eta : I_\eta \hookrightarrow I$ and $j : Y_\eta \hookrightarrow Y$ be the natural maps. We have a Leray spectral sequence

$$\mathrm{E}_2^{r,s} = \mathrm{H}^r(Y, \mathrm{R}^s j_*\underline{\mathbf{Z}}) \Rightarrow \mathrm{H}^{r+s}(Y_\eta, \underline{\mathbf{Z}}).$$

The relevance of homotopy-invariance, as we will see, is that when p is smooth along Y_0 then the $\mathrm{R}^s j_*\underline{\mathbf{Z}}$'s are easily computed and vanish for $s > 0$.

For any sheaf $\mathscr{F}$ on Y, its *qth vanishing-cycles sheaf* $\mathrm{R}^q\Psi_\eta(\mathscr{F})$ is defined to be $(\mathrm{R}^q j_*\mathscr{F})|_{Y_0}$. Since $(\mathrm{R}^q j_*\mathscr{F})|_{Y_\eta} = 0$ for $q > 0$, for positive q we see that $\mathrm{R}^q j_*(\mathscr{F})$ is the pushforward of $\mathrm{R}^q\Psi_\eta(\mathscr{F})$ along the closed embedding $Y_0 \hookrightarrow Y$. There is a natural map $\underline{\mathbf{Z}} \to \mathrm{R}^0\Psi_*(\underline{\mathbf{Z}})$ on Y_0 and we shall assume that this is an isomorphism, or equivalently that $\underline{\mathbf{Z}} \to j_*\underline{\mathbf{Z}}$ on Y is an isomorphism; intuitively, this says that the removal of Y_0 does not locally disconnect small connected opens in Y around points in Y_0, and it holds if p is smooth near points in Y_0. Thus, $\mathrm{E}_2^{r,0} = \mathrm{H}^r(Y, j_*\underline{\mathbf{Z}}) = \mathrm{H}^r(Y, \underline{\mathbf{Z}})$, and so the edge map $\mathrm{E}_2^{n,0} \to \mathrm{H}^n(Y_\eta, \underline{\mathbf{Z}})$ is the restriction map on cohomology. This implies that the obstruction to restriction being an isomorphism is concentrated in the terms $\mathrm{H}^r(Y, \mathrm{R}^s j_*\underline{\mathbf{Z}}) \simeq \mathrm{H}^r(Y_0, \mathrm{R}^s\Psi_\eta\underline{\mathbf{Z}})$ with $s > 0$. For example, if the vanishing-cycles sheaves vanish in positive degrees, then there is no obstruction.

For any $y \in Y_0$, the stalk $(\mathrm{R}^s\Psi_\eta\underline{\mathbf{Z}})_y = (\mathrm{R}^s j_*\underline{\mathbf{Z}})_y$ is the direct limit of the cohomologies $\mathrm{H}^s(U - U_0, \underline{\mathbf{Z}})$ over small opens $U \subseteq Y$ around y, so these y-stalks vanish if p is smooth near y. Indeed, if y has a neighborhood of the form $[0,t] \times B$ with $B = [0,1]^n$ then for $s > 0$ we see via homotopy-invariance that

$$\mathrm{H}^s(U - U_0, \underline{\mathbf{Z}}) = \mathrm{H}^s((0,t], \underline{\mathbf{Z}}) = 0$$

for a cofinal system of U's around y; equivalently, we may use topological smooth base change to identify $\mathrm{R}^s j_* \underline{\mathbf{Z}}$ near such a point with $p^*\mathrm{R}^s\eta_*\underline{\mathbf{Z}}$, and $(\mathrm{R}^s\eta_*\underline{\mathbf{Z}})_0 = \varinjlim \mathrm{H}^s((0,t],\underline{\mathbf{Z}}) = 0$ for $s>0$. In brief, this says that the cohomology of the total space of a family $p : Y \to [0,1]$ is unaffected by the removal of Y_0 when p is smooth along Y_0, and the reason is that the vanishing-cycles sheaves $\mathrm{R}^s\Psi_\eta\underline{\mathbf{Z}}$ for such a family are $\underline{\mathbf{Z}}$ for $s=0$ and zero for $s>0$.

Here is the étale analogue of homotopy-invariance for topological cohomology.

THEOREM 1.3.5.2 (Smooth base change). *Let S be a scheme, $f : X \to S$ a morphism, and $\mathscr{F}$ a torsion abelian étale sheaf on X with torsion-orders invertible on S. Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

with $S' = \varprojlim S_\lambda$ where $\{S_\lambda\}$ is an inverse system of smooth S-schemes such that the transition maps $S_{\lambda'} \to S_\lambda$ are affine. The natural base-change maps $g^*\mathrm{R}^i f_*\mathscr{F} \to \mathrm{R}^i f'_*(g'^*\mathscr{F})$ are isomorphisms.

REMARK 1.3.5.3. An interesting case of a limit of smooth maps is a separable algebraic extension or a purely transcendental extension $K \to K'$ of fields. Combining this with the remarks at the end of §1.1.6, we conclude if K/k is an extension of separably closed fields and X is a k-scheme, then $\mathrm{H}^i_{\text{ét}}(X,\mathscr{F}) \to \mathrm{H}^i_{\text{ét}}(X_K,\mathscr{F}_K)$ is an isomorphism for any torsion sheaf $\mathscr{F}$ whose torsion orders are relatively prime to the characteristic. For our purposes, the most important example of such an extension is $\overline{\mathbf{Q}} \to \mathbf{C}$.

To prove the smooth base change theorem, consider the stalks at a geometric point $\overline{s}'$ of S' over a geometric point $\overline{s}$ of S. We may replace S and S' with their strict henselizations at these points, and we need to prove that the natural map $\mathrm{H}^i_{\text{ét}}(X,\mathscr{F}) \to \mathrm{H}^i_{\text{ét}}(X',\mathscr{F}')$ is an isomorphism. The compatibility of cohomology and limits reduces us to the case when S' is a strict henselization at a geometric point on the closed fiber of a smooth affine S-scheme T. The smooth affine pair (S,T) may be realized as the base change of a smooth affine pair (S_0,T_0) with S_0 a noetherian strictly henselian local ring. Thus, we can assume S and S' are noetherian (and affine). The spectral sequence for an affine covering reduces the problem to the case when $X = \operatorname{Spec} B$ is affine. By expressing B as a rising union of finitely generated subalgebras over the coordinate ring of S, noetherian descent allows us to assume that X is finite type over S and $\mathscr{F}$ is constructible. This reduces the general case to the special case when S' is S-smooth and $\mathscr{F}$ is a $\mathbf{Z}/n\mathbf{Z}$-sheaf.

We refer the reader to [**6**, Arcata, Ch. V] for a proof in this case, based on the principles of homotopy-invariance; the role of the opens $U - U_0$ for small U around $y \in Y_0$ in our topological analysis is replaced by *vanishing-cycles schemes*

$$\widetilde{Y}_\eta^{\overline{y}} \stackrel{\text{def}}{=} \operatorname{Spec} \mathscr{O}^{\text{sh}}_{Y,\overline{y}} \times_{\operatorname{Spec} \mathscr{O}^{\text{sh}}_{S,\overline{s}}} \eta$$

where $Y \to S$ is a map of schemes, $\overline{y}$ is a geometric point of Y over a geometric point $\overline{s}$ of S, and η is a geometric point of $\operatorname{Spec} \mathscr{O}^{\text{sh}}_{S,\overline{s}}$. The intuition is that the (strictly henselian) irreducible component in $\operatorname{Spec} \mathscr{O}^{\text{sh}}_{S,\overline{s}}$ dominated by η replaces

$[0,t]$ for small t, and η replaces $(0,t]$. With this point of view, the analogue of our local analysis of vanishing cycles in the smooth topological case (that rested on homotopy-invariance) is the claim that when n is invertible on S and Y is S-smooth, then for all $\overline{s}$, $\overline{y}$, and η as above, the cohomology $\mathrm{H}^q_{\text{ét}}(\widetilde{Y}^{\overline{y}}_{\eta}, \mathbf{Z}/n\mathbf{Z})$ vanishes for $q > 0$ and is equal to $\mathbf{Z}/n\mathbf{Z}$ (via the canonical map) when $q = 0$. This local triviality property for constant-coefficient cohomology on the S-scheme Y is called *local acyclicity*, and the local acyclicity of smooth morphisms is the central result that underlies the smooth base change theorem.

Since we are not saying anything about the development of the theory of locally acyclic morphisms, let us at least explain easily why it is natural to restrict to torsion-orders that are invertible on the base. As a first example, if we consider p-torsion sheaves in characteristic $p > 0$ then the invariance of cohomology with respect to an extension between separably closed fields can fail. If K/k is an extension of fields of characteristic p, then Artin–Scheier theory identifies the natural map

$$\mathrm{H}^1_{\text{ét}}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to \mathrm{H}^1_{\text{ét}}(\mathbf{A}^1_K, \mathbf{Z}/p\mathbf{Z})$$

with the natural map

$$k[x]/\{f^p - f \mid f \in k[x]\} \to K[x]/\{f^p - f \mid f \in K[x]\}$$

that is obviously not an isomorphism when $k \neq K$.

The proof of smooth base change breaks down for torsion-orders dividing residue characteristics because local acyclicity can fail in such situations. To illustrate this, let R be a strictly henselian discrete valuation ring (*e.g.*, $R = k[\![t]\!]$ for a separably closed field k), and let K_s be a separable closure of the fraction field K of R. Consider the origin in the closed fiber on the affine line over $\operatorname{Spec} R$. The scheme of vanishing-cycles with respect to $\eta : \operatorname{Spec} K_s \to \operatorname{Spec} R$ is $R\{x\} \otimes_R K_s$, where $R\{x\} = \mathscr{O}^{\mathrm{h}}_{\mathbf{A}^1_R, 0}$ is the (strictly) henselian local ring at the origin. We claim that this has non-vanishing $\mathrm{H}^1_{\text{ét}}$ with $\mathbf{Z}/p\mathbf{Z}$-coefficients. In the equicharacteristic case, this non-vanishing follows (by Artin–Schreier theory) from the fact that the equation $f^p - f = x/r$ cannot be solved in $R\{x\} \otimes_R K_s$ when r is a nonzero element in the maximal ideal of R (there are no solutions in the extension ring $R[\![x]\!] \otimes_R K_s$ because the solutions in $K_s[\![x]\!]$ have unbounded denominators). In the mixed characteristic case, this non-vanishing follows (by Kummer theory) from the fact that the equation $f^p = 1 + x$ cannot be solved in $R\{x\} \otimes_R K_s$ (the pth roots of $1+x$ in $K_s[\![x]\!]$ have unbounded denominators).

1.3.6. Cohomology with proper support. Consider a commutative diagram of topological spaces

(1.3.6.1)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

with $j : X \hookrightarrow \overline{X}$ an open immersion and $\overline{f} : \overline{X} \to S$ proper. Let $\mathscr{F}$ be an abelian sheaf on $X_{\text{ét}}$. Define the sheaf $f_!\mathscr{F}$ on S by

$$(f_!\mathscr{F})(U) \stackrel{\text{def}}{=} \{s \in \Gamma'(f^{-1}(U), \mathscr{F}) \mid \operatorname{supp}(s) \text{ proper over } S\}.$$

When $f = j$, we recover extension-by-zero. It is easy to see that

$$f_! \simeq \overline{f}_* \circ j_!,$$

since for any closed set $Z \hookrightarrow X$ with Z proper over S, the embedding $Z \hookrightarrow \overline{X}$ is proper and hence is a closed embedding.

If X is paracompact Hausdorff, then by [**10**, Ch. II, Thm. 3.5.5(c)] the functor $j_!$ takes soft (*e.g.*, injective or flabby) abelian sheaves to soft sheaves, and these are acyclic for cohomology. If S has a base of paracompact Hausdorff opens (*e.g.*, if S is an analytic space), then the restriction of the proper $\overline{X}$ over these opens is paracompact Hausdorff. Thus, universal δ-functor arguments (and the exactness of $j_!$) yield a δ-functorial isomorphism

$$\mathrm{R}^\bullet \overline{f}_* \circ j_! \simeq \mathrm{R}^\bullet(\overline{f}_* \circ j_!) \simeq \mathrm{R}^\bullet f_!. \tag{1.3.6.2}$$

The significance of this isomorphism is that the acyclicity property for $j_!$ on injectives does not hold (even for smooth affine curves over algebraically closed fields), and in fact the equality (1.3.6.2) breaks down in the étale topology. It is the left side that provides the right definition in algebraic geometry:

Definition 1.3.6.1. Let $f : X \to S$ be a separated and finite type map of schemes, with S quasi-compact and quasi-separated. Let $j : X \to \overline{X}$ be an open immersion into a proper S-scheme $\overline{f} : \overline{X} \to S$. The *higher direct images with proper support* are the terms in the δ-functor $\mathrm{R}^\bullet f_! \stackrel{\mathrm{def}}{=} \mathrm{R}^\bullet \overline{f}_* \circ j_!$ on the category of torsion étale sheaves.

The existence of a j as in Definition 1.3.6.1 is the Nagata compactification theorem [**20**] when S is noetherian, and the general case reduces to the noetherian case via noetherian descent (Theorem 1.3.2.2(1),(2)). The exactness of $j_!$ makes $\mathrm{R}^\bullet f_!$ a δ-functor in the obvious manner. This δ-functor is naturally independent of the choice of j [**9**, Ch. I, 8.4–8.6], but the proof rests on the proper base change theorem and so necessitates the torsion condition in the definition. When $S = \mathrm{Spec}\, k$ for a separably closed field k, we write $\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}}$ to denote the corresponding groups. That is, for a finite-type separated scheme X over a separably closed field k and a torsion abelian sheaf $\mathscr{F}$ on X, we have

$$\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}}(X, \mathscr{F}) \stackrel{\mathrm{def}}{=} \mathrm{H}^\bullet_{\mathrm{\acute{e}t}}(\overline{X}, j_!\mathscr{F})$$

where $j : X \hookrightarrow \overline{X}$ is an open immersion into a proper k-scheme. This is canonically independent of the choice of j. The general independence of compactification allows us to use gluing over open affines in the base to define the δ-functor $\mathrm{R}^\bullet f_!$ for any scheme S (not required to be quasi-compact or quasi-separated).

Remark 1.3.6.2. The δ-functors $\mathrm{R}^\bullet f_!$ admit structures much like in topology. Let us summarize the essential ones:

- (Leray spectral sequence) Let $f' : X' \to X$ be a second separated map of finite type. There is a spectral sequence
$$\mathrm{E}_2^{p,q} = \mathrm{R}^p f'_! \circ \mathrm{R}^q f_! \Rightarrow \mathrm{R}^{p+q}(f' \circ f)_!.$$
- (proper pullback) If $h : Y \to X$ is proper, there is a canonical pullback map $\mathrm{R}^\bullet f_! \to \mathrm{R}^\bullet h_! \circ h^*$ as δ-functors.

- (base change) For any cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

  there is a δ-functorial *base change morphism* $g^* \circ \mathrm{R}^\bullet f_! \to \mathrm{R}^\bullet f'_! \circ g'^*$; in contrast to the case of higher direct images, we do not have such maps for general commutative squares because (as in topology) pullback is only defined with respect to proper maps.
- (excision) Let $Z \hookrightarrow X$ be a closed subscheme with open complement U. Let $f_Z : Z \to S$ and $f_U : U \to S$ be the structure maps. There is a long-exact *excision sequence*

$$\cdots \xrightarrow{\delta} \mathrm{R}^i f_{U!}(\mathscr{F}|_U) \to \mathrm{R}^i f_! \mathscr{F} \to \mathrm{R}^i f_{Z!}(\mathscr{F}|_Z) \xrightarrow{\delta} \dots \tag{1.3.6.3}$$

  for torsion sheaves $\mathscr{F}$ on X; this is compatible with the base-change morphisms.

The general construction-principle in all cases is to first treat the case of quasi-compact and quasi-separated S by choosing compatible compactifications and carrying out a construction with ordinary higher direct images of extension-by-zero sheaves on compactification. The choice of compactifications is proved to not matter, and so globalization over general S is obtained by gluing over open affines.

Since $\mathrm{R}^\bullet f_!$ is defined in terms of higher-direct images of an extension-by-zero on a compactification, it follows from the proper base change theorem and the compatibility of extension-by-zero with respect to base change that the base-change maps for higher direct images with proper support are always isomorphisms. This fact, together with the nature of the excision sequence (with all three terms of the same type, in contrast to the intervention of cohomology-with-supports in the excision sequence for ordinary higher direct images), makes the δ-functor $\mathrm{R}^\bullet f_!$ much better-behaved than ordinary higher direct images.

THEOREM 1.3.6.3. [**9**, Ch. I, 8.8, 8.10] *Let $f : X \to S$ be a finite type separated map to a scheme S, and let $\mathscr{F}$ be a torsion abelian sheaf on $X_{\text{ét}}$. Then the abelian sheaf $\mathrm{R}^i f_!(\mathscr{F})$ on $S_{\text{ét}}$ vanishes for $i > 2\sup_{s\in S} \dim X_s$, and if S is noetherian then these sheaves are constructible for all i when $\mathscr{F}$ is constructible.*

In particular, if $S = \operatorname{Spec} k$ with k a separably closed field, then $\mathrm{H}^i_{\mathrm{c,ét}}(X, \mathscr{F})$ is a finite group and it vanishes for $i > 2\dim(X)$.

For ordinary higher direct images, Deligne proved base-change and finiteness properties over a dense open in the base (under a mild restriction):

THEOREM 1.3.6.4. [**6**, Th. finitude, 1.1, 1.9] *Let $f : X \to S$ be a separated map between schemes of finite type over regular base of dimension ≤ 1. Let $\mathscr{F}$ be a constructible abelian sheaf on $X_{\text{ét}}$ whose torsion-orders are invertible on S. The sheaves $\mathrm{R}^i f_* \mathscr{F}$ are constructible, and they vanish for $i > \dim S + 2\dim X$. Also, there exists a dense open $U \subseteq S$, depending on $\mathscr{F}$, such that the formation of $\mathrm{R}^i f_* \mathscr{F}|_U$ commutes with arbitrary base change on U for all i.*

1.3.7. Ehresmann's fibration theorem in the étale topology. We shall now apply smooth and proper base change to prove the étale version of the fact (Theorem **??**) that if $f : X \to S$ is a proper smooth map between analytic spaces, then $\mathrm{R}^\bullet f_*$ carries local systems to local systems (for sheaves of finite modules over a noetherian ring). Recall that the topological case ultimately reduced to Ehresmann's fibration theorem.

THEOREM 1.3.7.1. *Let $f : X \to S$ be smooth and proper, with S an arbitrary scheme, and let $\mathscr{F}$ be an lcc abelian sheaf on $X_{\text{ét}}$ whose torsion-orders are invertible on S. The sheaf $\mathrm{R}^n f_* \mathscr{F}$ is lcc on S for all n, and its formation commutes with arbitrary base change on S.*

The reader should compare the similarities between the proof we give and the proof of the analytic analogue. Specialization plays the same role in the proof, and the role of Ehresmann's fibration theorem is replaced with smooth base change and vanishing-cycles. The use of vanishing-cycles clarifies the role of smoothness, and provides a technique of analysis that is useful in the presence of singularities (see Theorem **??**).

PROOF. By properness, $\mathrm{R}^n f_* \mathscr{F} = \mathrm{R}^n f_! \mathscr{F}$ is constructible and its formation commutes with base change. Thus, the specialization criterion for local constancy reduces us to checking that the specialization maps for these sheaves are isomorphisms. Observe that $\mathscr{F}$ is represented by a finite étale X-scheme X'. Since X and X' are finitely presented over S, by working over an open affine in S we may use noetherian descent to realize the given geometric setup as a base-change from a setup over a noetherian base. Thus, proper base change allows us to assume S is noetherian.

For any pair of physical points $\{s, \eta\}$ on a noetherian scheme S with s in the closure of η, there is a discrete valuation ring R and a map $\operatorname{Spec} R \to S$ hitting s and η. Another application of proper base change allows us to assume $S = \operatorname{Spec} R$, where R is a strictly henselian discrete valuation ring with fraction field K. In this case, the specialization mapping is identified with the natural map

$$\mathrm{H}^n_{\text{ét}}(X_s, \mathscr{F}_s) \simeq \mathrm{H}^n_{\text{ét}}(X, \mathscr{F}) \to \mathrm{H}^n_{\text{ét}}(X_{K_s}, \mathscr{F}_{K_s}),$$

where K_s/K is a separable closure. We want this map to be an isomorphism. This will be proved via the technique of vanishing-cycles. Letting $j : X_{K_s} \to X$ denote the canonical map, we have $\mathscr{F}_{K_s} = j^* \mathscr{F}$ and there is a Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(X, \mathrm{R}^q j_*(j^* \mathscr{F})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_{K_s}, \mathscr{F}_{K_s}).$$

Applying the proper base change theorem to the (generally non-constructible) torsion sheaves $\mathrm{R}^q j_*(j^* \mathscr{F})$ yields

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(X_s, i^* \mathrm{R}^q j_*(j^* \mathscr{F})),$$

where $i : X_s \hookrightarrow X$ is the inclusion. The sheaves $\mathrm{R}^q \Psi_\eta(\mathscr{F}) \stackrel{\text{def}}{=} i^* \mathrm{R}^q j_*(j^* \mathscr{F})$ are the *vanishing-cycles sheaves*, and the above spectral sequence is the *vanishing-cycles spectral sequence*. The vanishing-cycles functors $\mathrm{R}^q \Psi_\eta : \mathrm{Ab}(X) \to \mathrm{Ab}(X_s)$ may be defined with R replaced by any henselian discrete valuation ring and X replaced by any finite-type R-scheme.

The specialization mapping is the composite

$$\mathrm{H}^p_{\text{ét}}(X_s, \mathscr{F}_s) \to \mathrm{H}^p_{\text{ét}}(X, \mathrm{R}^0 \Psi_\eta(\mathscr{F})) = \mathrm{E}_2^{p,0} \to \mathrm{H}^p_{\text{ét}}(X_{K_s}, \mathscr{F}_{K_s}),$$

where the first step is the closed-fiber pullback of the natural map $\mathscr{F} \to j_* j^* \mathscr{F}$ and the final step is the edge map in the spectral sequence. Thus, specialization is an isomorphism if:

- the natural map $\mathscr{F}_s \to \mathrm{R}^0\Psi_\eta(\mathscr{F})$ is an isomorphism;
- for $q > 0$, $\mathrm{R}^q\Psi_\eta(\mathscr{F})$ vanishes.

These properties are étale-local on X near X_s, and we claim they hold for any smooth R-scheme X. Working locally allows us to assume $\mathscr{F} = \mathbf{Z}/n\mathbf{Z}$, and so for $U = \mathrm{Spec}\, \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ the claims are

$$\mathbf{Z}/n \simeq \mathrm{H}^0_{\text{ét}}(U_{K_s}, \mathbf{Z}/n\mathbf{Z}), \quad \mathrm{H}^q_{\text{ét}}(U_{K_s}, \mathbf{Z}/n\mathbf{Z}) = 0 \text{ for } q > 0.$$

These claims are precisely the local-acyclicity results that are the key facts shown in the proof of the smooth base change theorem. □

1.3.8. Trace maps and Poincaré duality. If M is an oriented paracompact Hausdorff topological manifold with pure dimension d, Poincaré-duality defines a natural trace map

$$\mathrm{H}^d_{\mathrm{c}}(M, \underline{\mathbf{Z}}) \simeq \mathrm{H}_0(M, \mathbf{Z}) \to \mathbf{Z},$$

and the resulting cup-product pairing

$$\mathrm{H}^j(M, \underline{\mathbf{Q}}) \times \mathrm{H}^{d-j}_{\mathrm{c}}(M, \underline{\mathbf{Q}}) \to \mathrm{H}^d_{\mathrm{c}}(M, \underline{\mathbf{Q}}) \to \mathbf{Q}$$

identifies $\mathrm{H}^j(M, \underline{\mathbf{Q}})$ with the linear dual of $\mathrm{H}^{d-j}_{\mathrm{c}}(M, \underline{\mathbf{Q}})$ (and *vice-versa* if these cohomologies are finite-dimensional). If M is a complex manifold with pure complex dimension n, these constructions can be given without choosing an orientation: the trace map acquires the form

$$\mathrm{H}^{2n}_{\mathrm{c}}(M, \underline{\mathbf{Z}}(n)) \simeq \mathrm{H}_0(M, \mathbf{Z}) \to \mathbf{Z}$$

and the cup-product pairing is

$$\mathrm{H}^j(M, \underline{\mathbf{Q}}(n)) \times \mathrm{H}^{2n-j}_{\mathrm{c}}(M, \underline{\mathbf{Q}}) \to \mathrm{H}^{2n}_{\mathrm{c}}(M, \underline{\mathbf{Q}}(n)) \to \mathbf{Q}.$$

There is an analogous theory of duality for $\mathbf{Z}/n\mathbf{Z}$-module sheaves in étale cohomology, resting on cup products and trace maps. Cup products are constructed (and uniquely characterized) by a procedure that is formally the same as in ordinary derived-functor sheaf cohomology, and the usual skew-commutativity properties hold. The unique characterization of cohomological cup products ensures agreement with Galois-cohomology cup products in the case of étale cohomology for a field, and the formal nature of the construction ensures that the comparison isomorphisms with topological cohomology (as in §1.3.10) are compatible with cup products. We omit further discussion of cup products, and instead address the construction of trace maps because the agreement of these and topological traces requires non-formal input.

The construction of étale-cohomology trace maps begins in the quasi-finite case. For any quasi-finite separated flat map $f : X \to Y$ of finite presentation, there is a *trace map* for abelian sheaves $\mathscr{F}$ on $Y_{\text{ét}}$:

$$\mathrm{tr}_f : f_! f^* \mathscr{F} \to \mathscr{F}. \tag{1.3.8.1}$$

This map is uniquely determined on by the formula

$$\bigoplus_{x \in f^{-1}(y)} \mathscr{F}_y = (f_! f^* \mathscr{F})_y \xrightarrow{(\mathrm{tr}_f)_y} \mathscr{F}_y \qquad (1.3.8.2)$$

$$(s_x) \mapsto \sum n_x s_x$$

on stalks, where n_x is the length of the artinian local ring $\mathscr{O}_{f^{-1}(y),x}$. The existence of the map (1.3.8.1) is not obvious; see [**15**, Exp. XVII, §6.2] for details.

We now address the case of curves before discussing more generalities. Let Z be a compact Riemann surface, and consider the diagram

$$\begin{tikzcd} \mathbf{Z}/n\mathbf{Z} & \mathrm{H}^2(Z, \mathbf{Z}(1)/n\mathbf{Z}(1)) \arrow[l, "\simeq"'] \arrow[d] \\ \operatorname{coker}\big(\operatorname{Pic}(Z) \xrightarrow{n} \operatorname{Pic}(Z)\big) \arrow[u, "\deg"] \arrow[r, "\simeq"'] & \mathrm{H}^2(Z, \mu_n) \end{tikzcd} \qquad (1.3.8.3)$$

where the top row is the reduction of the canonical isomorphism $\mathrm{H}^2(Z, \underline{\mathbf{Z}}(1)) \simeq \mathbf{Z}$ and the bottom row is induced by the analytic Kummer sequence

$$1 \to \mu_n \to \mathscr{O}_Z^\times \xrightarrow{t^n} \mathscr{O}_Z^\times \to 1.$$

In §**??**, it is proved that this diagram commutes. The terms in the bottom and left sides of this square make sense in algebraic geometry, as does the Kummer sequence, and this motivates the following construction for algebraic curves.

Consider a proper smooth curve Y over a separably closed field k with characteristic not dividing n. Let us check that the bottom and left sides of (1.3.8.3) do carry over in étale cohomology. In Example 1.2.7.1, we used the Kummer sequence to construct an isomorphism

$$\mathrm{H}^1_{\text{ét}}(Y, \mathbf{G}_m) \simeq \operatorname{Pic}(Y).$$

The mod-n reduction of $\mathrm{H}^1_{\text{ét}}(Y, \mathbf{G}_m)$ is identified with $\mathrm{H}^2_{\text{ét}}(Y, \mu_n)$ via the Kummer sequence, and so we have a natural isomorphism $\operatorname{Pic}(Y)/n\operatorname{Pic}(Y) \simeq \mathrm{H}^2_{\text{ét}}(Y, \mu_n)$. The degree-map $\operatorname{Pic}(Y) \to \mathbf{Z}$ is surjective, and its kernel $\operatorname{Pic}^0(Y) = \operatorname{Pic}^0_{Y/k}(k)$ is the group of rational points on the Picard (or Jacobian) variety $\operatorname{Pic}^0_{Y/k}$ over the separably closed field k. This Picard variety is an abelian variety, and so its group of k-rational points is n-divisible. Thus, the mod-n reduction of the degree map $\deg : \operatorname{Pic}(Y) \to \mathbf{Z}$ defines a map

$$\deg : \operatorname{coker}\big(\operatorname{Pic}(Y) \xrightarrow{n} \operatorname{Pic}(Y)\big) \to \mathbf{Z}/n\mathbf{Z}$$

that plays the role of the left side of (1.3.8.3); this map is an isomorphism when Y is connected. We conclude that the composite

$$\mathrm{H}^2_{\text{ét}}(Y, \mu_n) \simeq \operatorname{coker}\big(\operatorname{Pic}(Y) \xrightarrow{n} \operatorname{Pic}(Y)\big) \xrightarrow{\deg} \mathbf{Z}/n\mathbf{Z}, \qquad (1.3.8.4)$$

makes sense and provides a good candidate for a Poincaré-duality trace map on Y.

We have proposed reasonable trace maps in low dimensions, and now we turn to the topological theory as a guide for what to expect in general. For any paracompact Hausdorff complex manifold Z with pure dimension d, the mod-n reduction on the trace map over $\mathbf{Z}$ is a canonical map

$$\mathrm{H}^{2d}_{\mathrm{c}}(Z, \mu_n^{\otimes d}) \simeq \mathbf{Z}(d) \otimes \mathrm{H}^{2d}_{\mathrm{c}}(Z, \underline{\mathbf{Z}/n\mathbf{Z}}) \to \mathbf{Z}(d) \otimes (\mathbf{Z}/n\mathbf{Z})(-d) = \mathbf{Z}/n\mathbf{Z}.$$

This map does not depend on a choice of orientation, and it is an isomorphism when Z is connected. The appearance of $\mu_n^{\otimes d}$ in this intrinsic description on the topological side motivates the following considerations on the algebraic side.

In étale cohomology over a $\mathbf{Z}[1/n]$-scheme S, the theory of the trace-map aims to construct a canonical map

$$\mathrm{tr}_f : \mathrm{R}^{2d} f_!(\mu_n^{\otimes d}) \to \mathbf{Z}/n\mathbf{Z} \tag{1.3.8.5}$$

for any smooth separated finite-type map $f : Y \to S$ with pure relative dimension d, and the following properties are desired:

- compatibility with base change, with étale-localization on Y, and with reduction modulo a divisor of n;
- transitivity in f (via a Leray-degeneration in top degree);
- tr_f is an isomorphism when f has geometrically connected fibers;
- when $d = 0$, tr_f agrees with (1.3.8.1);
- when $d = 1$, $S = \mathrm{Spec}\, k$ for a separably closed field k, and Y is a proper smooth curve, tr_f equals (1.3.8.4).

There is a unique theory that satisfies these axioms; see [**9**, Ch. II, §1]. An elegant generalization is given for finitely presented flat maps in [**15**, Exp. XVIII, §2], and the axiom for smooth proper curves may be replaced with the axiom that for any smooth separated curve Y over an algebraically closed field, the composite map from local cohomology

$$\mathrm{H}^2_{\{y\},\text{ét}}(Y, \mu_n) \to \mathrm{H}^2_{\text{c},\text{ét}}(Y, \mu_n) = \mathrm{H}^2_{\text{ét}}(Y, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$$

sends a local Chern class $c_1(\mathscr{O}(y), 1)$ to 1, exactly in accordance with the analytic case in §**??**.

In [**26**, 4.8], Verdier explains Grothendieck's proof that this trace map gives a good duality theory in the smooth case; we only require a special case of the general result. Let $(\Lambda, \mathfrak{m})$ be a complete discrete valuation ring with finite residue field of characteristic ℓ, and fraction field of characteristic zero. For $n \geq 0$, let $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$; this is a $\mathbf{Z}/\ell^{n+1}$-algebra. For any Λ_n-sheaf $\mathscr{G}$, we define

$$\mathscr{G}(d) = \mathscr{G} \otimes_{\mathbf{Z}/\ell^{n+1}} \mu_{\ell^{n+1}}^{\otimes d};$$

note that if we consider $\mathscr{G}$ as a Λ_{n+1}-sheaf, this definition is unaffect (replacing $n + 1$ with $n + 2$).

Theorem 1.3.8.1 (Poincaré duality). *Let $f : X \to S$ be a smooth separated map between noetherian $\mathbf{Z}[1/\ell]$-schemes, and let $\mathscr{F}$ and $\mathscr{G}$ be constructible sheaves of Λ_n-modules on X and S. There is a canonical isomorphism*

$$\mathrm{Ext}^i_X(\mathscr{F}, f^*\mathscr{G}(d)) \simeq \mathrm{Hom}_S(\mathrm{R}^{2d-i} f_!(\mathscr{F}), \mathscr{G})$$

that is compatible with base change on X and étale localization on X.

When $S = \mathrm{Spec}\, k$ is a geometric point and $\mathscr{F}$ is lcc with Λ_n-flat stalks, the isomorphism in the special case $\mathscr{G} = \Lambda_n$ is induced by the cup product pairing

$$\mathrm{H}^i_{\text{ét}}\big(X, \mathscr{F}^\vee(d)\big) \otimes \mathrm{H}^{2d-i}_{\text{c},\text{ét}}\big(X, \mathscr{F}\big) \longrightarrow \mathrm{H}^{2d}_{\text{c},\text{ét}}\big(X, \Lambda_n(d)\big) \xrightarrow[\mathrm{tr}]{\simeq} \Lambda_n$$

between finite Λ_n-modules; in particular, this latter pairing is perfect.

Remark 1.3.8.2. The relevance of the lcc and flatness assumptions on $\mathscr{F}$ is to ensure that $\mathrm{H}^\bullet_{\text{ét}}(X, \mathscr{F}^\vee \otimes (\cdot))$ and $\mathrm{Ext}^\bullet_X(\mathscr{F}, \cdot)$ are identified as δ-functors. Also, the definition of the trace map on top-degree cohomology rests on the right-exactness of $\mathrm{H}^{2d}_{\mathrm{c,\acute{e}t}}$ to move Λ_n outside of the cohomology (leaving $\mu_{\ell^{n+1}}^{\otimes d}$ inside the cohomology).

The proof of Poincaré duality involves two separate aspects: abstract derived-category arguments are used to reduce the general case to a special case, and a calculation is necessary in the special case. In [**26**], Verdier explains Grothendieck's abstract methods that reduce the general case to the assertion that if X is a proper smooth connected curve over an algebraically closed field k, then the pairing

$$\mathrm{H}^1_{\acute{e}t}(X, \Lambda_n(1)) \otimes_{\Lambda_n} \mathrm{H}^1_{\acute{e}t}(X, \Lambda_n) \xrightarrow{\cup} \mathrm{H}^2_{\acute{e}t}(X, \Lambda_n) \simeq \Lambda_n \tag{1.3.8.6}$$

is perfect when ℓ is not the characteristic of k. Strictly speaking, Verdier works with Λ_n replaced by $\mathbf{Z}/n\mathbf{Z}$, but the arguments carry over *verbatim* . Unfortunately, the argument for the special case (1.3.8.6) is not provided in [**26**].

There are several ways to treat this special case. Let us first briefly sketch the topological method. Since proper and smooth curves can be deformed into characteristic zero ([**13**, Exp. III, 7.2], [**16**, 5.4.5]), the proper and smooth base-change theorems for étale cohomology reduce the problem to the case of characteristic zero. Further application of limit methods and the base-change theorems reduce the problem in characteristic zero to the case $k = \mathbf{C}$ (Lefschetz principle, or more specifically, smooth base change), and so comparison isomorphisms with topological cohomology (see §1.3.10) thereby reduce the problem to the known topological case.

An alternative approach is to express Λ_n as a finite free module over $\mathbf{Z}/m\mathbf{Z}$ for a suitable power m of ℓ, to use this to reduce to the case of $\mathbf{Z}/m\mathbf{Z}$-coefficients, and to then identify the μ_m-twisted pairing

$$\mathrm{H}^1_{\acute{e}t}(X, \mu_m) \otimes \mathrm{H}^1_{\acute{e}t}(X, \mu_m) \to \mathrm{H}^2_{\acute{e}t}(X, \mu_m^{\otimes 2}) \simeq \mu_m \tag{1.3.8.7}$$

with the m-torsion Weil pairing for the principally polarized Picard variety $\mathrm{Pic}^0_{X/k}$; here, we use the Kummer-sequence isomorphism

$$\mathrm{H}^1_{\acute{e}t}(X, \mu_m) \simeq \mathrm{Pic}(X)[m] = \mathrm{Pic}^0_{X/k}[m].$$

By once again using the Lefschetz principle, our work with compact Riemann surfaces (see Theorem **??**) establishes this compatibility. This gives another topological proof, and unfortunately I do not know an algebraic method to directly relate Weil pairings and cup products.

A second approach along these lines that does succeed algebraically is to use the Albanese property of $\mathrm{Pic}^0_{X/k}$. We may rewrite (1.3.8.6) in the form

$$\mathrm{Pic}^0_{X/k}[m] \times \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, e), \mathbf{Z}/m\mathbf{Z}) \to \mathbf{Z}/m\mathbf{Z},$$

where $e \in X$ is a point. In geometric class field theory for curves, it is proved that the Jacobian classifies geometrically-connected finite abelian étale covers of curves; more precisely, the map $\phi_e : X \to \mathrm{Pic}^0_{X/k}$ defined by $x \mapsto \mathscr{O}(e) \otimes \mathscr{O}(x)^{-1}$ induces an isomorphism on abelianized π_1's. Since the π_1 of an abelian variety over a separably closed field is identified with its total Tate module (Serre–Lang theorem [**22**, §18]), we may express (1.3.8.6) as a pairing

$$\mathrm{Pic}^0_{X/k}[m] \times \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Pic}^0_{X/k}[m], \mathbf{Z}/m\mathbf{Z}) \to \mathbf{Z}/m\mathbf{Z}.$$

In [**6**, pp.161-5], Deligne uses local Chern classes and cycle maps in étale cohomology to prove that this pairing is equal to the visibly perfect evaluation-pairing. Deligne's formulation of his result involves the intervention of a bothersome sign, but this is ultimately due to Deligne's decision to carry out a calculation that rests on $-\phi_e$ and not on ϕ_e.

1.3.9. Künneth formula. The computation of cohomology on products requires an analogue of the topological Künneth formula that relates the cohomology ring of a product with the cohomology rings of the factor spaces. Let us first review the Künneth formula in the simplest topological situation. If M and M' are compact Hausdorff manifolds, then there is a short-exact *Künneth sequence*

$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(M, \underline{A}) \otimes_A \mathrm{H}^q(M', \underline{A}) \to \mathrm{H}^n(M \times M', \underline{A}) \to T_n \to 0$$

where the first map is $\sum c_p \otimes c'_q \mapsto \sum \mathrm{pr}_M^*(c_p) \cup \mathrm{pr}_{M'}^*(c'_q)$ and the final term is

$$T_n = \bigoplus_{p+q=n+1} \mathrm{Tor}_A^1(\mathrm{H}^p(M, \underline{A}), \mathrm{H}^q(M', \underline{A})) \to 0,$$

with A any principal ideal domain. The reason that this relationship between cohomology of the product and of the factors may be described by a short exact sequence (rather than a spectral sequence) is because a principal ideal domain is regular with dimension ≤ 1, and so Tor_A^i vanishes for $i > 1$.

In the case of field-coefficients, the Tor^1-term vanishes and we get the *Künneth formula* that expresses the cohomology ring of a product as a tensor product of the cohomology rings of the factors. With more general coefficient rings, such as $\mathbf{Z}/n\mathbf{Z}$, it may happen that there are non-vanishing higher Tor^i's beyond degree 1; this makes it impossible to formulate an analogue of the Künneth relations as a short exact sequence. For example, every non-regular local noetherian ring has non-vanishing Tor-bifunctors in arbitrarily large degrees (by Serre's characterization of regular local noetherian rings as those with finite global dimension [**21**, 19.2]); this applies to artin local rings such as $\mathbf{Z}/n\mathbf{Z}$. The simplest way to state the Künneth formula with general (or even just artinian) coefficients is in terms of derived categories, and this is also the case in étale cohomology.

Turning to the étale topology, fix a commutative ring Λ that is killed by a nonzero integer, so étale sheaves of Λ-modules are torsion. For any separated finite-type map $f : X \to S$, we may consider the δ-functor $\mathrm{R}^\bullet f_!$ between étale sheaves of Λ-modules on $X_{\text{ét}}$ and on $S_{\text{ét}}$. If $f' : X' \to S$ is a second such map, and $\mathscr{F}$ and $\mathscr{F}'$ are sheaves of Λ-modules on $X_{\text{ét}}$ and $X'_{\text{ét}}$, we want to relate the cohomologies of $\mathscr{F}$ and $\mathscr{F}'$ with the cohomology of $\pi^*\mathscr{F} \otimes_\Lambda \pi'^*\mathscr{F}'$, where $\pi : X \times_S X' \to X$ and $\pi' : X \times_S X' \to X'$ are the projections. We cannot define pullback along π or π' for compactly-supported cohomology, as these maps might not be proper. However, we can still define natural maps

$$\mathrm{R}^p f_!(\mathscr{F}) \otimes_\Lambda \mathrm{R}^q f'_!(\mathscr{F}') \to \mathrm{R}^{p+q}(f \times f')_!(\pi^*\mathscr{F} \otimes_\Lambda \pi'^*\mathscr{F}') \tag{1.3.9.1}$$

by working locally over S, as follows.

We first assume that S is quasi-compact and quasi-separated (*e.g.*, affine), so we may choose open immersions $j : X \hookrightarrow \overline{X}$ and $j' : X' \hookrightarrow \overline{X}'$ into proper S-schemes $\overline{f} : \overline{X} \to S$ and $\overline{f}' : \overline{X}' \to S$. Thus, $j \times j'$ compactifies $X \times_S X'$ over S, and projection-pullback and cup product on $\overline{X} \times_S \overline{X}'$ yield maps

$$\mathrm{R}^p \overline{f}_*(j_!\mathscr{F}) \otimes_\Lambda \mathrm{R}^q \overline{f}'_*(j'_!\mathscr{F}') \to \mathrm{R}^{p+q}(\overline{f} \times \overline{f}')_*(\overline{\pi}^*(j_!\mathscr{F}) \otimes_\Lambda \overline{\pi}'^*(j'_!\mathscr{F}')).$$

Since $\overline{\pi}^*(j_!\mathscr{F}) \otimes_\Lambda \overline{\pi}'^*(j'_!\mathscr{F}') = (j \times j')_!(\pi^*\mathscr{F} \otimes_\Lambda \pi'^*\mathscr{F}')$, we obtain the desired map in (1.3.9.1); this is independent of the compactifications, and so it globalizes over an arbitrary base. Forming the direct sum over all (p,q) with $p+q=n$ gives the bifunctorial degree-n *Künneth morphism*

$$\bigoplus_{p+q=n} \mathrm{R}^p f_!(\mathscr{F}) \otimes_\Lambda \mathrm{R}^q f'_!(\mathscr{F}') \to \mathrm{R}^n(f \times f')_!(\pi^*\mathscr{F} \otimes_\Lambda \pi'^*\mathscr{F}').$$

The key problem is to determine if the Künneth morphism is an isomorphism, and to measure the obstructions when it is not. The obstructions are located in many Tor_Λ-sheaves: there are the higher Tor_Λ-sheaves for $\pi^*\mathscr{F}$ and $\pi'^*\mathscr{F}'$ on $X \times_S X'$, as well as Tor_Λ's among the $\mathrm{R}^p f_!\mathscr{F}$'s and $\mathrm{R}^q f'_!\mathscr{F}'$'s on S. This wealth of information is hard to analyze with the language of exact sequences or spectral sequences, and so it is necessary to work systematically in derived categories of sheaves of Λ-modules. This additional layer of abstraction and generality has the remarkable consequence that it yields an easy isomorphism result via the proper base change theorem. We shall now state this abstract isomorphism, and then explain how it relates to the Künneth morphism as constructed above.

Using the derived-category formalism and the proper base change theorem in derived categories of étale sheaves, for any bounded-above complexes of Λ-modules $\mathscr{F}^\bullet$ and $\mathscr{F}'^\bullet$ on X and X' there is a general *abstract Künneth isomorphism*

$$\mathbf{R}f_!(\mathscr{F}^\bullet) \otimes^{\mathbf{L}} \mathbf{R}f'_!(\mathscr{F}'^\bullet) \simeq \mathbf{R}(f \times f')_!((\pi^*\mathscr{F}^\bullet) \otimes^{\mathbf{L}} (\pi'^*\mathscr{F}'^\bullet))$$

relating total compactly-supported higher direct images and total tensor products; see [**9**, Ch. I, §8] for an efficient construction of this map and a quick proof that it is an isomorphism. This isomorphism is compatible with base change on S, and with natural change-of-coefficients functors from derived categories of Λ-sheaves to derived categories of Λ'-sheaves for any map $\Lambda \to \Lambda'$ between torsion rings.

If we take $\mathscr{F}^\bullet = \mathscr{F}[0]$ and $\mathscr{F}'^\bullet = \mathscr{F}'[0]$ to be complexes concentrated in degree 0, and we use spectral sequences that compute homologies of $\otimes^{\mathbf{L}}$'s in terms of ordinary Tor_Λ-sheaves, then we may define a composite map

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \mathrm{R}^p f_!(\mathscr{F}) \otimes_\Lambda \mathrm{R}^q f'_!(\mathscr{F}') & \xrightarrow{\alpha} & \mathrm{H}^n(\mathbf{R}f_!(\mathscr{F}[0]) \otimes^{\mathbf{L}} \mathbf{R}f'_!(\mathscr{F}'[0])) \\
 & & \kappa \downarrow \simeq \\
\mathrm{R}^n(f \times f')_!((\pi^*\mathscr{F}^\bullet) \otimes (\pi'^*\mathscr{F}'^\bullet)) & \xleftarrow[\beta]{} & \mathrm{H}^n(\mathbf{R}(f \times f')_!((\pi^*\mathscr{F}[0]) \otimes^{\mathbf{L}} (\pi'^*\mathscr{F}'[0])))
\end{array}$$

where α is the edge map from $\mathrm{E}_2^{0,n}$ in the *hyper-Tor spectral sequence*

$$\mathrm{E}_2^{r,s} = \bigoplus_{a+a'=s} \mathscr{T}or^r_{\underline{\Lambda}}(\mathrm{R}^a f_!\mathscr{F}, \mathrm{R}^{a'} f'_!\mathscr{F}') \Rightarrow \mathrm{H}^{r+s}(\mathbf{R}f_!(\mathscr{F}[0]) \otimes^{\mathbf{L}} \mathbf{R}f'_!(\mathscr{F}'[0])),$$

β is the canonical map, and the isomorphism κ linking them is the degree-n homology map induced by the abstract Künneth isomorphism. The map β is an isomorphism if either $\mathscr{F}$ or $\mathscr{F}'$ has Λ-flat stalks, for then the higher sheaf-Tor's between $\pi^*\mathscr{F}$ and $\pi'^*\mathscr{F}'$ vanish.

The crucial fact that relates this abstract nonsense to concrete sheaf-theoretic constructions is:

LEMMA 1.3.9.1. *The composite map $\beta \circ \kappa \circ \alpha$ agrees with the degree-n Künneth morphism that is defined via cup products.*

PROOF. We may use base change to reduce to the case when S is a geometric point. In this case, the abstract construction produces bifunctorial maps

$$\mathrm{H}^p_{\mathrm{c,\acute{e}t}}(X,\mathscr{F})\otimes_\Lambda \mathrm{H}^q_{\mathrm{c,\acute{e}t}}(X',\mathscr{F}')\to \mathrm{H}^{p+q}_{\mathrm{c,\acute{e}t}}(X\times X',\pi^*\mathscr{F}\otimes_\Lambda {\pi'}^*\mathscr{F}')$$

that must be proved to agree with the cup-product construction. This problem may be reduced to the proper case with ordinary cup products. Since cup products in ordinary cohomology (with a fixed coefficient-ring Λ) are uniquely characterized as a family of bifunctors satisfying a short list of exactness and skew-commutativity properties, it suffices to check that these conditions are satisfied by the abstract construction. The verification is a tedious exercise that rests solely on general properties of derived tensor products (such as their signed-commutativity). □

The concrete conclusion from these considerations is:

THEOREM 1.3.9.2 (Künneth formula). *Let $\mathscr{F}$ and $\mathscr{F}'$ be sheaves of Λ-modules with flat stalks. The Künneth morphism is the edge map from $\mathrm{E}_2^{0,n}$ in a bifunctorial spectral sequence*

$$\bigoplus_{a+a'=s}\mathscr{T}or^r_{\underline{\Lambda}}(\mathrm{R}^a f_!\mathscr{F},\mathrm{R}^{a'}f'_!\mathscr{F}')\Rightarrow \mathrm{R}^{r+s}(f\times f')_!(\pi^*\mathscr{F}\otimes_\Lambda {\pi'}^*\mathscr{F}').$$

For any extension of scalars $\Lambda\to\Lambda'$, this identification is compatible with the natural change-of-coefficients maps on Tor-sheaves.

1.3.10. Comparison with topological cohomology. Let X be an algebraic $\mathbf{C}$-scheme. In §1.1.1 we defined the *topological étale site* on $X(\mathbf{C})$ and we noted that $X(\mathbf{C})_{\text{ét}}$ is equivalent to the category $(X^{\mathrm{an}})_{\text{ét}}$ of analytic spaces equipped with a local isomorphism to X^{an}; thus, we may likewise define the *analytic étale site* on X^{an}. Lemma 1.1.2.2 implies that the associated topos $\text{Ét}(X^{\mathrm{an}})$ is equivalent to the category $\mathrm{Top}(X(\mathbf{C}))$ of sheaves of sets on $X(\mathbf{C})$, and so the category of abelian sheaves on the analytic étale site is equivalent to the category of abelian sheaves on $X(\mathbf{C})$. Thus, topological cohomology functors on $X(\mathbf{C})$ are identified with cohomology functors on the analytic étale site.

The comparison isomorphisms between étale and topological cohomology shall be formulated as a comparison between cohomology functors for abelian sheaves on $X_{\text{ét}}$ and $(X^{\mathrm{an}})_{\text{ét}}$, so the equivalence between $\text{Ét}(X^{\mathrm{an}})$ and $\mathrm{Top}(X(\mathbf{C}))$ will provide the identification with cohomology on $X(\mathbf{C})$. In §1.2.4, we constructed a δ-functorial Zariski-étale comparison morphism

$$\mathrm{H}^\bullet(S,\mathscr{F})\to \mathrm{H}^\bullet_{\text{ét}}(S,\iota^*\mathscr{F})$$

for abelian sheaves $\mathscr{F}$ on the Zariski topology of a scheme S. We saw that these maps are rarely isomorphisms, though in the case of $\mathscr{O}_S$-modules $\mathscr{F}$ we defined a refined $\mathscr{O}_{S_{\text{ét}}}$-module $\mathscr{F}_{\text{ét}}$ equipped with a map $\iota^*\mathscr{F}\to\mathscr{F}_{\text{ét}}$, and we proved (Theorem 1.2.6.2) that composition with this map yields an isomorphism $\mathrm{H}^\bullet(S,\mathscr{F})\simeq \mathrm{H}^\bullet_{\text{ét}}(S,\mathscr{F}_{\text{ét}})$ when $\mathscr{F}$ is quasi-coherent. The comparison between étale and topological cohomologies works out more nicely, as we now explain.

The first step is to define a pair of adjoint functors

$$i_{X*}:\text{Ét}(X^{\mathrm{an}})\to\text{Ét}(X),\quad i_X^*:\text{Ét}(X)\to\text{Ét}(X^{\mathrm{an}})$$

that will define maps between cohomology on the analytic and algebraic étale sites. Analytification defines a functor $X_{\text{ét}}\to(X^{\mathrm{an}})_{\text{ét}}$ that is compatible with fiber products. Thus, given a sheaf of sets $\mathscr{G}$ on $(X^{\mathrm{an}})_{\text{ét}}$ we get a sheaf of sets $i_{X*}\mathscr{G}$ on $X_{\text{ét}}$

by the formula $U' \rightsquigarrow \mathscr{G}(U'^{\mathrm{an}})$. There is a left adjoint $i_X^* : \acute{\mathrm{E}}\mathrm{t}(X) \to \acute{\mathrm{E}}\mathrm{t}(X^{\mathrm{an}})$, the *analytification* functor on sheaves; it is denoted $\mathscr{F} \rightsquigarrow \mathscr{F}^{\mathrm{an}}$. The construction of i_X^* goes exactly as for étale-sheaf pullbacks induced by maps between schemes: for any analytic étale map $U \to X^{\mathrm{an}}$, we first form a direct limit of $\mathscr{F}(U' \to X)$'s over commutative diagrams of analytic spaces

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ X^{\mathrm{an}} \arrow[r, "i_X"'] & X \end{tikzcd}$$

This defines a presheaf on $(X^{\mathrm{an}})_{\acute{\mathrm{e}}\mathrm{t}}$ whose sheafification is called $\mathscr{F}^{\mathrm{an}}$. The verification of adjointness between i_{X*} and i_X^* is straightforward, and we thereby see that the analytification functor $\acute{\mathrm{E}}\mathrm{t}(X) \to \acute{\mathrm{E}}\mathrm{t}(X^{\mathrm{an}})$ is exact. Similar exactness holds for the induced functor between the subcategories of abelian-group objects.

EXAMPLE 1.3.10.1. To avoid confusion, we address a compatibility with another analytification procedure for sheaves. Let $\mathscr{F}$ be an abelian sheaf for the Zariski topology on an algebraic $\mathbf{C}$-scheme X. In §1.2.7, we defined an associated sheaf $i^*\mathscr{F}$ on $X_{\acute{\mathrm{e}}\mathrm{t}}$. We can analytify this sheaf to get a sheaf $(i^*\mathscr{F})^{\mathrm{an}}$ on the analytic étale site of X^{an}. On the other hand, by using the map of topological spaces $X^{\mathrm{an}} \to X$, we can form the ordinary topological pullback of $\mathscr{F}$ to get a sheaf $\mathscr{F}^{\mathrm{an}}$ on the topological space $X(\mathbf{C})$. It is a simple exercise with adjointness to check that $\mathscr{F}^{\mathrm{an}}$ corresponds to $(i^*\mathscr{F})^{\mathrm{an}}$ under the equivalence between sheaf theory on the topological space $X(\mathbf{C})$ and on the analytic étale site.

We now turn to the interaction among these functors and maps between algebraic $\mathbf{C}$-schemes and analytic spaces. The idea is that the pair (i_{X*}, i_X^*) is to be considered as a morphism $i_X : X^{\mathrm{an}} \to X_{\acute{\mathrm{e}}\mathrm{t}}$ (in fact, a *morphism of sites* is defined to be a pair of adjoint functors, with an exact left-adjoint term and a left-exact right-adjoint term), and for any map $f : X \to S$ between algebraic $\mathbf{C}$-schemes we want to have a commutative diagram

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "i_X"] \arrow[d, "f^{\mathrm{an}}"'] & X \arrow[d, "f"] \\ S^{\mathrm{an}} \arrow[r, "i_S"'] & S \end{tikzcd}$$

To make precise sense of this idea without digressing into the generality of morphisms of sites, we will work with pairs of adjoint functors between sheaf-categories.

For sheaves of sets $\mathscr{F}$ on $X_{\acute{\mathrm{e}}\mathrm{t}}$, the definition of $\mathscr{F}^{\mathrm{an}}$ provides an evident natural map $\mathscr{F}(V) \to \mathscr{F}^{\mathrm{an}}(V^{\mathrm{an}})$ for V in $X_{\acute{\mathrm{e}}\mathrm{t}}$, and so taking $V = f^{-1}(U)$ for étale $U \to S$ defines a natural map $f_*\mathscr{F} \to \iota_{X*}(f_*^{\mathrm{an}}\mathscr{F}^{\mathrm{an}})$ on $S_{\acute{\mathrm{e}}\mathrm{t}}$. Adjointness thereby yields a degree-0 comparison morphism

$$(f_*\mathscr{F})^{\mathrm{an}} \to f_*^{\mathrm{an}}(\mathscr{F}^{\mathrm{an}}).$$

Restricting attention to the case of abelian $\mathscr{F}$, the exactness of analytification enables us to use a universal δ-functor argument to uniquely extend the degree-0 comparison morphism to a *comparison morphism* of δ-functors

$$(\mathrm{R}^\bullet f_*(\mathscr{F}))^{\mathrm{an}} \to \mathrm{R}^\bullet f_*^{\mathrm{an}}(\mathscr{F}^{\mathrm{an}}).$$

When f is separated and finite type, a similar construction using the comparison morphism for extension-by-zero sheaves on compactifications (over quasi-compact opens in S) defines a δ-functorial *comparison morphism*

$$(\mathrm{R}^\bullet f_!(\mathscr{F}))^{\mathrm{an}} \to \mathrm{R}^\bullet f_!^{\mathrm{an}}(\mathscr{F}^{\mathrm{an}}).$$

EXAMPLE 1.3.10.2. If X is an algebraic $\mathbf{C}$-scheme, there is a natural map $\mathbf{G}_{m/X} \to i_{X*}\mathbf{G}_{m/X^{\mathrm{an}}}$ whose adjoint defines a natural map $\xi_X : \mathbf{G}_{m/X}^{\mathrm{an}} \to \mathbf{G}_{m/X^{\mathrm{an}}}$. This yields $\mathrm{H}^i_{\text{ét}}(X, \mathbf{G}_{m/X}) \to \mathrm{H}^i(X^{\mathrm{an}}, \mathbf{G}_{m/X}^{\mathrm{an}}) \to \mathrm{H}^i(X^{\mathrm{an}}, \mathbf{G}_{m/X^{\mathrm{an}}})$. The first step is the comparison morphism and the second step is ξ_X. In the special case $i = 1$, this composite is identified with the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{an}})$ that sends the isomorphism class of a line bundle $\mathscr{L}$ on X to the isomorphism class of the analytic line bundle $\mathscr{L}^{\mathrm{an}}$ defined by analytification for coherent sheaves.

The comparison morphisms may also be defined for cohomology with supports along a closed subset of X, and are compatible with base-change morphisms, Leray spectral sequences, excision, and cup products. An important additional compatibility is that the comparison morphism for higher direct images with proper support is compatible with Poincaré-duality trace maps in the smooth case. In concrete terms, this says that if X is a smooth d-dimensional separated $\mathbf{C}$-scheme of finite type, then the comparison morphism

$$\mathrm{H}^{2d}_{\mathrm{c},\text{ét}}(X, \mu_n^{\otimes d}) \to \mathrm{H}^{2d}_{\mathrm{c}}(X^{\mathrm{an}}, \mathbf{Z}(d)/n\mathbf{Z}(d))$$

is compatible with trace maps to $\mathbf{Z}/n\mathbf{Z}$.

When $d = 1$, this compatibility is an immediate consequence of the fact that mod-n traces for proper smooth algebraic curves are computed with the Kummer sequence whereas mod-n topological traces for compact Riemann surfaces may be computed with the analytic Kummer sequence (as is proved in §**??***ff.*). In our earlier discussion of the algebraic trace, we listed four axioms that uniquely characterize it. The strategy to prove the compatibility between the comparison morphism and traces is to show that composing the topological trace with the comparison morphism provides a trace-theory for algebraic $\mathbf{C}$-schemes that satisfies the four axioms. We omit the details, but note that the key calculation in this analysis is the agreement for proper smooth curves.

Here is the main comparison theorem, due to Artin.

THEOREM 1.3.10.3. [**15**, Exp. XVI, §4], [**9**, Ch. I, 11.6] *Let $f : X \to S$ be a separated finite-type map between algebraic $\mathbf{C}$-schemes. On the category of torsion abelian étale sheaves on X, the comparison morphism for $\mathrm{R}^\bullet f_!$'s is an isomorphism. On the category of constructible abelian étale sheaves on X, the comparison morphism for $\mathrm{R}^\bullet f_*$'s is an isomorphism.*

In the deeper case of ordinary higher direct images, the original proof by Artin used resolution of singularities for varieties of dimension at most $\dim X$. An elegant resolution-free proof in all dimensions is given by Berkovich in [**2**, 7.5.1] in the context of analytification for algebraic schemes over non-archimedean fields. Berkovich's proof carries over essentially *verbatim* to the case of analytification for algebraic $\mathbf{C}$-schemes, and so gives a resolution-free proof of the comparison isomorphism over $\mathbf{C}$; Berkovich's "elementary fibrations of pure dimension 1" have to be interpreted to mean "locally a product of the base with a unit disc" in the complex-analytic case. The only point requiring some care is that the argument uses a consequence [**2**, 7.4.9] of Berkovich's general Poincaré duality [**2**, 7.3.1] for non-archimedean analytic spaces, but fortunately Berkovich's proof of analytic Poincaré duality adapts with essentially no change to the complex analytic case (using sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules or F-vector spaces for a field F), with the same re-interpretation of "elementary fibration of pure codimension 1".

1.4. The adic formalism

The work of Deuring, Weil, and Tate showed that in the study of abelian varieties over fields, ℓ-adic homology is a good subsitute for the integral homology that is used in the analytic theory over $\mathbf{C}$. In general, an inverse-limit mechanism on étale cohomology with torsion coefficients will yield an étale cohomology theory with values in vector spaces over ℓ-adic fields.

We proceed in three stages. In §1.4.1–§1.4.2, we focus on the case of modules. For a noetherian ring A that is separated and complete for the topology defined by an ideal I, there is a very elementary way to describe the category of finite A-modules in terms of projective systems of A/I^{n+1}-modules for $n \geq 0$. We explain why this description is insufficient to define ℓ-adic étale sheaves, and then we develop the Artin–Rees formalism that provides a more useful category of projective systems for describing finite A-modules. In §1.4.3–§1.4.4 we sheafify what has gone before so as to define ℓ-adic sheaves. These are the étale analogue of local systems of finite $\mathbf{Z}_\ell$-modules in topology. For clarity, we allow the coefficient-ring Λ to be any complete local noetherian ring with finite residue field. In §1.4.5–§1.4.6 we define $\mathbf{Q}_\ell$-sheaves and cohomology for ℓ-adic sheaves, and we extend the Artin–Rees formalism to cohomology functors so as to develop a theory of three cohomological operations (f^*, $\mathrm{R}^i f_*$, $\mathrm{R}^i f_!$) for ℓ-adic sheaves. Finally, in §1.4.7–§1.4.8 we study the analytification functor for ℓ-adic sheaves and discuss the proof of the comparison isomorphisms relating topological and étale cohomology for ℓ-adic sheaves on algebraic $\mathbf{C}$-schemes.

1.4.1. Artin–Rees categories. For a compact C^∞-manifold X and $r \geq 0$, considerations with Čech theory for a finite geodesically-convex open cover of X yield isomorphisms

$$\mathrm{H}^r(X, \underline{\mathbf{Z}}_\ell) \simeq \mathbf{Z}_\ell \otimes \mathrm{H}^r(X, \underline{\mathbf{Z}}) \simeq \varprojlim \mathrm{H}^r(X, \underline{\mathbf{Z}})/\ell^n \mathrm{H}^r(X, \underline{\mathbf{Z}}) \simeq \varprojlim \mathrm{H}^r(X, \underline{\mathbf{Z}/\ell^n}).$$

This motivates the expectation that inverse limits of torsion-sheaf cohomology are a reasonable way to construct the correct cohomology with characteristic-zero coefficients.

As a warm-up, let us consider the special case $r = 1$ and $X = A$ a complex torus. We have $\mathrm{H}^1(A, \underline{\mathbf{Z}/m}) \simeq \mathrm{H}_1(A, \mathbf{Z}/m)^\vee$, and the universal connected abelian covering of A with m-torsion covering group is the multiplication-map $m : A \to A$ whose kernel is denoted $A[m]$. Thus, we get $\mathrm{H}_1(A, \mathbf{Z}/m) \simeq A[m]$, and so

$$\mathrm{H}^1(A, \underline{\widehat{\mathbf{Z}}}) \simeq \varprojlim A[m]^\vee = T_{\widehat{\mathbf{Z}}}(A)^\vee, \tag{1.4.1.1}$$

where the transition maps in the inverse limit are dual to the inclusion maps $A[m] \hookrightarrow A[m']$ when $m|m'$; the ℓ-component $\mathrm{H}^1(A, \underline{\mathbf{Z}}_\ell)$ is similarly described as the dual to the ℓ-adic Tate module.

The right side of (1.4.1.1) makes sense in algebraic geometry, and its ℓ-part inspires the expectation that the ℓ-adic cohomology of an abelian variety A over an algebraically closed field k should be the linear dual of its ℓ-adic Tate module (and hence should be free of rank $2 \dim A$ when $\ell \neq \mathrm{char}(k)$). Since the Serre-Lang theorem [**22**, §18] identifies the étale fundamental group $\pi_1(A, 0)$ with the total Tate module $T_{\widehat{\mathbf{Z}}}(A)$, Theorem 1.2.5.4 yields

$$\varprojlim \mathrm{H}^1_{\text{ét}}(A, \mathbf{Z}/\ell^n\mathbf{Z}) = \varprojlim A[\ell^n]^\vee = T_\ell(A)^\vee$$

for any prime ℓ, recovering that $\mathrm{H}^1_{\text{ét}}(A, \mathbf{Z}_\ell)$ should agree with this inverse limit.

We must now develop a formalism for working with projective systems without passing to an inverse-limit object. It is instructive to first consider a simpler task: if R is a complete discrete valuation ring with fraction field K, can we describe the categories of finite R-modules and finite-dimensional K-vector spaces in terms of some categories of projective systems of finite-length R-modules?

Let us consider the category of finite modules over any noetherian ring A that is I-adically separated and complete for some ideal I. For each finite A-module M we get a projective system $(M_n)_{n\geq 0}$ with $M_n = M/I^{n+1}M$ a finite module over $A_n = A/I^{n+1}$. This projective system enjoys a special property: the natural maps $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ induced by the projective-system structure are isomorphisms for all n.

Definition 1.4.1.1. A projective system (M_n) of finite modules over the A_n's is *strictly I-adic* if $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism for all $n \geq 0$.

It is clear that replacing I with any ideal lying between some I^n and I^m (with $m \geq n > 0$) yields a concept that is functorially equivalent, and so it is the topology of A and not the choice of ideal I that is the relevant structure. It is a basic fact [**18**, 0_{I}, 7.2.9] that the functors $M \rightsquigarrow (M/I^{n+1}M)$ and $(M_n) \rightsquigarrow \varprojlim M_n$ define a categorical equivalence between the category of finite A-modules and the category of strictly I-adic projective systems (M_n).

Remark 1.4.1.2. Our terminology conflicts with Jouanolou's in [**14**, Exp. V]. What we are calling *strictly I-adic* is called I-adic by Jouanolou, and Jouanolou uses the terminology *strict* to indicate the weaker condition that transition maps are epimorphisms.

Example 1.4.1.3. Let us illustrate some deficiencies with the equivalence between finite A-modules and strict I-adic systems. Consider module operations on finite A-modules, such as formation of $\mathrm{Hom}_A(M, N)$ and $M \otimes_A N$, or $\ker T$ and $\operatorname{coker} T$ for a linear map $T : M \to N$. How do these operations translate into the language of projective systems introduced above?

If we wish to replace M and N with (M_n) and (N_n), we need to directly describe module operations in terms of projective systems without forming inverse limits as an intermediate step. The isomorphisms $(M \otimes_A N)_n \simeq M_n \otimes_{A_n} N_n$, and $\operatorname{coker}(T)_n \simeq \operatorname{coker}(T_n)$ show that the operations $\otimes$ and coker present no surprises. In contrast, $\mathrm{Hom}_A(M, N) \to \varprojlim \mathrm{Hom}_{A_n}(M_n, N_n)$ is an isomorphism [**18**, 0_{I}, 7.8.2] but $(\mathrm{Hom}_{A_n}(M_n, N_n))$ is usually not strictly I-adic, and likewise the $\ker(T_n)$'s usually do not form a strictly I-adic projective system.

To give a counterexample for kernels, suppose $(A, I) = (R, \mathfrak{m})$ with R a discrete valuation ring, and assume $T : M \to N$ is surjective (or equivalently, T_n is surjective for all n). When does it happen that $(\ker(T_n))$ is strictly I-adic, even just in large degrees? It is a pleasant exercise with Tor's to check that this is exactly the condition that the torsion submodule of M surjects onto that of N.

We must modify the category of projective systems if we are to translate basic module operations into term-by-term operations on projective systems. For example, we want a dictionary such that the abelian-category structure on the category of finite A-modules can be detected at the projective-system level; Example 1.4.1.3 shows that this is hopeless if we use the naive category of strictly I-adic projective systems. The solution is suggested by:

EXAMPLE 1.4.1.4. Suppose $T : M \to N$ is a map of finite modules over an I-adically separated and complete noetherian ring A. Consider the natural maps

$$f_n : \ker(T)_n \to \ker(T_n). \tag{1.4.1.2}$$

We claim that the resulting map $f_\bullet$ of projective systems is nearly an isomorphism in the sense that the kernels and cokernels of the f_n's are *null systems*: projective systems such that ν-fold composites of successive transition maps vanish, with ν some large integer. The that reason such a property makes $f_\bullet$ look like an isomorphism is that it yields canonical maps $h_n : \ker(T_n) \to \ker(T)_{n-\nu}$ (for $n \geq \nu - 1$) that collectively serve as an inverse to $f_\bullet$ in the sense that the composites $f_n \circ h_{n+\nu}$ and $h_n \circ f_n$ for $n \gg 0$ are each just the ν-fold composites of transition maps (that induce the identity on inverse limits).

To see what is going on, we decompose $T : M \to N$ into two exact sequences

$$0 \to K \to M \to M' \to 0, \;\; 0 \to M' \to N \to M'' \to 0.$$

We get exact sequences

$$\mathrm{Tor}^1_A(M', A_n) \to K_n \to M_n \to M'_n \to 0$$

and

$$\mathrm{Tor}^1_A(M'', A_n) \to M'_n \to N_n \to M''_n \to 0.$$

Let us assume for a moment that both projective systems of Tor^1's are null systems, say with vanishing e'-fold and e''-fold composites for successive transition maps.

For any $m_n \in \ker(T_n)$, its image $m'_n \in M'_n$ comes from $\mathrm{Tor}^1_A(M'', A_n)$. Hence, projecting m'_n down into $M'_{n-e''}$ must give zero. Thus, the image of m_n in $M_{n-e''}$ must come from $K_{n-e''} = \ker(T)_{n-e''}$. However, such a lift into $K_{n-e''}$ is only well-defined modulo the image of $\mathrm{Tor}^1_A(M', A_{n-e''})$ in $K_{n-e''}$. Thus, if we push down e' steps further then the ambiguity is killed. We thereby get well-defined maps $h_n : \ker(T_n) \to \ker(T)_{n-e'-e''}$ that are A-linear and enjoy the desired properties with respect to $f_\bullet$.

To prove the Tor^1's form null systems, we claim more generally that for any finite A-module M, the $\mathrm{Tor}^p_A(M, A_n)$'s form a null system for any $p \geq 0$. The case $p = 0$ is clear. For $p = 1$, use a presentation $M \simeq F/M'$ with F finite free over A to get $\mathrm{Tor}^1_A(M, A_n) \simeq (I^{n+1}F \cap M')/I^{n+1}M'$ compatibly with change in n. By the *Artin–Rees Lemma*, there is a positive integer e such that for $m > e$,

$$I^m F \cap M' = I^{m-e}(I^e F \cap M') \subseteq I^{m-e} M'.$$

Thus, the e-fold composites of transition maps on Tor^1's vanish in degrees $> e$. Using dimension shifting when $p > 1$ completes the proof.

We are now inspired to incorporate a shifting mechanism on projective systems so that, in the notation of the preceding example, $h_\bullet$ really is an inverse to $f_\bullet$.

Consider the category of projective systems $M_\bullet = (M_n)_{n\in\mathbf{Z}}$ of A-modules indexed by $\mathbf{Z}$ such that $M_n = 0$ if $n \ll 0$. We do not assume the M_n's are A/I^{n+1}-modules (when $n \geq 0$), or even that they are A-finite. We define the (functorial) shift $M_\bullet[d] \stackrel{\text{def}}{=} (M_{n+d})$ for $d \in \mathbf{Z}$. For $d \geq 0$, define the map $i_{M_\bullet,d} : M_\bullet[d] \to M_\bullet$ to be the d-fold composite of successive transition maps. This induces an isomorphism on inverse limits, and so the map $i_{M_\bullet,d}$ should be promoted to an isomorphism. We create an inverse $i_{M_\bullet,d}^{-1}$ by changing the morphism groups:

DEFINITION 1.4.1.5. The *Artin–Rees category* of A-modules has objects $M_\bullet = (M_n)_{n\in\mathbf{Z}}$ that are projective systems of A-modules with $M_n = 0$ for $n \ll 0$, and the *Artin–Rees morphism groups* are

$$\mathrm{Hom}_{\mathrm{A-R}}(M_\bullet, N_\bullet) = \varinjlim \mathrm{Hom}(M_\bullet[d], N_\bullet),$$

where $\mathrm{Hom}(M_\bullet[d], N_\bullet)$ is the A-module of maps of projective systems, and the transition maps in the direct limit are composition with $i_{M_\bullet[d],1} : M_\bullet[d+1] \to M_\bullet[d]$. Composition of Artin–Rees morphisms is defined in the evident manner via shifts.

EXAMPLE 1.4.1.6. The map $i_{M_\bullet,d}$ for $d \geq 0$ is an isomorphism in the Artin–Rees category, with the inverse $j_{M_\bullet,d} : M_\bullet \to M_\bullet[d]$ represented by the term-by-term identity map on $M_\bullet[d]$, since $i_{M_\bullet,d} \circ j_{M_\bullet,d} = \mathrm{id}_{M_\bullet}$ and $j_{M_\bullet,d} \circ i_{M_\bullet,d} = \mathrm{id}_{M_\bullet[d]}$.

The Artin–Rees category is also abelian, and operations such as image and kernel may be formed termwise after applying a shift on the source. To see this, consider an Artin–Rees morphism $f_\bullet : M_\bullet \to N_\bullet$. Pick $d \geq 0$ so that $F_\bullet = f_\bullet \circ i_{M_\bullet,d}$ is induced by a map of projective systems. The projective systems of $\ker F_n$'s and $\operatorname{coker} F_n$'s enjoy the universal properties to be $\ker f_\bullet$ and $\operatorname{coker} f_\bullet$ in the Artin–Rees category.

1.4.2. Artin–Rees adic objects. The Artin–Rees category is visibly an A-linear category, and up to canonical equivalence it is unaffected by replacing I with any open and topologically nilpotent ideal. We emphasize that a morphism $M_\bullet \to N_\bullet$ in the Artin–Rees category is usually not given by a term-by-term collection of maps $M_n \to N_n$. Instead, we have to compose back to some shift $M_\bullet[d]$ with $d \geq 0$ and define compatible maps $M_{n+d} \to N_n$ for all n, and we identify two such systems of maps that coincide upon composition back to some $M_\bullet[d']$ with large d'. Working in the Artin–Rees category amounts to systematically invoking the Artin–Rees lemma; the shifting mechanism corresponds to the uniform constant that appears in the Artin–Rees lemma, as in Example 1.4.1.4.

An object $M_\bullet$ in the Artin–Rees category is called a *null system* if there exists $d \geq 0$ such that the map $M_{n+d} \to M_n$ vanishes for all n. This implies that $i_{M_\bullet,d}$ vanishes as an Artin–Rees morphism, and conversely if $i_{M_\bullet,d}$ vanishes as an Artin–Rees morphism then $M_{n+d+d'} \to M_n$ vanishes for all n with some large d'. Since $i_{M_\bullet,d}$ is an isomorphism for all $d \geq 0$, it vanishes if and only if $M_\bullet$ is a zero object. Thus, the zero objects in the Artin–Rees category are precisely the null systems. This has an important consequence: to check if a map $f : M_\bullet \to N_\bullet$ in the abelian Artin–Rees category is a monomorphism (resp. epimorphism), we may compose with an isomorphism $i_{M_\bullet,d}$ such that $f \circ i_{M_\bullet,d}$ is represented by compatible termwise maps $M_{n+d} \to N_n$, and then the resulting projective system of kernels (resp. cokernels) is a null system if and only if f is monic (resp. epic).

DEFINITION 1.4.2.1. Assume A/I is artinian. An object $M_\bullet$ in the Artin–Rees category *Artin–Rees I-adic* if it is Artin–Rees isomorphic to $M'_\bullet$ with $M'_n = 0$ for $n < 0$, M'_n finite over $A_n = A/I^{n+1}$ for all $n \geq 0$, and $A_n \otimes_{A_{n+1}} M'_{n+1} \to M'_n$ an isomorphism for all $n \geq 0$. Such projective systems $M'_\bullet$ are *strictly I-adic.*

The reason we impose the artinian property on A/I is to force the terms M'_n in a strictly I-adic object to be of finite length. We could have defined strict I-adicness without a finiteness condition, but such a notion is not useful.

EXAMPLE 1.4.2.2. Artin–Rees I-adics need not be strictly I-adic, even in large degrees. For an example, let M be a finite A-module with A/I artinian and pick $d > 1$. Define $M_\bullet = (M/I^{n+1}M)$. Define $M'_n = A_{n-d}^{\oplus(n-d)} \oplus M/I^{n-d}M$ for $n > d$ and $M'_n = 0$ for $n \leq d$, with transition maps given by 0 on the $A_{n-d}^{\oplus(n-d)}$'s and projection on the $M/I^{n-d}M$'s. The object $M'_\bullet$ is not strictly I-adic, but it is Artin–Rees I-adic: compose the Artin–Rees isomorphism $M'_\bullet \to M_\bullet[-d]$ with $i^{-1}_{M_\bullet[-d],d}$.

EXAMPLE 1.4.2.3. Consider the category whose objects are the strictly I-adics and whose morphisms are morphisms of projective systems (*i.e.*, compatible termwise maps in all degrees). The evident functor from this category to the Artin–Rees category is *fully faithful* onto the full subcategory of strict I-adics. This amounts to the fact that if $M_\bullet$ and $M'_\bullet$ are strict I-adics, then any ordinary map of projective systems $M_\bullet[d] \to M'_\bullet$ with $d \geq 0$ uniquely factors through $i_{M_\bullet,d}$ in the category of ordinary ($\mathbf{Z}$-indexed) projective systems. To prove this, we just have to observe that (for $n \geq 0$) any map from M_{n+d} to an A_n-module uniquely factors through projection to $A_n \otimes_{A_{n+d}} M_{n+d} \simeq M_n$.

Although the Artin–Rees category does not depend on I, the property of being Artin–Rees I-adic does depend on I. For example, we claim that the projective system $(A/I^{2n})_{n\in\mathbf{Z}}$ is not Artin–Rees I-adic when $I \neq 0$ (define $A/I^{2n} = 0$ for $n \leq 0$). This is an immediate consequence of the following lemma that serves to motivate a hypothesis in Theorem 1.4.2.5. We omit the easy proof.

LEMMA 1.4.2.4. *If $M_\bullet$ is Artin–Rees I-adic, then there exist $e, \nu \in \mathbf{Z}$ with $e \geq 0$ so that for $n \gg 0$, the image of $I^{n+1+\nu}M_n$ in M_{n-e} vanishes.*

THEOREM 1.4.2.5 (Stability properties of adic systems). *Assume A/I is artinian. The full subcategory of Artin–Rees I-adic objects is stable under formation of kernels and cokernels, and hence forms an abelian subcategory. The functor $N_\bullet \rightsquigarrow \varprojlim N_n$ is an equivalence of categories from the category of Artin–Rees I-adics to the category of finite A-modules.*

If $0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$ is a short exact sequence in the Artin–Rees category with Artin–Rees I-adic outer terms and $I^{n+1+\nu}M_n$ has vanishing image in M_{n-e} for all $n \gg 0$ and some $e, \nu \in \mathbf{Z}$ with $e \geq 0$, then $M_\bullet$ is Artin–Rees I-adic.

The hypothesis on the M_n's in the second part of the theorem is certainly satisfied if $I^{n+1+\nu}M_n = 0$ for $n \gg 0$ and some $\nu \in \mathbf{Z}$, a condition that is essentially always satisfied in practice.

PROOF. Suppose $f_\bullet : N_\bullet \to N'_\bullet$ is an Artin–Rees morphism of Artin–Rees I-adic objects. To show $\ker f_\bullet$ and $\operatorname{coker} f_\bullet$ are Artin–Rees I-adic, we may compose with isomorphisms on the source and target so that $N'_\bullet$ and $N_\bullet$ are strictly I-adic. By Example 1.4.2.3, $f_\bullet$ is induced by an ordinary map of projective systems. This nearly translates the first part of the theorem into the known equivalence of categories between strictly I-adic projective systems and finite A-modules. The only subtle aspect was handled by Example 1.4.1.4 via the *Artin–Rees lemma*: kernels formed in the category of strict I-adic objects (*i.e.*, finite A-modules) are kernels in the Artin–Rees category. Thus, $N_\bullet \rightsquigarrow \varprojlim N_n$ is an equivalence.

It remains to show that if $0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$ is Artin–Rees short exact with $M'_\bullet$ and $M''_\bullet$ both Artin–Rees I-adic and $I^{n+1+\nu}M_n$ has vanishing image in M_{n-e} for $n \gg 0$, then $M_\bullet$ is Artin–Rees I-adic. If we define $I^r = A$ for $r \leq 0$, the hypothesis on $M_\bullet$ implies that the natural map $M_\bullet \to (M_n/I^{n+1+\nu}M_n)$ has null kernel and cokernel systems, so by shifting and composing with Artin–Rees isomorphisms we may assume that $M''_\bullet$ is a strictly I-adic projective system, that $I^{n+1+d}M_n = 0$ for all n and some $d \geq 0$, and that $M'_\bullet$ is strictly I-adic. Thus, we can assume that the short exact sequence

$$0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$$

is represented by maps of projective systems $f_\bullet : M'_\bullet[d] \to M_\bullet$ and $h_\bullet : M_\bullet \to M''_\bullet$ such that $h_\bullet \circ f_\bullet = 0$ and both $(\ker f_n)$ and $(\operatorname{coker} h_n)$ are null. The exactness hypothesis implies that the kernel and cokernel systems for the map $M'_\bullet \to (\ker h_n)$ are null systems.

Define $M' = \varprojlim M'_n$, $M = \varprojlim M_n$, $M'' = \varprojlim M''_n$. By strictness, M' and M'' are finite A-modules. Consider the complex

$$0 \to M' \xrightarrow{f} M \xrightarrow{h} M'' \to 0. \tag{1.4.2.1}$$

We claim this is a short exact sequence, so in particular M is a finite A-module. The system of short exact sequences

$$0 \to \operatorname{im}(h_n) \to M''_n \to \operatorname{coker}(h_n) \to 0 \tag{1.4.2.2}$$

has left terms of finite length since each M''_n is of finite length. Recall the general *Mittag-Leffler criterion* that an inverse limit of short exact sequences of modules is short exact if the left term (N'_n) has the property that for each n, the decreasing system of images $N'_{n+\nu} \to N'_n$ stabilizes for large ν (perhaps depending on n); this criterion is always satisfied when the N'_n's have finite length, and so passage to the inverse limit on (1.4.2.2) preserves short-exactness.

We claim that passing to the inverse limit on the system of short exact sequences

$$0 \to \ker(h_n) \to M_n \to \operatorname{im}(h_n) \to 0$$

also yields a short exact. It suffices to check that the $\ker(h_n)$'s satisfy the Mittag-Leffler criterion. Since the maps $M'_n \to \ker(h_n)$ have null kernel and cokernel systems and the M'_n's have surjective transition maps (and so satisfy the Mittag-Leffler criterion), the Mittag-Leffler criterion is inherited by the $\ker(h_n)$'s, as desired.

We also need to prove that $\varprojlim M'_n \to \varprojlim \ker(h_n)$ is an isomorphism. This follows from:

LEMMA 1.4.2.6. *Let $X_\bullet = (X_n)$ and $Y_\bullet = (Y_n)$ be projective systems of modules over a ring A, with $X_n = Y_n = 0$ for $n \ll 0$, and let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of projective systems with $(\ker f_n)$ and $(\operatorname{coker} f_n)$ null systems. The induced map on inverse limits is an isomorphism of A-modules.*

PROOF. Suppose $e > 0$ is a positive integer such that all e-fold composites of successive transition maps on kernels vanish, and let $e' > 0$ serve a similar role for cokernels. It suffices to construct compatible A-linear maps $Y_n \to X_{n-e-e'}$ for $n \geq e + e'$ such that the composites

$$X_\bullet[e+e'] \to Y_\bullet[e+e'] \to X_\bullet, \;\; Y_\bullet[e+e'] \to X_\bullet \to Y_\bullet$$

are the natural maps (that induce isomorphisms on inverse limits). Since each $y_n \in Y_n$ projects to an element in $\operatorname{coker}(f_n)$ that vanishes in $\operatorname{coker}(f_{n-e'})$, we can find $x_{n-e'} \in X_{n-e'}$ whose image in $Y_{n-e'}$ is that of y_n, but $x_{n-e'}$ is ambiguous up to $\ker f_{n-e'}$. Pushing $x_{n-e'}$ into $X_{n-e-e'}$ kills the ambiguity since $\ker f_{n-e'} \to \ker f_{n-e-e'}$ is zero. This provides well-defined maps $Y_n \to X_{n-e-e'}$ that are readily checked to satisfy the desired properties. □

Returning to the proof, we may combine our inverse-limit constructions to recover (1.4.2.1) as a short exact sequence. Since $f_\bullet : M'_\bullet[d] \to M_\bullet$ and $h_\bullet : M_\bullet \to M''_\bullet$ are maps of projective systems such that the nth term is an A_n-module for all $n \geq 0$, we arrive at a commutative diagram in the Artin–Rees category

$$\begin{tikzcd}
0 \arrow[r] & (M'/I^{n+1+d}) \arrow[r] \arrow[d] & (M/I^{n+1+d}) \arrow[r] \arrow[d] & (M''/I^{n+1+d}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_\bullet[d] \arrow[r, "f_\bullet"'] & M_\bullet \arrow[r, "h_\bullet"'] & M''_\bullet \arrow[r] & 0
\end{tikzcd}$$

with outer vertical maps that are Artin–Rees isomorphisms. The bottom row is short exact in the Artin–Rees category by hypothesis, and the top row is short exact in the Artin–Rees category since it corresponds (under the equivalence between the strict I-adics and finite A-modules) to a short exact sequence of finite A-modules. Thus, the snake lemma in the abelian Artin–Rees category ensures that the middle vertical arrow in the above diagram is an Artin–Rees isomorphism; this proves that $M_\bullet$ is Artin–Rees I-adic. □

Example 1.4.2.7. Suppose A is local noetherian with maximal ideal $\mathfrak{m}$, and consider an Artin–Rees $\mathfrak{m}$-adic (M_n) with M_n a finite A_n-module for all $n \geq 0$ and $M_n = 0$ for $n < 0$. It is an instructive exercise to construct the maximal $\mathfrak{m}$-primary submodule (*i.e.*, finite-length submodule) in the associated finite A-module $M = \varprojlim M_n$ by working directly with the projective system (M_n) and not using the crutch of passage to inverse limits.

When A/I is artinian, we have now realized our goal of finding a useful categorical description of finite A-modules in terms of projective systems of finite-length modules equipped with a suitably enhanced notion of morphism. The utility of this description is due to the fact that kernels and cokernels can be formed in a termwise manner (up making suitable shifts).

Now assume A is a domain with fraction field K (retaining the condition that A/I is artinian, so A must be local). If we can describe the category of finite-dimensional K-vector spaces in terms of the category of finite A-modules, then we will get a description of the category of finite-dimensional K-vector spaces in terms of the category of Artin–Rees I-adic projective systems. Observe that every finite-dimensional K-vector space V has the form $V = K \otimes_A L$ for a finite (free) A-module L, and

$$K \otimes_A \mathrm{Hom}_A(L, L') \to \mathrm{Hom}_K(K \otimes_A L, K \otimes_A L') \tag{1.4.2.3}$$

is an isomorphism for finite A-modules L and L'. This is formally analogous to the isogeny category of complex tori, and it motivates:

Definition 1.4.2.8. The *Artin–Rees category of K-vector spaces* consists of the Artin–Rees I-adics with $\mathrm{Hom}_{\mathrm{A-R},K}(M_\bullet, N_\bullet) \overset{\mathrm{def}}{=} K \otimes_A \mathrm{Hom}_{\mathrm{A-R}}(M_\bullet, N_\bullet)$.

For clarity, if $M_\bullet$ is an Artin–Rees I-adic object then $K \otimes M_\bullet$ denotes the same object viewed in the Artin–Rees category of K-vector spaces. Using the isomorphism (1.4.2.3), we see via Theorem 1.4.2.5 that the Artin–Rees category of K-vector spaces admits a fully faithful and essentially surjective functor to the category of finite-dimensional K-vector spaces, namely $K \otimes M_\bullet \rightsquigarrow K \otimes_A \varprojlim M_n$. In Theorem 1.4.2.5 we had a natural functor from finite modules into the Artin–Rees category, namely $M \rightsquigarrow (M/I^{n+1}M)$, but in the vector-space situation there is no natural functor analogous to this (*i.e.*, carrying vector spaces to projective systems) since K-vector spaces are not spanned by natural finite A-submodules.

1.4.3. ℓ-adic sheaves. Let Λ be a complete local noetherian ring with maximal ideal $\mathfrak{m}$. Define $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$ for $n \geq 0$. Our aim in this section is to study the étale analogues of the local systems of Λ-modules in ordinary topology. For finite rings Λ (such as $\mathbf{Z}/\ell^n\mathbf{Z}$), the correct étale notion is that of an lcc Λ-module on the étale site of a scheme. For more general Λ we must leave the category of ordinary étale sheaves and develop a systematic framework for using projective systems of sheaves. We will be guided by our success in carrying out such a program for the category of finite Λ-modules.

Definition 1.4.1.5 makes sense in any abelian category whatsoever, and in such generality always yields an abelian category with kernels and cokernels formed in the evident termwise manner (upon applying a suitable shift). Since we have only discussed constructible sheaves on noetherian schemes, and the direct limit in the definition of Hom groups in the Artin–Rees category is not a good notion for sheaf categories over non-quasi-compact topologies, for the remainder of this chapter we shall require X to be noetherian. We use Definition 1.4.1.5 to define the *Artin–Rees category of* Λ*-sheaves* on $X_{\text{ét}}$. The adic objects are defined as follows:

DEFINITION 1.4.3.1. Let X be a noetherian scheme. A *strictly* $\mathfrak{m}$*-adic sheaf* on $X_{\text{ét}}$ is an object $\mathscr{F}_\bullet = (\mathscr{F}_n)_{n\in\mathbf{Z}}$ in the Artin–Rees category of Λ-sheaves on $X_{\text{ét}}$ such that $\mathscr{F}_n = 0$ for $n < 0$, $\mathscr{F}_n$ is a Λ_n-module for all $n \geq 0$, and the natural map $\mathscr{F}_{n+1}/\mathfrak{m}^{n+1}\mathscr{F}_{n+1} \to \mathscr{F}_n$ is an isomorphism for all $n \geq 0$. If these $\mathscr{F}_n$'s are all constructible, we say $\mathscr{F}_\bullet$ is a *constructible strictly* $\mathfrak{m}$*-adic sheaf.* An $\mathfrak{m}$*-adic sheaf* on $X_{\text{ét}}$ is an Artin–Rees Λ-module $\mathscr{F}_\bullet$ on $X_{\text{ét}}$ that is Artin–Rees isomorphic to a strictly $\mathfrak{m}$-adic sheaf $\mathscr{F}'_\bullet$. If $\mathscr{F}'_\bullet$ can be chosen with each $\mathscr{F}'_n$ constructible, then $\mathscr{F}_\bullet$ is a *constructible* $\mathfrak{m}$*-adic sheaf.*

Applying additive functors termwise extends ordinary sheaf constructions to these Artin–Rees categories. An elementary example is the tensor product (over Λ): form sheaf tensor-products termwise. We likewise define extension-of-scalars relative to a local map $\Lambda \to \Lambda'$, as well as the bifunctor $\mathscr{H}om_\bullet(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ when $\mathscr{F}_\bullet$ is constructible $\mathfrak{m}$-adic and $\mathscr{G}_n$ is a Λ_{n+d}-sheaf for all $n \geq 0$ and some $d \geq 0$.

We must not forget that a constructible $\mathfrak{m}$-adic sheaf is rarely strict: most cohomological functors of interest will (by non-trivial theorems) carry constructible $\mathfrak{m}$-adic sheaves to constructible $\mathfrak{m}$-adic sheaves, but the strictness property will nearly always be destroyed. From now on, we require Λ to have *finite residue field* Λ_0, since we are generally interested in constructible $\mathfrak{m}$-adic sheaves and such nonzero objects can only exist when Λ_0 is finite.

REMARK 1.4.3.2. In the literature, what we are calling a constructible $\mathfrak{m}$-adic sheaf is often called a *constructible* Λ*-sheaf.*

EXAMPLE 1.4.3.3. The category of constructible Λ-modules on $X_{\text{ét}}$ (*i.e.*, constructible abelian sheaves endowed with an action of Λ) fully faithfully embeds into the category of constructible $\mathfrak{m}$-adic sheaves on $X_{\text{ét}}$. More specifically, if $\mathscr{F}$ is a constructible Λ-module then $\mathscr{F}$ is a Λ_ν-module for some $\nu \geq 0$. The associated strict Artin–Rees object has nth term $\mathscr{F}/\mathfrak{m}^{n+1}\mathscr{F}$ for all $n \geq 0$ (so this is $\mathscr{F}$ for $n \geq \nu$).

This functor is an equivalence of categories onto the full subcategory of constructible $\mathfrak{m}$-adic sheaves $\mathscr{F}_\bullet$ such that $\mathscr{F}_\bullet[\mathfrak{m}^{\nu+1}] \to \mathscr{F}_\bullet$ is an Artin–Rees isomorphism for some large ν, with $(\cdot)[\mathfrak{m}^{\nu+1}]$ denoting $\mathfrak{m}^{\nu+1}$-torsion. Indeed, for any $\mathscr{F}_\bullet$ enjoying this latter property we may pass to an isomorphic object so that $\mathscr{F}_\bullet$ is strictly $\mathfrak{m}$-adic; in this case, the isomorphism condition says that $\mathscr{F}_n$ is a Λ_ν-module for all n, and so strictness implies $\mathscr{F}_n = \mathscr{F}_\nu$ for all $n \geq \nu$.

For $\Lambda = \mathbf{Z}_\ell$ with ℓ a unit on X, an important example of a constructible ℓ-adic sheaf is $\mathbf{Z}_\ell(1)$, the strict object whose nth term is $\mu_{\ell^{n+1}}$ for $n \geq 0$. Its dual sheaf $\mathscr{H}om_{\mathbf{Z}_\ell}(\mathbf{Z}_\ell(1), \mathbf{Z}_\ell)$ is denoted $\mathbf{Z}_\ell(-1)$. More generally, we can form tensor powers $\mathbf{Z}_\ell(r)$ for all $r \in \mathbf{Z}$. Extension of scalars defines the $\mathfrak{m}$-adic sheaves $\Lambda(r)$ when Λ has residue characteristic ℓ.

EXAMPLE 1.4.3.4. As an illustration of the difference between working in the Artin–Rees category and working with inverse-limit sheaves on the étale site, consider the inverse-limit sheaf $\mathscr{F} = \varprojlim \mu_{\ell^{n+1}}$ on $(\operatorname{Spec} \mathbf{Q})_{\text{ét}}$. Since any number field has only finitely many roots of unity, and a finite abelian group has trivial maximal ℓ-divisible subgroup, clearly $\mathscr{F}(\operatorname{Spec} K) = \{1\}$ for every number field K. That is, $\mathscr{F}$ vanishes. In constrast, the constructible ℓ-adic sheaf $\mathbf{Z}_\ell(1)$ is non-zero.

Let us say that $\mathscr{G}_\bullet$ in the Artin–Rees category has *stable images* if there is a $d_0 \geq 0$ so that for each n the image of $\mathscr{G}_{n+d}$ in $\mathscr{G}_n$ is the same for all $d \geq d_0$ (this is called the *Mittag-Leffler Artin–Rees condition* in [**14**, Exp. V, 2.1.1]). Although the operation $\mathscr{G} \rightsquigarrow \mathscr{G}_n$ is not a functor on the Artin–Rees category (think about the inverse of $i_{\mathscr{G}_\bullet,d}$ for $d > 0$), the property of having stable images is invariant under passage from $\mathscr{G}_\bullet$ to $\mathscr{G}_\bullet[d]$ for any d, and hence if $\mathscr{G}_\bullet$ and $\mathscr{G}'_\bullet$ are Artin–Rees isomorphic then $\mathscr{G}_\bullet$ has stable images if and only if $\mathscr{G}'_\bullet$ does.

The formation of the stable image in a fixed degree is not functorial in $\mathscr{G}_\bullet$ unless we restrict attention to strictly $\mathfrak{m}$-adic objects; the problem is caused by shifting. However, in one case the shifting does not cause difficulties: if $\mathscr{G}_\bullet$ has stable images, say $\overline{\mathscr{G}}_n$ in degree n, then the condition that the natural maps $\overline{\mathscr{G}}_{n+1} \to \overline{\mathscr{G}}_n$ be isomorphisms for all $n \gg 0$ is invariant under Artin–Rees isomorphisms. Let us say that such a $\mathscr{G}_\bullet$ has *terminal stable images* (and the common $\overline{\mathscr{G}}_n$ for large n is called the *terminal stable image* of $\mathscr{G}_\bullet$). On the full subcategory of $\mathscr{G}_\bullet$'s with terminal stable images, the terminal stable image of $\mathscr{G}_\bullet$ is functorial in $\mathscr{G}_\bullet$.

The inclusion of the (ordinary) constructible Λ-modules into the Artin–Rees category is an equivalence onto the full subcategory of those Artin–Rees objects with terminal stables images such that the terminal stable image is constructible. Formation of the terminal stable image provides a quasi-inverse to this inclusion functor.

1.4.4. Properties of constructible $\mathfrak{m}$-adic sheaves.

THEOREM 1.4.4.1 (Local nature of constructibility). *Let $\{S_i\}$ be a stratification of X, and let $\{U_j\}$ be an étale cover. If $\mathscr{F}_\bullet$ is an object in the Artin–Rees category* *of Λ-modules on $X_{\text{ét}}$, then $\mathscr{F}_\bullet$ is constructible $\mathfrak{m}$-adic if and only if all $\mathscr{F}_\bullet|_{S_i}$ or all $\mathscr{F}_\bullet|_{U_i}$ are constructible $\mathfrak{m}$-adic.*

PROOF. We can assume the U_i's are quasi-compact. For $n \geq 0$, define $\widetilde{\mathscr{F}}_\bullet^n = \mathscr{F}_\bullet/\mathfrak{m}^{n+1}\mathscr{F}_\bullet$. Assume that restricting $\mathscr{F}_\bullet$ to either all the S_i's or to all the U_j's gives a constructible $\mathfrak{m}$-adic sheaf. We claim that $\widetilde{\mathscr{F}}_\bullet^n$ has terminal stable images and its terminal stable image is a constructible Λ_n-module. These properties are obvious when $\mathscr{F}_\bullet$ is a constructible strictly $\mathfrak{m}$-adic sheaf, and it suffices to check these properties for the restriction of $\mathscr{F}_\bullet$ to all U_i's or all S_j's; invariance under passage to an isomorphic object in the Artin–Rees category thereby settles the claim.

Now let $\mathscr{F}'_n$ denote the functorial terminal stable image of $\widetilde{\mathscr{F}}_\bullet^n$; functoriality provides us with maps $\mathscr{F}'_{n+1} \to \mathscr{F}'_n$ over $\Lambda_{n+1} \to \Lambda_n$, and the induced maps $\mathscr{F}'_{n+1}/\mathfrak{m}^{n+1}\mathscr{F}'_{n+1} \to \mathscr{F}'_n$ are isomorphisms: this may be checked either étale-locally or over a stratification. Thus, if we define $\mathscr{F}'_n = 0$ for $n < 0$ then $\mathscr{F}'_\bullet = (\mathscr{F}'_n)$ is a constructible strictly $\mathfrak{m}$-adic object. It remains to construct an Artin–Rees isomorphism $\mathscr{F}'_\bullet \simeq \mathscr{F}_\bullet$. The key point is that there is a large integer e independent of n so that the terminal stable image in $\mathscr{F}_\bullet/\mathfrak{m}^{n+1}\mathscr{F}_\bullet$ is achieved in degree $n+e$. The existence of such an integer (though not its value) is unaffected by replacing $\mathscr{F}_\bullet$ with an Artin–Rees isomorphic object, and so to prove existence we may work étale-locally on X or over the strata of a stratification of X. Thus, the existence problem is reduced to the trivial constructible strict case.

With the integer e in hand, we get a morphism $\mathscr{F}_\bullet[e] \to \mathscr{F}'_\bullet$ that we claim is an isomorphism. To prove this, it is harmless to pass to a larger e. We can then work either étale-locally or on the strata of a stratification to reduce to the case when $\mathscr{F}_\bullet$ is Artin–Rees isomorphic to a strict object. Since the problem is unaffected by enlarging e, and the construction of $\mathscr{F}'_\bullet$ is functorial in $\mathscr{F}_\bullet$, we reduce to the trivial case when $\mathscr{F}_\bullet$ is a constructible strictly $\mathfrak{m}$-adic sheaf. □

REMARK 1.4.4.2. An important consequence of the preceding proof is that if $\mathscr{F}_\bullet$ is a constructible $\mathfrak{m}$-adic sheaf, the (constructible) strict object $\mathscr{F}'_\bullet$ constructed via terminal-images is functorial in $\mathscr{F}_\bullet$. This provides a quasi-inverse to the inclusion functor from the category of constructible strict $\mathfrak{m}$-adic projective systems to the category of constructible $\mathfrak{m}$-adic sheaves.

DEFINITION 1.4.4.3. Let $\mathscr{F}_\bullet$ be a constructible $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$. The *stalk* of $\mathscr{F}_\bullet$ at a geometric point $\overline{x}$ is the finite Λ-module $\varprojlim (\mathscr{F}_n)_{\overline{x}}$.

By Lemma 1.4.2.6, formation of the stalk at a geometric point is a functor. An alternative definition of the stalk that makes this obvious is to pull back to a geometric point and apply Theorem 1.4.2.5 to the Artin–Rees category of constructible $\mathfrak{m}$-adic sheaves on a geometric point.

DEFINITION 1.4.4.4. A constructible $\mathfrak{m}$-adic sheaf $\mathscr{F}_\bullet$ on $X_{\text{ét}}$ is a *lisse* $\mathfrak{m}$*-adic sheaf* if it is Artin–Rees isomorphic to a strictly $\mathfrak{m}$-adic sheaf $\mathscr{F}'_\bullet$ with $\mathscr{F}'_n$ an lcc Λ_n-module for all $n \geq 0$.

By Theorem 1.4.4.1 and Remark 1.4.4.2, this definition is a local property for the étale topology on X. In general, the property of being lisse $\mathfrak{m}$-adic is preserved under formation of tensor products and $\mathscr{H}om$'s. The sheaves $\Lambda(r)$ are lisse $\mathfrak{m}$-adic when $\mathrm{char}(\Lambda_0)$ is a unit on X. As in Example 1.4.3.3, lisse $\mathfrak{m}$-adic sheaves rarely become constant over an étale cover.

EXAMPLE 1.4.4.5. Let $\mathscr{F}_\bullet$ be a lisse $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$. Consider the stalk $(\mathscr{F}_\bullet)_{\overline{x}}$ at a geometric point $\overline{x}$ of X. By Remark 1.4.4.2, we can functorially find a strict object $\mathscr{F}'_\bullet$ with lcc $\mathscr{F}'_n$'s and an Artin–Rees isomorphism $\mathscr{F}_\bullet \simeq \mathscr{F}'_\bullet$.

Assume that X is connected. By Grothendieck's equivalence between lcc sheaves and finite monodromy representations (Theorem 1.2.2.3), the stalk $(\mathscr{F}_\bullet)_{\overline{x}}$ is a finite Λ-module equipped with a continuous linear action of $\pi_1(X, \overline{x})$, and this construction is functorial in $\mathscr{F}_\bullet$. Since any finite Λ-module M with a continuous linear action of $\pi_1(X, \overline{x})$ can be canonically expressed as $\varprojlim M_n$ where the modules $M_n = M/\mathfrak{m}^{n+1}M$ are endowed with compatible continuous linear actions of π_1, the stalk functor at $\overline{x}$ is an equivalence of categories between lisse $\mathfrak{m}$-adic sheaves and continuous linear representations of $\pi_1(X, \overline{x})$ on finite Λ-modules. This is an analogue of the dictionary in §**??** between local systems and π_1-representations for reasonable connected topological spaces.

One important consequence of this description is that, without connectivity requirements, the category of lisse $\mathfrak{m}$-adic sheaves on $X_{\text{ét}}$ is an abelian category and the inclusion of this category into the abelian Artin–Rees category of $\mathfrak{m}$-adic sheaves is compatible with the formation of kernels and cokernels. To prove this claim we may assume X is connected, and then the equivalence with continuous linear representations of $\pi_1(X, \overline{x})$ on finite Λ-modules shows that the category of lisse $\mathfrak{m}$-adic sheaves on $X_{\text{ét}}$ is abelian. The torsion-level description of module operations in the Artin–Rees theory for ordinary modules immediately implies that the kernels and cokernels formed in the category of lisse $\mathfrak{m}$-adic sheaves agree with kernels and cokernels formed in the category of Artin–Rees $\mathfrak{m}$-adic sheaves.

EXAMPLE 1.4.4.6. There is a specialization criterion for constructible $\mathfrak{m}$-adic sheaves $\mathscr{F}_\bullet$ to be lisse $\mathfrak{m}$-adic: it is necessary and sufficient that the natural specializaton map $\mathscr{F}_{\bullet,\overline{s}} \to \mathscr{F}_{\bullet,\overline{\eta}}$ is an isomorphism whenever $\overline{\eta}$ generizes $\overline{s}$. Indeed, we may assume $\mathscr{F}_\bullet$ is a strict $\mathfrak{m}$-adic system with constructible $\mathscr{F}_n$'s, so $\mathscr{F}_{n,\overline{s}} = \mathscr{F}_{\bullet,\overline{s}}/\mathfrak{m}^{n+1}\mathscr{F}_{\bullet,\overline{s}}$, and similarly for $\overline{\eta}$. It follows that the specialization maps for the constructible $\mathscr{F}_n$'s are isomorphisms, so each $\mathscr{F}_n$ is lcc.

Ordinary constructible sheaves on a noetherian scheme X become lcc upon restriction to the strata of a stratification of X. This result also has an $\mathfrak{m}$-adic analogue in terms of lisse $\mathfrak{m}$-adic sheaves as defined above. We now prove this result and record some of its consequences:

THEOREM 1.4.4.7. *Let $\mathscr{F}_\bullet$ be a constructible $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$. There exists a stratification of X such that the restriction of $\mathscr{F}_\bullet$ to each stratum is lisse. In particular, $\mathscr{F}_\bullet$ has vanishing stalks at all geometric points if and only if $\mathscr{F}_\bullet = 0$, and a complex of constructible $\mathfrak{m}$-adic sheaves is exact if and only if it is exact on all stalks.*

PROOF. It is harmless to pass to Artin–Rees isomorphic objects, so we may assume $\mathscr{F}_\bullet$ is a constructible strictly $\mathfrak{m}$-adic sheaf. Our module work proved that inverse limits set up an equivalence between the Artin–Rees category of $\mathfrak{m}$-adic Λ-modules and the category of finite Λ-modules. The zero objects are the null systems, so the stratification claim implies the stalk claims.

By noetherian induction, we just have to find a Zariski-dense open U in X such that the constructible restrictions $\mathscr{F}_n|_U$ are lcc. The finite filtration of $\mathscr{F}_n$ by $\mathfrak{m}^m\mathscr{F}_n$'s for $0 \le m \le n+1$ has successive quotients $\mathfrak{m}^m\mathscr{F}_m$ due to strictness. By the specialization criterion a constructible sheaf to be lcc, the middle term in a short exact sequence of constructible abelian sheaves on $X_{\text{ét}}$ is lcc if the outer terms are lcc. Thus, it suffices to find a dense open U such that $\mathfrak{m}^m\mathscr{F}_m|_U$ is lcc for all $m \geq 0$.

Let $B = \mathrm{gr}_{\mathfrak{m}}\Lambda \stackrel{\text{def}}{=} \oplus_{m\geq 0}\mathfrak{m}^m/\mathfrak{m}^{m+1}$; this is an $\mathbf{N}_0$-graded finitely generated Λ_0-algebra ($\mathbf{N}_0 = \mathbf{N} \cup \{0\}$). We may view $\mathscr{G} = \oplus_{m\geq 0}\mathfrak{m}^m\mathscr{F}_m$ as an $\mathbf{N}_0$-graded sheaf of B-modules, and as such we have a surjection

$$\pi : B \otimes_{\Lambda_0} \mathscr{F}_0 = \bigoplus_{m\geq 0}(\mathfrak{m}^m/\mathfrak{m}^{m+1}) \otimes_{\Lambda_0} \mathscr{F}_0 \twoheadrightarrow \mathscr{G}.$$

If we let $\mathscr{K} = \oplus\mathscr{K}_j$ denote the $\mathbf{N}_0$-graded kernel of this B-linear surjection, then all $\mathscr{K}_j$'s are constructible. The isomorphism

$$((\mathfrak{m}^m/\mathfrak{m}^{m+1}) \otimes_{\Lambda_0} \mathscr{F}_0)/\mathscr{K}_m \simeq \mathfrak{m}^m\mathscr{F}_m$$

implies that $\mathfrak{m}^m\mathscr{F}_m|_U$ is lcc if $\mathscr{K}_m|_U$ and $\mathscr{F}_0|_U$ are lcc ($U \subseteq X$ open).

The key claim (proof below) is that $B\otimes_{\Lambda_0}\mathscr{F}_0$ is a graded-noetherian B-module in the sense that rising chains of B-stable $\mathbf{N}_0$-graded subsheaves terminate. This implies that any $\mathbf{N}_0$-graded B-submodule of $B \otimes_{\Lambda_0} \mathscr{F}_0$ is generated over B by its graded components in degrees $j \leq m_0$ for some (variable) m_0. Apply this to $\mathscr{K}$. For $m > m_0$, the subsheaf $\mathscr{K}_m$ in $(\mathfrak{m}^m/\mathfrak{m}^{m+1}) \otimes_{\Lambda_0} \mathscr{F}_0$ is the image of the multiplication map

$$\phi_m : \bigoplus_{j=0}^{m_0}(\mathfrak{m}^{m-j}/\mathfrak{m}^{m+1-j}) \otimes_{\Lambda_0} \mathscr{K}_j \to (\mathfrak{m}^m/\mathfrak{m}^{m+1}) \otimes_{\Lambda_0} \mathscr{F}_0.$$

Thus, if we pick a dense open U such that $\mathscr{K}_j|_U$ is lcc for all $0 \leq j \leq m_0$ and $\mathscr{F}_0|_U$ is lcc, then $\phi_m|_U$ is a map between lcc sheaves for all $m > m_0$. Hence, its image $\mathscr{K}_m|_U$ is lcc for all m, as desired.

To prove that $B\otimes_{\Lambda_0}\mathscr{F}_0$ is a graded-noetherian B-module, we must use the fact that constructible sheaves on a noetherian scheme are (precisely the) noetherian objects in the category of abelian étale sheaves on any noetherian scheme. Applying this to $\mathscr{F}_0$ and recalling that B is a finitely generated $\mathbf{N}_0$-graded Λ_0-algebra, we reduce to the following sheafified Hilbert basis theorem: if A is any commutative ring, B is an $\mathbf{N}_0$-graded finite-type A-algebra, and $\mathscr{F}$ is a noetherian A-module, then $B \otimes_A \mathscr{F}$ is a graded-noetherian B-module. Such a result can be proved in any A-linear abelian category admitting tensor products [**14**, Exp. V, 5.1.4], but in our special case the argument goes as follows. We may reduce to the case $B = A[t_1, \ldots, t_r]$ with the t_j's assigned various degrees, and so

$$B \otimes_A \mathscr{F} = \bigoplus_{e_1,\ldots,e_r\geq 0} \mathscr{F} t_1^{e_1} \cdots t_r^{e_r}.$$

Using multiplication by monomials, a graded B-submodule $\mathscr{M}$ of $B \otimes_A \mathscr{F}$ is a direct sum of A-submodules $\mathscr{M}_{\underline{e}} \subseteq \mathscr{F} \simeq \mathscr{F} \cdot t^{\underline{e}}$ for each multi-index $\underline{e}$, with the requirement that $\mathscr{M}_{\underline{e}} \subseteq \mathscr{M}_{\underline{e}'}$ inside $\mathscr{F}$ whenever $e_i \leq e_i'$ for all $1 \leq i \leq r$ (this inclusion corresponds to multiplication by $\prod t_i^{e_i'-e_i}$ on $\mathscr{M}$).

We claim that the set of distinct $\mathscr{M}_{\underline{e}}$'s (as subobjects of $\mathscr{F}$) is finite, so finitely many $\mathscr{M}_{\underline{e}}$'s generate $\mathscr{M}$ over B. Granting this, suppose $\mathscr{M}^{(1)} \subseteq \mathscr{M}^{(2)} \subseteq \ldots$ is a rising sequence of graded B-submodules of $B\otimes_A\mathscr{F}$, with $\mathscr{M}^{(\alpha)}$ having component $\mathscr{M}_{\underline{e}}^{(\alpha)}$ for the monomial $t^{\underline{e}}$. If we define $\mathscr{M} = \varinjlim \mathscr{M}^{(\alpha)}$ then we want $\mathscr{M}^{(\alpha')} = \mathscr{M}$ for some α'. Clearly $\mathscr{M}$ is a graded B-submodule of $B \otimes_A \mathscr{F}$ with $\underline{e}$th component

$\varinjlim \mathscr{M}_{\underline{e}}^{(\alpha)}$, so by the noetherian property of $\mathscr{F}$ we get $\mathscr{M}_{\underline{e}} = \mathscr{M}_{\underline{e}}^{(\alpha_{\underline{e}})}$ for some $\alpha_{\underline{e}}$. By hypothesis there exist only finitely many distinct $\mathscr{M}_{\underline{e}}$'s, say associated to multi-indices $\underline{e}^{(1)}, \dots \underline{e}^{(\mu)}$, and these generate $\mathscr{M}$ over B. We may take $\alpha' = \max \alpha_{\underline{e}^{(j)}}$.

To establish in general that there are only finitely many distinct $\mathscr{M}_{\underline{e}}$'s, consider the more general situation of a map of partially ordered sets $\phi : \mathbf{N}_0^r \to \mathscr{P}$, where $r \geq 1$ and $\mathbf{N}_0^r$ is partially ordered by the condition $\underline{e} \leq \underline{e}'$ whenever $e_i \leq e_i'$ for all i. Assume $\mathscr{P}$ is noetherian in the sense that any monotonically increasing sequence in $\mathscr{P}$ terminates. For example, $\mathscr{P}$ could be the set of subobjects $\mathscr{M}_{\underline{e}}$ (partially ordered by inclusion) inside of the noetherian object $\mathscr{F}$. We claim that any such ϕ has finite image. If not, then by selecting ϕ-preimages of an infinite sequence of distinct points, we get an infinite sequence σ of distinct points in $\mathbf{N}_0^r$ such that (by the noetherian property of $\mathscr{P}$) there is no subsequence that is strictly increasing relative to the partial ordering on $\mathbf{N}_0^r$. A simple pigeonhole argument via induction on r shows that no such sequence can exist. $\square$

Corollary 1.4.4.8. *Assume Λ is a discrete valuation ring, and let $\mathscr{F}_\bullet$ be a constructible $\mathfrak{m}$-adic sheaf. There is a unique ordinary constructible Λ-subsheaf $\mathscr{F}$ contained in $\mathscr{F}_\bullet$ such that $\mathscr{F}_\bullet/\mathscr{F}$ has Λ-flat stalks.*

Proof. Let $\mathscr{F}_\bullet(n)$ be the $\mathfrak{m}^{n+1}$-torsion subobject in $\mathscr{F}_\bullet$. This is a constructible $\mathfrak{m}$-adic sheaf that is killed by a power of $\mathfrak{m}$, and so it is an ordinary constructible Λ_n-sheaf. The $\mathscr{F}_\bullet(n)$'s are a rising chain of subobjects of $\mathscr{F}_\bullet$ whose stalks are the corresponding $\mathfrak{m}$-power torsion-levels in stalks of $\mathscr{F}_\bullet$ (due to exactness of stalk functors). Thus, it suffices to show that this chain terminates. By using a suitable stratification of X, we may assume $\mathscr{F}_\bullet$ is lisse. In this case, we can assume X is connected, and then we can inspect the associated finite Λ-module with continuous π_1-action to see the termination property. $\square$

Theorem 1.4.4.9 (Stability properties of adic systems). *The full subcategory of constructible $\mathfrak{m}$-adic sheaves is stable under the formation of kernels and cokernels within the larger Artin–Rees category of Λ-modules on $X_{\text{ét}}$.*

Moreover, if

$$0 \to \mathscr{F}'_\bullet \to \mathscr{F}_\bullet \to \mathscr{F}''_\bullet \to 0$$

is a short exact sequence in the Artin–Rees category of Λ-modules such that $\mathscr{F}'_\bullet$ and $\mathscr{F}''_\bullet$ are constructible $\mathfrak{m}$-adic, and $\mathfrak{m}^{n+1+\nu}\mathscr{F}_n$ has vanishing image in $\mathscr{F}_{n-e}$ for $n \gg 0$ and some $e, \nu \in \mathbf{Z}$ with $e \geq 0$, then $\mathscr{F}_\bullet$ is a constructible $\mathfrak{m}$-adic sheaf.

The hypothesis on $\mathscr{F}_\bullet$ is satisfied if $\mathfrak{m}^{n+1+\nu}\mathscr{F}_n = 0$ for $n \gg 0$ and some $\nu \in \mathbf{Z}$.

Proof. To establish the first part, let $\phi : \mathscr{G}'_\bullet \to \mathscr{G}_\bullet$ be a map between constructible $\mathfrak{m}$-adic sheaves. We can find a stratification of X such that $\mathscr{G}'_\bullet$ and $\mathscr{G}_\bullet$ have lisse restriction on the strata. To prove that $\ker \phi$ and $\operatorname{coker} \phi$ are constructible $\mathfrak{m}$-adic, it suffices to work on these strata, and so we may assume $\mathscr{G}'_\bullet$ and $\mathscr{G}_\bullet$ are lisse. This case was settled in Example 1.4.4.5.

Now consider a short exact sequence

$$0 \to \mathscr{F}'_\bullet \to \mathscr{F}_\bullet \to \mathscr{F}''_\bullet \to 0$$

in the Artin–Rees category of Λ-modules on $X_{\text{ét}}$ such that the outer terms are constructible $\mathfrak{m}$-adic and $\mathfrak{m}^{n+1+\nu}\mathscr{F}_n$ has vanishing image in $\mathscr{F}_{n-e}$ for $n \gg 0$ and some $e, \nu \in \mathbf{Z}$ with $e \geq 0$. We want to prove that $\mathscr{F}_\bullet$ is constructible $\mathfrak{m}$-adic. We may assume that the given short exact sequence is short-exact in the category of projective systems. Let $\phi'' : \widetilde{\mathscr{F}}''_\bullet[d''] \to \mathscr{F}_\bullet$ be a map of projective systems that is an Artin–Rees isomorphism with $\widetilde{\mathscr{F}}''_\bullet$ a strictly $\mathfrak{m}$-adic system whose terms are constructible. We may form the ϕ''-pullback of the given short exact sequence to reduce to the case when $\mathscr{F}''_\bullet$ is a shift on a constructible stricly $\mathfrak{m}$-adic projective system, and the shift can be eliminated by applying an auxiliary shift: in the Artin–Rees category, this procedure only changes terms up to isomorphism.

There is likewise a map $\phi' : \mathscr{F}'_\bullet[d'] \to \widetilde{\mathscr{F}}'_\bullet$ to a constructible strictly $\mathfrak{m}$-adic system with ϕ' an Artin–Rees isomorphism, and forming a $\phi'[-d']$-pushout on our short exact sequence of projective systems yields $\mathscr{F}''_\bullet[-d']$ as the kernel term. Thus, may assume that our short exact sequence is short-exact as a diagram of projective systems and that the outer terms are constructible strictly $\mathfrak{m}$-adic. Each abelian sheaf $\mathscr{F}_n$ is therefore an extension of one noetherian object by another, and so $\mathscr{F}_n$ is a noetherian object in $\mathrm{Ab}(X)$. Thus, all $\mathscr{F}_n$'s are constructible. Passing to a stratification allows us to assume that outer terms $\mathscr{F}'_n$ and $\mathscr{F}''_n$ are all lcc. Thus, each constructible $\mathscr{F}_n$ satisfies the specialization criterion for local constancy. We may assume X is connected, say with $\overline{x}$ a geometric point, so all of the sheaf-theoretic information can be transferred into the category of finite Λ-modules equipped with a continuous linear action of $\pi_1(X, \overline{x})$. The proof of Theorem 1.4.2.5 may now be carried over with the additional data of π_1-actions carried along in the constructions. $\square$

1.4.5. $\mathbf{Q}_\ell$-sheaves. To make a theory with coefficients in a field, we employ the same localization device on Hom-groups as near the end of §1.4.2:

Definition 1.4.5.1. Let $(\Lambda, \mathfrak{m})$ be a complete local noetherian domain with finite residue field; let K be its fraction field. The category of *constructible K-sheaves* on a noetherian scheme X is the localization of the category of constructible $\mathfrak{m}$-adic sheaves at the action of nonzero elements of Λ. That is, its objects are the same as those of the category of constructible $\mathfrak{m}$-adic sheaves, but we apply $K \otimes_\Lambda (\cdot)$ to the Hom-modules. If $\mathscr{F}_\bullet$ is a constructible $\mathfrak{m}$-adic sheaf, then $K \otimes \mathscr{F}_\bullet$ denotes the same object viewed in the category of constructible K-sheaves.

As an example, $K(1) = K \otimes \Lambda(1)$ denotes the Tate object when the residue characteristic of Λ is a unit on X. In general, the category of constructible K-sheaves is an abelian category and the functor $\mathscr{F}_\bullet \rightsquigarrow K \otimes \mathscr{F}_\bullet$ from constructible $\mathfrak{m}$-adic sheaves to constructible K-sheaves is exact.

If $(\Lambda', \mathfrak{m}')$ is a finite extension of Λ, the functor $\Lambda' \otimes_\Lambda (\cdot)$ from constructible $\mathfrak{m}$-adic sheaves to constructible $\mathfrak{m}'$-adic sheaves defines the extension-of-scalars functor $K' \otimes_K (\cdot)$ from constructible K-sheaves to constructible K'-sheaves.

Example 1.4.5.2. In the special case when Λ is a discrete valuation ring, every constructible K-sheaf is isomorphic to $K \otimes \mathscr{F}_\bullet$ where $\mathscr{F}_\bullet$ is a constructible $\mathfrak{m}$-adic sheaf that has Λ-flat stalks. Indeed, $\mathscr{F}_\bullet$ contains a constructible Λ_r-subsheaf $\mathscr{F}$ such that $\mathscr{F}_\bullet/\mathscr{F}$ has Λ-flat stalks (Theorem 1.4.4.8), so the vanishing of $K \otimes \mathscr{F}$ gives the result.

The *stalk* of a constructible K-sheaf $K \otimes \mathscr{F}_\bullet$ at a geometric point $\overline{x}$ is the finite-dimensional K-vector space $K \otimes_\Lambda \mathscr{F}_{\bullet, \overline{x}}$. A constructible K-sheaf vanishes if and only if its stalks (in the K-sense) all vanish: this ultimately comes down to the fact that $K \otimes \mathscr{F}_\bullet = 0$ if and only if some nonzero element in Λ kills $\mathscr{F}_\bullet$ in the Artin–Rees category, and fact is an obvious result on the module side that transfers to the constructible $\mathfrak{m}$-adic sheaf side because we can find a stratification of X such that $\mathscr{F}_\bullet$ has lisse restrictions on the strata. Similarly, exactness of a complex of constructible K-sheaves can be checked on K-vector space stalks.

DEFINITION 1.4.5.3. A *lisse* K-sheaf on $X_{\text{ét}}$ is a constructible K-sheaf that is isomorphic to $K \otimes \mathscr{F}_\bullet$ for a lisse $\mathfrak{m}$-adic sheaf $\mathscr{F}_\bullet$.

THEOREM 1.4.5.4. *Let X be a connected noetherian scheme with geometric point $\overline{x}$. The stalk at $\overline{x}$ defines a fully faithful and essentially surjective functor from lisse K-sheaves on $X_{\text{ét}}$ to continuous K-linear representations of $\pi_1(X, \overline{x})$ on finite-dimensional K vector spaces.*

PROOF. The key is to reduce to the integral case in Example 1.4.4.5. To make this work, we just need to know that any continuous linear representation of π_1 on a finite-dimensional K-vector space V stabilizes some Λ-submodule that spans V over K. Pick any finite Λ-submodule L_0 in V that spans V over K. Since L_0 is open in V and π_1 is compact, some finite-index open subgroup H in π_1 carries L_0 into itself. Replacing L_0 with the slightly larger lattice $L = \sum g(L_0)$ as g runs over representatives of π_1/H, we get a π_1-stable submodule as desired. $\square$

REMARK 1.4.5.5. Let us mention some interesting applications of Theorem 1.4.5.4, though we will not use them and so we omit the proofs. Assume that X is normal and noetherian. Theorem 1.4.5.4 can be used to prove that the property of a K-sheaf being lisse is local for the étale topology on X, and that a constructible K-sheaf $\mathscr{F}_\bullet$ on $X_{\text{ét}}$ is lisse if and only if for any pair of geometric points $\overline{s}$, $\overline{\eta}$ on X with $\overline{s}$ specializing $\overline{\eta}$, the specialization map $(\mathscr{F}_\bullet)_{\overline{s}} \to (\mathscr{F}_\bullet)_{\overline{\eta}}$ is an isomorphism. The role of normality in the proofs is the surjectivity aspect in Example 1.2.2.2.

Étale descent also works for lisse K-sheaves on normal noetherian schemes X. To make this precise, let $f : X' \to X$ be an étale surjection with X' noetherian, and let $\mathscr{F}'$ be a lisse K-sheaf on X'. *Decent data* on $\mathscr{F}'$ relative to f is an isomorphism $\varphi : p_2^*\mathscr{F}' \simeq p_1^*\mathscr{F}'$ on $X' \times_X X'$ that satisfies the cocycle condition with respect to pullbacks on the triple product. If $\mathscr{F}' = f^*\mathscr{F}$ for a lisse K-sheaf $\mathscr{F}$ on X, then there is obvious descent data φ on $\mathscr{F}'$ relative to f, and the theorem that can be proved via Theorem 1.4.5.4 is that the functor $\mathscr{F} \rightsquigarrow (\mathscr{F}', \varphi)$ is an equivalence of categories from the category of lisse K-sheaves on X to the category of lisse K-sheaves on X' equipped with descent data relative to f.

DEFINITION 1.4.5.6. Let X be a noetherian scheme, and fix a prime ℓ and an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$. The category of *constructible $\overline{\mathbf{Q}}_\ell$-sheaves* on $X_{\text{ét}}$ is the $\overline{\mathbf{Q}}_\ell$-linear category whose objects are triples $(\mathscr{F}, K, i)$ where $i : K \hookrightarrow \overline{\mathbf{Q}}_\ell$ is an embedding of a finite extension of $\mathbf{Q}_\ell$ into $\overline{\mathbf{Q}}_\ell$ and $\mathscr{F}$ is a constructible K-sheaf, and the morphism-groups are

$$\mathrm{Hom}((\mathscr{F}, K, i), (\mathscr{F}', K', i')) \stackrel{\text{def}}{=} \overline{\mathbf{Q}}_\ell \otimes_{K''} \mathrm{Hom}(K'' \otimes_K \mathscr{F}, K'' \otimes_{K'} \mathscr{F}'),$$

where K'' is any finite extension of $\mathbf{Q}_\ell$ inside of $\overline{\mathbf{Q}}_\ell$ containing both $i(K)$ and $i'(K')$. A constructible $\overline{\mathbf{Q}}_\ell$-sheaf is *lisse* if it is $\overline{\mathbf{Q}}_\ell$-isomorphic to a triple $(\mathscr{F}, K, i)$ with $\mathscr{F}$ a lisse K-sheaf for some $K \subseteq \overline{\mathbf{Q}}_\ell$ of finite degree over $\mathbf{Q}_\ell$.

It is mechanical to carry over the K-sheaf results to the case of $\overline{\mathbf{Q}}_\ell$-sheaves; this ultimately comes down to the fact that the constructible adic theory is compatible with the extension-of-scalars functors $\Lambda \to \Lambda'$ induced by finite extensions on discrete valuation rings.

THEOREM 1.4.5.7. *If X is a connected noetherian scheme with geometric point $\overline{x}$, then the stalk functor at $\overline{x}$ defines a fully faithful and essentially surjective functor from the category of lisse $\overline{\mathbf{Q}}_\ell$-sheaves to the category of continuous linear representations of $\pi_1(X,\overline{x})$ on finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces.*

PROOF. Since any map of finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces may be defined over a finite extension of any specified subfield of definition of the two vector spaces, the only non-obvious fact we have to prove is that for any affine algebraic group G over a finite extension K of $\mathbf{Q}_\ell$ (*e.g.*, $G = \mathrm{GL}_n$), any continuous representation $\rho : \pi_1(X,\overline{x}) \to G(\overline{\mathbf{Q}}_\ell)$ factors through $G(K')$ for K' some subfield of $\overline{\mathbf{Q}}_\ell$ that is finite over K; this is false if $\overline{\mathbf{Q}}_\ell$ is replaced with its completion. The group $G(\overline{\mathbf{Q}}_\ell)$ is Hausdorff since G is separated, and the image of ρ is a compact subgroup Γ. Thus, it suffices to prove that any compact subgroup Γ in $G(\overline{\mathbf{Q}}_\ell) = \cup G(K')$ is contained in some $G(K')$. By chasing coset representatives, there is no harm in replacing Γ with a (finite index) open subgroup. Hence, it suffices to show that one of the closed subgroups $\Gamma \cap G(K')$ has non-empty interior in Γ. These subgroups form a countable directed system of closed subsets of the compact space Γ and their union is Γ, so the Baire category theorem would solve our problem if Γ is metrizable. Since G is affine, $G(\overline{\mathbf{Q}}_\ell)$ is metrizable and hence so is the subset Γ. □

1.4.6. ℓ-adic cohomology. As usual, fix Λ to be a complete local noetherian ring with finite residue field and maximal ideal $\mathfrak{m}$. The cohomological functors are defined on Artin–Rees categories of $\mathfrak{m}$-adic sheaves by applying the functors termwise in projective systems. For example, if $f : X \to S$ is a map between noetherian schemes then we define $\mathrm{R}^i f_*(\mathscr{F}_\bullet)$ to be the projective system $(\mathrm{R}^i f_*(\mathscr{F}_n))$, and similarly for $\mathrm{R}^i f_!$ when f is separated and finite type (and all $\mathscr{F}_n$'s are torsion sheaves). The Leray spectral sequences, excision, and cup products carry over. The cohomological functors carry null systems to null systems, and so it is readily checked that a short exact sequence of Artin–Rees $\mathfrak{m}$-adic sheaves gives rise to a long exact cohomology sequence of Artin–Rees sheaves. The proper and smooth base change theorems likewise carry over.

THEOREM 1.4.6.1. *Let $f : X \to S$ be a separated finite-type morphism between noetherian schemes. Let $\mathscr{F}_\bullet$ be a constructible $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$. The projective system $\mathrm{R}^i f_!(\mathscr{F}_\bullet)$ is a constructible $\mathfrak{m}$-adic sheaf for all i. If S is of finite type over a regular base of dimension ≤ 1 and the residue characteristic of Λ is invertible on S, then the projective system $\mathrm{R}^i f_*(\mathscr{F}_\bullet)$ is a constructible $\mathfrak{m}$-adic sheaf for all i.*

PROOF. Observe that in our definition of the Artin–Rees category of projective systems of Λ-modules, we could have uses an arbitrary Λ-linear abelian category $\mathscr{A}$ in the role of the category of sheaves of Λ-modules. In the special case when $\mathscr{A}$ is the category of of sheaves of Λ-modules, note that an object killed by $\mathfrak{m}^{n+1}$ for some $n \geq 0$ is a noetherian object in $\mathscr{A}$ if and only if its underlying abelian sheaf is constructible. Thus, we can generalize the concept of a constructible $\mathfrak{m}$-adic sheaf as follows: for any Λ-linear abelian category $\mathscr{A}$, a projective system $\mathscr{F}_\bullet$ in $\mathscr{A}$ with $\mathscr{F}_n = 0$ for $n \ll 0$ is *AR-$\mathfrak{m}$-adic noetherian* if it is Artin–Rees isomorphic to a strictly $\mathfrak{m}$-adic system $\mathscr{F}'_\bullet$ such that $\mathscr{F}'_n$ is a noetherian object in $\mathscr{A}$ for all n.

In [**14**, Exp. V], a theory of Artin–Rees projective systems is developed for any Λ-linear abelian category $\mathscr{A}$, with Λ any commutative ring. A key result in the theory [**14**, Exp. V, 5.3.1] is that if $F = (F^i) : \mathscr{A} \to \mathscr{A}'$ is a (Λ-linear) δ-functor between such categories then the analgous termwise functors F^i on categories of projective systems preserve the subcategory of AR-$\mathfrak{m}$-adic noetherian objects if:

(1) $F^i(\mathscr{F})$ is noetherian for any noetherian $\mathscr{F}$ that is killed by $\mathfrak{m}$;
(2) $F^i = 0$ for sufficiently large i.

Strictly speaking, the criterion in [**14**, Exp. V, 5.3.1] replaces (2) with a condition involving graded sheaves of modules over finite-type graded Λ-algebras, but it can be proved in an elementary manner that this condition is implied by (1) and (2).

It remains to verify (1) and (2) for the δ-functors $\mathrm{R}^\bullet f_!$ and $\mathrm{R}^\bullet f_*$ on categories of Λ-sheaves. Theorems 1.3.6.3 and 1.3.6.4 provide these properties. □

Example 1.4.6.2. If X is separated of finite type over a separably closed field and $\mathscr{F}_\bullet$ is a constructible $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$, we conclude that the projective system of cohomologies $\mathrm{H}^i_{\mathrm{c,\acute{e}t}}(X, \mathscr{F}_n)$ with fixed i is Artin–Rees isomorphic to the strict object on the (not obviously) finite Λ-module $H^i = \varprojlim \mathrm{H}^i_{\mathrm{c,\acute{e}t}}(X, \mathscr{F}_n)$. That is, if we fix i then the maps $H^i/\mathfrak{m}^{n+1}H^i \to \mathrm{H}^i_{\mathrm{c,\acute{e}t}}(X, \mathscr{F}_n)$ have cokernels and kernels that form null systems. There is no claim made about kernels or cokernels being of bounded length as $n \to \infty$.

Let us now assume that Λ be a discrete valuation ring (with fraction field K). This restriction is needed to prove most of the interesting cohomology theorems, partly due to the simple nature of Λ-flatness and partly due to the simple nature of duality theory for Λ_n-modules (*e.g.*, Λ_n is injective over itself). Here is an $\mathfrak{m}$-adic version of the generic base-change theorem (Theorem 1.3.6.4):

Theorem 1.4.6.3. *Let $f : X \to S$ be a separated map between schemes of finite type over regular base of dimension ≤ 1. Let $\mathscr{F}_\bullet$ be a constructible $\mathfrak{m}$-adic sheaf on X and assume that $\Lambda/\mathfrak{m}$ has characteristic that is invertible on S. There exists a dense open $U \subseteq S$, depending on $\mathscr{F}_\bullet$, such that the formation of $\mathrm{R}^i f_* \mathscr{F}|_U$ commutes with arbitrary base change on U.*

Proof. In principle, the result at finite level only produces a dense open U_n such that higher direct images of $\mathscr{F}_n$ commute with base change over U_n, and U_n depends on $\mathscr{F}_n$. Thus, we need an additional argument to find a U that works for $\mathscr{F}_\bullet$. For a short exact sequence of constructible $\mathfrak{m}$-adic sheaves

$$0 \to \mathscr{G}'_\bullet \to \mathscr{G}_\bullet \to \mathscr{G}''_\bullet \to 0,$$

the long exact cohomology sequence shows that a dense open that works for two of these also works for the third. Thus, we may reduce to the case where $\mathscr{F}_\bullet$ is a constructible Λ_n-sheaf for some n or is a strict $\mathfrak{m}$-adic sheaf whose terms $\mathscr{F}_n$ are constructible and Λ_n-flat for all n.

The constructible Λ_n-sheaf case is already known, and if $\mathscr{F}_\bullet$ is a strict $\mathfrak{m}$-adic sheaf with each $\mathscr{F}_n$ constructible and Λ_n-flat, then we claim that U_1 works for each $\mathscr{F}_n$, and so U_1 works for $\mathscr{F}_\bullet$. Indeed, we have short exact sequences

$$0 \to \mathscr{F}_1 \to \mathscr{F}_{n+1} \to \mathscr{F}_n \to 0$$

due to the hypotheses on $\mathscr{F}_\bullet$, so induction on n follows readily via the long exact cohomology sequence. □

For constructible K-sheaves, we define cohomology (and higher direct images with and without proper supports) by carrying out constructions in the Artin–Rees $\mathfrak{m}$-adic categories and then embedding the (constructible $\mathfrak{m}$-adic) result into the corresponding K-sheaf categories. The stalks of such higher direct images (with and without proper supports) are K-vector spaces of finite dimension. The same goes for constructible $\overline{K}$-sheaves. The preceding results all immediately carry over to the categories of constructible K-sheaves and constructible $\overline{K}$-sheaves.

THEOREM 1.4.6.4 (Künneth formula revisited). *Let S be a noetherian scheme and $f : X \to S$ and $f' : X' \to S'$ separated maps of finite type. For constructible K-sheaves $\mathscr{F}$ and $\mathscr{F}'$ on X and X', the natural cup-product pairing*

$$\bigoplus_{p+q=r} \mathrm{R}^p f_! \mathscr{F} \otimes_K \mathrm{R}^q f_! \mathscr{F}' \to \mathrm{R}^r (f \times f')_! (\pi^* \mathscr{F} \otimes_K {\pi'}^* \mathscr{F}')$$

is an isomorphism, where π and π' are the projections for $X \times_S X'$.

PROOF. We may assume S is a geometric point. As in the previous argument, we may find strict $\mathfrak{m}$-adic sheaves $\mathscr{G} = (\mathscr{G}_n)$ and $\mathscr{G}' = (\mathscr{G}'_n)$ on X and X' with $\mathscr{F} \simeq K \otimes \mathscr{G}$ and $\mathscr{F}' \simeq K \otimes \mathscr{G}'$ such that $\mathscr{G}_n$ and $\mathscr{G}'_n$ are Λ_n-flat for all n. The map of interest is the extension-of-scalars to K on the inverse limit of the Künneth morphisms

$$\bigoplus_{p+q=r} \mathrm{H}^p_{\mathrm{c,ét}}(X, \mathscr{G}_n) \otimes_{\Lambda_n} \mathrm{H}^q_{\mathrm{c,ét}}(X', \mathscr{G}'_n) \to \mathrm{H}^r_{\mathrm{c,ét}}(X \times X', \pi^* \mathscr{G}_n \otimes_{\Lambda_n} {\pi'}^* \mathscr{G}'_n);$$

the Künneth formula in the form of Theorem 1.3.9.2 says that this is a degree-r edge map in a spectral sequence with terms

$$\mathrm{E}_2^{p,q} = \bigoplus_{a+a'=q} \mathrm{Tor}^p_{\Lambda_n}(\mathrm{H}^a_{\mathrm{c,ét}}(X, \mathscr{G}_n), \mathrm{H}^{a'}_{\mathrm{c,ét}}(X', \mathscr{G}'_n)),$$

and that this identification is compatible with reduction maps $\Lambda_{m+1} \to \Lambda_m$ and change-of-coefficients maps on Tor's.

Since the E_2-terms at finite level have finite length, we may pass to the inverse limit on n to get a similar spectral sequence at the $\mathfrak{m}$-adic level, where the E_2-terms are the inverse limits of the finite-level E_2-terms. Thus, it suffices to show that for fixed p and q with $p > 0$, the inverse limit of the finite-level $\mathrm{E}_2^{p,q}$'s is a torsion Λ-module (as then tensoring with K kills such terms and leaves us with the desired map as an isomorphism). Taking into account the Artin–Rees formalism, we are reduced to the general claim that if M and M' are finite Λ-modules, then

$$\varprojlim \mathrm{Tor}^p_{\Lambda_n}(M/\mathfrak{m}^{n+1}, M'/\mathfrak{m}^{n+1})$$

is a torsion Λ-module for $p > 0$. The structure theorem for finite Λ-modules reduces us to the cases when each module is either Λ or some Λ_m. The case when either is Λ is trivial, and so we may assume $M = \Lambda_m$ and $M' = \Lambda_{m'}$ for some $m, m' \geq 0$. In this case, any Λ-bilinear bifunctor of M and M' is killed by $\mathfrak{m}^{\max(m,m')+1}$. $\square$

THEOREM 1.4.6.5 (Poincaré duality revisited). *Let k be a separably closed field with characteristic not divisible by the residue characteristic of Λ. Let $\mathscr{F}$ be a lisse K-sheaf on X. Assume X is smooth with dimension d. The pairing*

$$\mathrm{H}^i_{\mathrm{ét}}(X, \mathscr{F}^\vee(d)) \otimes \mathrm{H}^{2d-i}_{\mathrm{c,ét}}(X, \mathscr{F}) \longrightarrow \mathrm{H}^{2d}_{\mathrm{c}}(X, K(d)) \xrightarrow[\mathrm{tr}]{\simeq} K$$

is a perfect duality between finite-dimensional K-vector spaces.

PROOF. We may write $\mathscr{F} = K \otimes \mathscr{G}$ for a strictly $\mathfrak{m}$-adic sheaf $\mathscr{G} = (\mathscr{G}_n)$, where each $\mathscr{G}_n$ is lcc and Λ_n-flat. It suffices to prove that the pairing of finite Λ-modules

$$\mathrm{H}^i_{\text{ét}}(X, \mathscr{G}^\vee(d)) \times \mathrm{H}^{2d-i}_{\text{c,ét}}(X, \mathscr{G}) \to \Lambda(d)$$

is a non-degenerate modulo torsion (and so it is perfect upon extending scalars to K). The mod-$\mathfrak{m}^{n+1}$-reduction of this pairing factors as

$$\mathrm{H}^i_{\text{ét}}(X, \mathscr{G}^\vee(d))_n \times \mathrm{H}^{2d-i}_{\text{c,ét}}(X, \mathscr{G})_n \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{G}_n^\vee(d)) \times \mathrm{H}^{2d-i}_{\text{c,ét}}(X, \mathscr{G}_n) \to \Lambda_n$$

where the final step is a perfect pairing by Poincaré duality at finite level (Theorem 1.3.8.1). The maps in the first step are isomorphisms up to kernel and cokernel terms that are killed by a uniform power $\mathfrak{m}^N$, and so failure of non-degeneracy in the Λ-adic pairing is entirely due to torsion elements in the inverse limit. $\square$

1.4.7. Analytification of ℓ-adic sheaves. Let $(\Lambda, \mathfrak{m})$ be a complete local noetherian ring with finite residue field. Let X be a finite type $\mathbf{C}$-scheme. If $\mathscr{F}_\bullet$ is a constructible $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$, its *analytification* is the abelian sheaf

$$\mathscr{F}_\bullet^{\text{an}} \stackrel{\text{def}}{=} \varprojlim \mathscr{F}_n^{\text{an}}$$

on $X^{\text{an}} = X(\mathbf{C})$. This is a functor because shifting induces an isomorphism on inverse limits.

THEOREM 1.4.7.1. *The functor $\mathscr{F}_\bullet \rightsquigarrow \mathscr{F}_\bullet^{\text{an}}$ is exact. It also commutes with formation of stalks, with extension by zero from an open subset, and with pushforward from a closed set. Moreover, the analytification of a lisse $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$ is a local system of finite Λ-modules on X^{an}.*

PROOF. The compatibility of analytification with pushforward under closed immersions has been verified at the level of constructible abelian sheaves, and so it holds in the $\mathfrak{m}$-adic case because pushforward commutes with formation of inverse limits of sheaves on ordinary topological spaces.

Assume the compatibility of analytification and stalks; this will be checked shortly. Exactness of analytification now follows immediately, as does compatibility with extension by zero (this comes down to proving vanishing of certain stalks of an analytified extension by zero). To prove that if $\mathscr{F}_\bullet$ is lisse $\mathfrak{m}$-adic then $\mathscr{F}_\bullet^{\text{an}}$ is a local system on X^{an}, we may assume $\mathscr{F}_\bullet$ is strict with the $\mathscr{F}_n$'s lcc, and we can assume X is connected. In particular, all stalks of $\mathscr{F}_\bullet$ at geometric points are abstractly isomorphic as finite Λ-modules. The same therefore holds for the stalks of $\mathscr{F}_\bullet^{\text{an}}$, due to the assumed stalk-compatibility of analytification. Pick $x \in X(\mathbf{C})$. The isomorphism $(\mathscr{F}_\bullet^{\text{an}})_x \simeq \varprojlim \mathscr{F}_{n,x}$ and the strictly $\mathfrak{m}$-adic property of $(\mathscr{F}_{n,x})$ yield an isomorphism

$$(\mathscr{F}_\bullet^{\text{an}})_x / \mathfrak{m}^{n+1} (\mathscr{F}_\bullet^{\text{an}})_x \simeq \mathscr{F}_{n,x}$$

compatibly with change in n. Thus, we may choose local sections $s_1, \dots, s_r$ of $\mathscr{F}_\bullet^{\text{an}}$ near x with the s_j's projecting to a Λ_0-basis of $\mathscr{F}_{0,x}^{\text{an}}$.

Since $\mathscr{F}_0^{\text{an}}$ is locally constant, the s_j's project to a basis of $\mathscr{F}_0^{\text{an}}$ near x, and since $\mathscr{F}_{\bullet,y}^{\text{an}}$ is a finite Λ-module with mod-$\mathfrak{m}$ reduction $\mathscr{F}_{0,y}^{\text{an}}$ we deduce by Nakayama's lemma that the s_j's generate $\mathscr{F}_\bullet^{\text{an}}$ near x. Finitely many generating relations on the s_j's in $\mathscr{F}_{\bullet,x}^{\text{an}}$ also kill the image of the s_j's in $\mathscr{F}_{\bullet,y}^{\text{an}}$ for y near x. We therefore arrive at a surjection $\pi : M \to \mathscr{F}_\bullet^{\text{an}}|_U$ over some open U around x, where M is a finite Λ-module and π_x an isomorphism. All stalks of $\mathscr{F}_\bullet^{\text{an}}$ are abstractly isomorphic as

Λ-modules, so the surjective π_y is an isomorphism for all y near x. Hence, π is an isomorphism near x, and so $\mathscr{F}_\bullet^{\mathrm{an}}$ is a local system of finite Λ-modules.

Now we turn to the essential part: for a constructible $\mathfrak{m}$-adic $\mathscr{F}_\bullet$ on $X_{\text{ét}}$ and any $x \in X(\mathbf{C})$, we want to prove that the map $(\varprojlim \mathscr{F}_n^{\mathrm{an}})_x \to \varprojlim \mathscr{F}_{n,x}^{\mathrm{an}}$ is an isomorphism. We shall use an elegant argument due to Deligne. The problem is local, so we may assume X is separated and finite-type, and $\mathscr{F}_\bullet$ is strict.

Consider a decreasing sequence of closed sets $X = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_\nu = \emptyset$ such that each $\mathscr{F}_n|_{Z_j - Z_{j-1}}$ is lcc. We shall write $\mathscr{S}$ to denote the collection of locally closed sets $U_j = Z_j - Z_{j-1}$. A sheaf of sets on $X(\mathbf{C})$ is *compatible* with $\mathscr{S}$ if its restriction to each $U_j(\mathbf{C})$ is locally constant. On the topological side, the sheaves $\mathscr{F}_n^{\mathrm{an}}$ on $X(\mathbf{C})$ restrict to local systems (of finite abelian groups) on the locally closed subsets $U_j(\mathbf{C})$ that disjointly cover $X(\mathbf{C})$.

LEMMA 1.4.7.2. *Let Y be a Zariski-closed set in X. For any open $U \subseteq X(\mathbf{C})$ around $Y(\mathbf{C})$, there exists an open $V \subseteq U$ around $Y(\mathbf{C})$ such that* $\mathrm{image}(\mathscr{G}(U) \to \mathscr{G}(V)) \to \Gamma(Y(\mathbf{C}), \mathscr{G})$ *is injective for all sheaves of sets $\mathscr{G}$ on $X(\mathbf{C})$ that are compatible with $\mathscr{S}$.*

PROOF. To prepare for what we will prove, write $Y = Y_1 \cup Y_2$ where Y_1 is the union of the irreducible components of Y that are irreducible components of X, and Y_2 is the union of the other irreducible components of Y (that are nowhere dense in X). For each generic point η of X, there is some least $j > 0$ such that $\eta \notin Z_j$. Let U_η be an irreducible dense open in $\overline{\{\eta\}}$, so U_η is contained in U_{j-1}. The U_η's are pairwise-disjoint opens in X with dense union U, and it is clear that any $\mathscr{G}$ compatible with $\mathscr{S}$ has locally constant restriction to U. We may therefore replace Z_j with $Z_j \cap (X - U)$ for all $j > 0$ to reduce to the case with Z_1 nowhere dense in X. If the nowhere-dense Y_2 is not contained in Z_1, then we replace Z_1 with $Z_1 \cup Y_2$ and replace Z_{j+1} with Z_j for all $j > 1$. This brings us to the case when Z_1 contains Y_2 and Z_1 is nowhere-dense in X.

Let $\widetilde{X}$ be the normalization of X_{red}, and let $\widetilde{Y}$ (resp. $\widetilde{Y}_j$) be the preimage of Y (resp. Y_j) in $\widetilde{Y}$. Clearly $\widetilde{Y}_1$ is a union of connected components of $\widetilde{X}$, and $\widetilde{Y}_2$ is nowhere-dense in the other connected components. Let $X' \to \widetilde{X}$ denote the blow-up along $\widetilde{Y}_2$ (say with its reduced structure), and write Y' and Y'_j to denote the preimages of Y and Y_j in X'. Let Z'_j be the preimage of Z_j in X'.

The connected components of X' are irreducible, and $Y' = Y'_1 \cup Y'_2$ with Y'_1 a union of connected components of X' and Y'_2 a closed set of pure codimension 1 in the other connected components of X'; keep in mind that these other connected components are irreducible. These properties are preserved under pullback to any disjoint union of blow-ups along proper closed subsets of irreducible components of X'. Thus, if we next blow-up along Z'_1, and then blow-up along the preimage of Z'_2 in this new blow-up, *etc.*, and rename the final result as X', we lose none of the preceding properties but we gain the condition that each non-empty Z_j for $j > 0$ has preimage Z'_j in X' of pure codimension 1. That is, the Z'_j's are a decreasing sequence of hypersurfaces in X'. Let $\mathscr{S}'$ be the collection of locally closed sets $U'_j = Z'_j - Z'_{j+1}$ in X'.

Applying de Jong's alterations theorem [**5**, Thm. 4.1] to each connected (*i.e.*, irreducible) component of X' and the locus where this component meets Z'_1, we get a smooth quasi-projective $\mathbf{C}$-scheme X_0 and a generically-finite proper map $f : X_0 \to X$ such that $f^{-1}(Z'_1)$ is a strict normal crossings divisor D_0 (*i.e.*, a union of smooth hypersurfaces that are mutually transverse at all $\mathbf{C}$-points where they meet). The preimage $Y_0 = f^{-1}(Y)$ contains some connected components of X_0 and meets the others in a pure codimension-1 subset that lies in D_0. Thus, Y_0 is a union of some connected components of X_0 and of some irreducible components of D_0 in the other connected components of X_0. Likewise, for codimension reasons, the preimage $Z_{j,0}$ of each Z_j in X_0 (for $j > 0$) is a union of irreducible components of D_0, and these form a decreasing sequence. Let $\mathscr{S}_0$ be the collection of locally closed sets $U_{j,0} = Z_{j,0} - Z_{j+1,0}$ in X_0.

Applying Theorem 1.4.7.4 to the manifold $X_0(\mathbf{C})$, the collection $\mathscr{S}_0$, and the part of $Y_0(\mathbf{C})$ lying over $Y_2(\mathbf{C})$ (*i.e.*, ignoring connected components of $X_0(\mathbf{C})$ that are contained in $Y_0(\mathbf{C})$), there exists a fundamental system of opens W in $X_0(\mathbf{C})$ around $Y_0(\mathbf{C})$ such that $\mathscr{G}_0(W) \to \mathscr{G}_0(Y_0(\mathbf{C}))$ is an isomorphism for all sheaves $\mathscr{G}_0$ on $X_0(\mathbf{C})$ that are compatible with $\mathscr{S}_0$ (the situation on connected components of $X_0(\mathbf{C})$ that are contained in $Y_0(\mathbf{C})$ is trivial). We pick such an open, call it V_0, and may assume $V_0 \subseteq f^{-1}(U)$. Since f is proper and V_0 contains $f^{-1}(Y(\mathbf{C})) = Y_0(\mathbf{C})$, there exists an open V around $Y(\mathbf{C})$ with $f^{-1}(V) \subseteq V_0$. Surjectivity of f therefore implies $V \subseteq U$. It remains to check that two sections in $\mathscr{G}(U)$ that are equal along $Y(\mathbf{C})$ must be equal over V. By surjectivity of f, it suffices to show that two sections in $\mathscr{G}_0(f^{-1}(U))$ that are equal along $Y_0(\mathbf{C})$ must coincide over $f^{-1}(V)$. Since $f^{-1}(V) \subseteq V_0$ and $\mathscr{G}_0(V_0)$ injects into $\mathscr{G}_0(Y_0(\mathbf{C}))$, we get the desired result. $\square$

LEMMA 1.4.7.3. *There exists an open V' around $Y(\mathbf{C})$ such that for $\mathscr{G}$ compatible with $\mathscr{S}$, $\mathscr{G}(V')$ surjects onto $\mathscr{G}(Y(\mathbf{C}))$.*

PROOF. Let $f : X_0 \to X$ be an alteration as in the proof of the preceding lemma (with $\mathscr{S} = \{Z_j\}$ modified as in that proof so that Z_1 is nowhere-dense and $Y_2 \subseteq Z_1$, without loss of generality). Define $Y_0 = f^{-1}(Y)$, $X_{00} = X_0 \times_X X_0$, and $Y_{00} = Y_0 \times_X X_0 = X_0 \times_X Y_0$. Let $\mathscr{S}_0$ and $\mathscr{S}_{00}$ denote the pullbacks of $\mathscr{S}$ to X_0 and X_{00}, so $\mathscr{S}_0$ is a decreasing chain of hypersurfaces in X_0. Note that the irreducible components of Y_0 that are not connected components of X_0 are contained in Z_1. Pick V_0 in $X_0(\mathbf{C})$ around $Y_0(\mathbf{C})$ as in Theorem 1.4.7.4 so that $\mathscr{G}_0(V_0) \to \mathscr{G}_0(Y_0(\mathbf{C}))$ is an isomorphism for all $\mathscr{G}_0$ on $X_0(\mathbf{C})$ compatible with $\mathscr{S}_0$ (this is trivial on the connected components of X_0 that are contained in Y_0).

Apply the preceding lemma to the nonempty closed set Y_{00} in X_{00} and the open $U_{00} \overset{\mathrm{def}}{=} \mathrm{pr}_1^{-1}(V_0) \cap \mathrm{pr}_2^{-1}(V_0)$ around $Y_{00}(\mathbf{C})$. We get an open $V_{00} \subseteq U_{00}$ around $Y_{00}(\mathbf{C})$ such that the map

$$\mathrm{image}(\mathscr{G}_{00}(U_{00}) \to \mathscr{G}_{00}(V_{00})) \to \mathscr{G}_{00}(Y_{00}(\mathbf{C})) \tag{1.4.7.1}$$

is injective. By properness we may find V' in $X(\mathbf{C})$ around $Y(\mathbf{C})$ such that the preimage V'_{00} of V' in $X_{00}(\mathbf{C})$ is contained in V_{00}, so the preimage V'_0 of V' in $X_0(\mathbf{C})$ lies in V_0. Since $\mathscr{G}_0(V_0) \to \mathscr{G}_0(Y_0(\mathbf{C}))$ is an isomorphism, every $s \in \mathscr{G}(Y(\mathbf{C}))$ has pullback to $Y_0(\mathbf{C})$ that uniquely extends to an $\widetilde{s}_0 \in \mathscr{G}_0(V_0)$.

We just need $\widetilde{s}_0|_{V'_0}$ to descend to V_0. The two pullbacks of $\widetilde{s}_0|_{V'_0}$ to sections over V'_{00} coincide, as these are both restrictions from sections of $\mathscr{G}_{00}(U_{00})$ that have the same restriction s_{00} on $Y_{00}(\mathbf{C})$ and hence coincide (by the injectivity in (1.4.7.1)). Since $V'_{00} = V'_0 \times_{V'} V'_0$, it suffices to show that if $T' \to T$ is a proper surjection of topological spaces and $\mathscr{G}$ is a sheaf on T with pullbacks $\mathscr{G}'$ and $\mathscr{G}''$ to T' and $T'' = T' \times_T T'$, then $\mathscr{G}(T)$ is the equalizer kernel of $\mathscr{G}'(T') \rightrightarrows \mathscr{G}''(T'')$. Sections of $\mathscr{G}$ may be functorially identified with continuous T-maps to a topological space

$\widetilde{\mathscr{G}}$ over T (using the pre-Grothendieck viewpoint of sheaves as topological spaces), and a continuous map $T' \to \widetilde{\mathscr{G}}$ that is constant on fibers over T uniquely factors continuously through the quotient map $T' \to T$, so we are done. $\square$

We can now apply the two lemmas to conclude that for sufficiently small open U in $X(\mathbf{C})$ around $Y(\mathbf{C})$, there exists an open $V \subseteq U$ around $Y(\mathbf{C})$ such that $\mathrm{image}(\mathscr{G}(U) \to \mathscr{G}(V)) \to \Gamma(Y(\mathbf{C}), \mathscr{G})$ is an isomorphism for all $\mathscr{G}$ compatible with the stratification $\mathscr{S}$. This general result will now be applied to the sheaves $\mathscr{F}_n^{\mathrm{an}}$ on $X(\mathbf{C})$ and $Y = \{x\}$. We conclude that for any sufficiently small open U in $X(\mathbf{C})$ around x, there exists a smaller open V around x so that

$$\mathrm{image}(\mathscr{F}_n^{\mathrm{an}}(U) \to \mathscr{F}_n^{\mathrm{an}}(V)) \to (\mathscr{F}_n^{\mathrm{an}})_x \simeq \mathscr{F}_{n,x} \tag{1.4.7.2}$$

is an isomorphism for all n.

Choose a sequence of pairs $U_m \supseteq V_m$ as above (for $m = 1, 2, \dots$), with the U_m's a base of opens around x and $U_{m+1} \subseteq V_m$. Letting $I_{n,m}$ denote the image of $\mathscr{F}_n^{\mathrm{an}}(U_m)$ in $\mathscr{F}_n^{\mathrm{an}}(V_m)$, for fixed m the $I_{n,m}$'s form a compatible system of subgroups of the $\mathscr{F}_n^{\mathrm{an}}(V_m)$'s with $I_{n,m}$ mapping isomorphically to $\mathscr{F}_{n,x}$, so the $I_{n,m}$'s are compatible with change in n. Hence, for each fixed $m \geq 1$, any element in $\mathscr{F}_{\bullet,x}$ arises from an element in $\varprojlim I_{n,m} \subseteq \mathscr{F}_\bullet^{\mathrm{an}}(V_m)$. This shows that $\iota_x : \mathscr{F}_{\bullet,x}^{\mathrm{an}} \to \mathscr{F}_{\bullet,x}$ is surjective. For injectivity of ι_x, pick an element s_x in the kernel and represent it by $s \in \mathscr{F}_\bullet^{\mathrm{an}}(U_m)$ for some large m. For each $n \geq 0$, the kernel of $\mathscr{F}_n^{\mathrm{an}}(U_m) \to \mathscr{F}_{n,x}$ vanishes under restriction to $\mathscr{F}_n^{\mathrm{an}}(V_m)$. Thus, $s|_{V_m} = 0$, so $s_x = 0$. This completes the proof of Theorem 1.4.7.1, conditional on Theorem 1.4.7.4. $\square$

Let X be a paracompact Hausdorff complex manifold and D a non-empty analytic set in X that is a normal crossings divisor (*i.e.*, the reduced analytic structure on D is locally described by the vanishing of a product of local coordinate functions). Consider a decreasing chain $Z_1 \supseteq Z_2 \supseteq \dots$ of analytic sets in $D = Z_1$ with each Z_j a union of intersections of irreducible components of D. Let $\mathscr{S}$ be the collection of locally closed sets $U_j = Z_j - Z_{j+1}$, where $Z_0 = X$. A sheaf of sets $\mathscr{G}$ on X is *compatible with* $\mathscr{S}$ if $\mathscr{G}|_{U_j}$ is locally constant for all j.

THEOREM 1.4.7.4. *Let Y be a non-empty union of irreducible components of D. There exists a base of opens V in X around Y such that $\mathscr{G}(V) \to \mathscr{G}(Y)$ is an isomorphism for all $\mathscr{G}$ that are compatible with $\mathscr{S}$.*

PROOF. By choosing an open around Y and renaming it as X, it suffices to produce one V. We can assume X is connected, hence of finite dimension, and so we may induct on the dimension of X. The case $\dim X \leq 1$ is clear, since D is a discrete closed subset in this case, so we now assume $\dim X > 1$.

Step 1. We shall reduce to the case $Y = D$ with Y connected. Since the paracompact Hausdorff manifold X is connected, it is a normal topological space with a countable base of opens. The set $\{Y_i\}$ of connected components of the analytic set Y is therefore countable, and since it is locally finite we see that any union of Y_i's is closed in X. By enumerating the Y_i's and using normality of X, we can find disjoint opens U and U' in X around Y_1 and $Y' = \cup_{i>1} Y_i$ respectively, with U connected. Repeating the same for Y' in the space U', and so on, we are reduced to the case of connected Y. Removing the connected components of D not containing Y does not disconnect X, by the Riemann extension theorem (see the proof of Theorem **??**). Thus, it can be assumed that D is connected.

With D connected, hence path-connected, we claim that $\mathscr{G}(D) \to \mathscr{G}(Y)$ is injective. To see this, choose a path $\sigma : [0,1] \to D$ joining a chosen $d \in D$ to Y, with $\sigma(t)$ lying in an overlap of irreducible components of D for only finitely many t. Pullback to $[0,1]$ thereby reduces us to the claim that if $\mathscr{H}$ is a sheaf of sets on $[0,1]$ that is locally constant on $(0,1]$, then $\mathscr{H}([0,1]) \to \mathscr{H}_0$ is injective. This claim is easy to prove, since a locally constant sheaf on an interval is constant.

If $Y \neq D$, pick an irreducible component H of D that meets Y but is not contained in Y. The restriction $\mathscr{G}|_H$ is compatible with the decomposition $\mathscr{S}_H$ induced by $\mathscr{S}$ on H. Since $\dim H = \dim X - 1$ and $H \cap Y$ is non-empty of pure codimension 1 in H, by induction on dimension (applied to the connected manifold H and the decomposition $\mathscr{S}_H$) there is an open W in H around $H \cap Y$ such that $\mathscr{G}'(W) \to \mathscr{G}'(H \cap Y)$ is an isomorphism for $\mathscr{G}'$ on H compatible with $\mathscr{S}_H$. Since $H \cap Y = W \cap Y$, by taking $\mathscr{G}' = \underline{\mathbf{Z}}$ we see that each connected component of W meets Y. Thus, $Y \cup W$ is connected. Discard $H - W$ from X, and if this operation disconnects D then discard the connected components of D not containing Y. Rename $Y \cup W$ as Y. If $Y \neq D$, the process can be repeated. The set of irreducible components of D is countable and locally finite in X, so this recursive process reduces us to the case $Y = D$ (so $\mathscr{G}|_{X-Y}$ is locally constant).

Step 2. For non-empty connected open V in X, the complement $V - (Y \cap V)$ is connected. Thus, $\mathscr{G}(V) \to \mathscr{G}(Y \cap V)$ is injective for all such V since $\mathscr{G}|_{X-Y}$ is locally constant. It is therefore enough to find one connected open V_0 around Y such that $\mathscr{G}(V_0) \to \mathscr{G}(Y)$ is surjective for all $\mathscr{G}$. We will first give an explicit V_0 for a local version of the problem, and then (in Step 3) we globalize the solution. The local problem is this: X is the open unit polydisc in $\mathbf{C}^n$, $j : Y \hookrightarrow X$ is cut out in X by $z_1 \cdots z_r = 0$ for some $r \leq n$, and we claim $\mathscr{G}(X) \hookrightarrow \mathscr{G}(Y)$ is an isomorphism for any sheaf of sets $\mathscr{G}$ on X that is locally constant on a fixed stratification defined by unions of intersections of irreducible components of Y.

The case $n = 1$ is clear, and we shall induct on n. The inductive hypothesis applies to the unit polydisc in the coordinate hyperplanes $z_j = 0$ for $r < j \leq n$, so we may assume $r = n$. Exhausting the open unit polydisc by closed polydiscs of polyradius $\rho \to 1^-$, we reduce ourselves to considering the situation when we replace X with the closed unit polydisc $\overline{B}$ and replace Y with the locus Z that is the union of the coordinate hyperplanes in $\overline{B}$; we consider sheaves of sets on $\overline{B}$ that are locally constant on $\overline{B} - Z$. For each $0 < \varepsilon \leq 1$, define $K_\varepsilon = \cup_{1 \leq j \leq n} \{|z_j| \leq \varepsilon\}$ in $\overline{B}$. By compactness, the K_ε's are a base of neighborhoods in $\overline{B}$ around Z. Thus, $\varinjlim \mathscr{G}(K_\varepsilon) \to \mathscr{G}(Z)$ is an isomorphism [**10**, II, 3.3.1], so it suffices to prove that $\mathscr{G}(K_1) \to \mathscr{G}(K_\varepsilon)$ is an isomorphism for all ε. This holds if $\mathscr{G}(K_1 - Z) \to \mathscr{G}(K_\varepsilon - Z)$ is an isomorphism for all ε.

Since $\mathscr{G}$ is locally constant on $K_1 - Z$, it is enough to show that $K_1 - Z$ admits a deformation retract onto $K_\varepsilon - Z$. We will construct a deformation retract of K_1 onto K_ε. The family of maps $\phi_t^\varepsilon : [0,1] \to [0,\varepsilon]$ defined by $u \mapsto \min(u, t\varepsilon + (1-t)u)$ is a deformation retract onto $[0,\varepsilon]$, so $(z_1, \ldots, z_n) \mapsto \phi_t^\varepsilon(\min |z_j|)(z_1, \ldots, z_n)$ defines a deformation retract of K_1 onto K_ε. This solves the local problem.

Step 3. For each $y \in Y$, choose a small connected open neighborhood $U_y \subseteq X$ solving the local problem as in Step 2, so $\mathscr{G}(U_y) \simeq \mathscr{G}(Y \cap U_y)$ for all $\mathscr{G}$. Thus, given $s \in \mathscr{G}(Y)$ we may uniquely extend $s|_{Y \cap U_y}$ to $s(y) \in \mathscr{G}(U_y)$. If the $s(y)$'s agree on overlaps then $V_0 = \cup U_y$ would be a connected open around Y such that

$\mathscr{G}(V_0) \to \mathscr{G}(Y)$ is an isomorphism. However, there may be monodromy obstructions to overlap compatibility; more concretely, the U_y's might have disconnected overlaps. In order to get around this problem, we pick a Riemannian metric ρ on X and consider small open metric balls $B_{r_y}(y) \subseteq U_y$ centered around each $y \in Y$. Provided r_y is small enough (depending on y), these balls are geodesically convex. Thus, the overlaps $B(y, y') = B_{r_y}(y) \cap B_{r_{y'}}(y')$ are geodesically convex and hence connected (even contractible) when non-empty. For any $s \in \mathscr{G}(Y)$, we would like to glue the restrictions $\widetilde{s}(y) = s(y)|_{B_{r_y}(y)}$.

For each non-empty $B(y, y')$, the sections $\widetilde{s}(y)|_{B(y,y')}$ and $\widetilde{s}(y')|_{B(y,y')}$ must be shown to coincide as sections of the local system $\mathscr{G}_{B(y,y')\cap(X-Y)}$ on the connected $B(y, y') \cap (X - Y)$. Since this latter connected set meets any neighborhood of any point in $B(y, y')$, it is enough that $\widetilde{s}(y)$ and $\widetilde{s}(y')$ agree at some point of $B(y, y')$ when $B(y, y')$ is non-empty. Take r_y so small that $B_{3r_y}(y) \subseteq U_y$ and $B_{3r_y}(y)$ is geodesically convex. Now suppose the open $B_{r_y}(y) \cap B_{r_{y'}}(y')$ contains a point x. Since Y has empty interior, we may take $x \notin Y$. By symmetry, we may assume $r_{y'} \leq r_y$. Thus, $B_{r_{y'}}(y') \subseteq B_{3r_y}(y)$. Using the local structure of Y near y and y', together with the fact that deleting Y does not disconnect any connected open in X, we may construct an embedded path σ (resp. σ') from y (resp. y') to x inside $B_{r_y}(y)$ (resp. $B_{r_{y'}}(y')$) such that the path lies in $X - Y$ past time $t = 0$. There is a unique continuation for a section of $\mathscr{G}$ along such an embedded path because any sheaf $\mathscr{H}$ on $[0, 1]$ that is locally constant on $(0, 1]$ enjoys the property that the natural map $\mathscr{H}([0, 1]) \to \mathscr{H}_0$ is an isomorphism.

The stalk $\widetilde{s}(y')_x \in \mathscr{G}_x$ is a continuation of $s_{y'} \in \mathscr{G}_{y'}$ along σ', and the stalk $\widetilde{s}(y)_x$ is a continuation of s_y along σ. Since $s(y) \in \mathscr{G}(U_y)$ restricts to $s|_{Y\cap U_y}$, so $s(y)|_{B_{3r_y}(y)} \in \mathscr{G}(B_{3r_y}(y))$ has y'-stalk $s_{y'}$ and y-stalk s_y, $s(y)|_{\sigma'}$ and $s(y)|_\sigma$ provide the unique continuations of $s_{y'}$ and s_y along the respective paths σ' and σ. This proves $\widetilde{s}(y')_x = s(y)_x = \widetilde{s}(y)_x$, so we may glue over the union V_0 of the $B_{r_y}(y)$'s. $\square$

1.4.8. Comparison isomorphism for adic sheaves. We now state the $\mathfrak{m}$-adic comparison isomorphism for both cohomology and compactly-supported cohomology, assuming Λ to be a complete discrete valuation ring with finite residue field and characteristic-zero fraction field.

THEOREM 1.4.8.1 (adic comparison isomorphism). *Let $f : X \to S$ be a separated map between finite type $\mathbf{C}$-schemes. Let $\mathscr{F}_\bullet$ be a constructible $\mathfrak{m}$-adic sheaf on X. The natural maps*

$$(\mathrm{R}^i f_! \mathscr{F}_\bullet)^{\mathrm{an}} \to \varprojlim \mathrm{R}^i f_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) \leftarrow \mathrm{R}^i f_!^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}})$$

and

$$(\mathrm{R}^i f_* \mathscr{F}_\bullet)^{\mathrm{an}} \to \varprojlim \mathrm{R}^i f_*^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) \leftarrow \mathrm{R}^i f_*^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}})$$

are isomorphisms for all i.

PROOF. In contrast to ordinary constructible sheaves, when studying constructible $\mathfrak{m}$-adic sheaves we cannot reduce to the case of constant sheaves and hence cannot assume our sheaves extend across a compactification (a technique used when proving the comparison isomorphism for ordinary constructible sheaves). This is the reason we cannot easily make proofs in the torsion case adapt to the $\mathfrak{m}$-adic case.

We may assume $\mathscr{F}_\bullet$ is strictly $\mathfrak{m}$-adic, and then Artin's comparison theorem (Theorem 1.3.10.3) implies that the first map in each row is an isomorphism. Thus, it remains to solve the problem of moving an inverse limit through a derived functor on the topological side.

The case of higher direct images with proper supports will be treated first, as it is easier. The analysis of this case requires several properties: compatibility of analytification with stalks and extension by zero, as well as exactness of analytification and the basic fact (Mittag-Leffler criterion) that inverse limits of short exact sequences of finite-length modules are short exact. These properties (the first three are in Theorem 1.4.7.1) allow us to use the same reduction arguments as in the case of torsion coefficients (via Leray, excision, *etc.*) to reduce to the case when X is a smooth curve and $\mathscr{F}_\bullet$ is lisse $\mathfrak{m}$-adic. The only delicate point in the reduction to curves is to systematically use the Mittag-Leffler criterion and the comparison isomorphism at finite levels to ensure that the inverse limits are compatible with formation of homologies in (Leray) spectral sequences.

We are now in the situation where X^{an} is the complement of finitely many points on a compact connected Riemann surface and $\mathscr{F}_\bullet$ is lisse $\mathfrak{m}$-adic. Theorem 1.4.7.1 ensures $\mathscr{F}_\bullet^{\mathrm{an}}$ is a local system of finite Λ-modules on X^{an}. Since we can always use excision to remove a point in case X happened to be proper, we are left with a problem for an open Riemann surface Z and a local system $\mathscr{F}$ of finite Λ-modules on Z: prove that

$$\mathrm{H}^i_{\mathrm{c}}(Z,\mathscr{F}) \to \varprojlim \ \mathrm{H}^i_{\mathrm{c}}(Z,\mathscr{F}_n)$$

is an isomorphism, where $\mathscr{F}_n = \mathscr{F}/\mathfrak{m}^{n+1}\mathscr{F}$.

If Δ is a union of small punctured discs around the points missing from Z, then $\mathrm{H}^\bullet_{\mathrm{c}}(Z,\cdot) = \varinjlim \mathrm{H}^\bullet_{Z-\Delta}(Z,\cdot)$ as δ-functors, where the limit is over shrinking Δ (*i.e.*, increasing compacts $Z-\Delta$). The excision sequence

$$\dots \to \mathrm{H}^i_{Z-\Delta}(Z,\cdot) \to \mathrm{H}^i(Z,\cdot) \to \mathrm{H}^i(\Delta,\cdot) \to \dots$$

shows that the transition maps in the direct limit are isomorphisms when computing on a local system, as the same holds for cohomology of a local system on a shrinking family of punctured discs (by Corollary **??**). This reduces us to proving

$$\mathrm{H}^i(Z,\mathscr{F}) \to \varprojlim \ \mathrm{H}^i(Z,\mathscr{F}_n), \ \ \mathrm{H}^i(\Delta,\mathscr{F}) \to \varprojlim \ \mathrm{H}^i(\Delta,\mathscr{F}_n)$$

are isomorphisms for all i, with the terms in the inverse limit of finite-length over Λ. Since Z and Δ admit deformation retracts onto bouquets of circles, using Corollary **??** and Mayer-Vietoris reduces us to the analogous problem for a local system $\mathscr{F}$ of finite Λ-modules on a circle C. We compute sheaf cohomology for local systems on C using the acyclic covering $\mathfrak{U} = \{C-\{x\}, C-\{y\}\}$ for two distinct points x and y. All terms in the Čech complex $\mathrm{C}^\bullet(\mathfrak{U},\mathscr{F}_n)$ are Λ_n-finite. Clearly $\mathrm{C}^\bullet(\mathfrak{U},\mathscr{F}) = \varprojlim \ \mathrm{C}^\bullet(\mathfrak{U},\mathscr{F}_n)$, so by Mittag-Leffler we are done.

The $\mathfrak{m}$-adic comparison isomorphism for ordinary direct images is much more difficult. As with the torsion case, the basic strategy of the proof is to reduce to the proper case. Artin's technique for doing this rests on the use of constant sheaves, and so it appears to encounter serious obstacles in the $\mathfrak{m}$-adic setting. Berkovich's resolution-free proof of the comparison isomorphism for higher direct images of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules in the non-archimedean case [**2**, Prop. 7.5.1] does not make crucial use of reduction to constant sheaves, and with some work it adapts to yield the desired result at the $\mathfrak{m}$-adic level. There is one serious complication, due to the essential use of derived categories in Berkovich's inductive method. The problem is that we cannot form inverse limits in derived categories. More precisely, we need a machinery of derived categories for $\mathfrak{m}$-adic sheaves. Quite recently, a completely general formalism for arbitrary noetherian schemes has been developed by Behrend [**1**] (earlier theories were developed by Deligne, and later by Ekedahl, imposing finiteness conditions on Galois cohomology of residue fields); the basic idea in Berhend's approach is to build a theory in which projective systems are equipped with a specified stratification on which all terms are lcc. The resulting derived category of $\mathfrak{m}$-adic sheaves admits a well-behaved cohomological formalism that is closely connected to the formalism at finite level. Using this formalism, the arguments of Berkovich can be pushed through at the $\mathfrak{m}$-adic level. $\square$

The adic comparison isomorphism carries over to the category of constructible K-sheaves (resp. constructible $\overline{K}$-sheaves), where the topological side becomes cohomology of sheaves of finite-dimensional K-vector spaces (resp. $\overline{K}$-vector spaces). The only subtle point in the proof is to verify that we may move a tensor product $K \otimes_\Lambda (\cdot)$ or $\overline{K} \otimes_\Lambda (\cdot)$ through a cohomological functor on the topological side. For higher direct images with proper supports, we may use extension by zero to reduce to the proper case and then (by topological proper base change) we can reduce to the case of ordinary cohomology on a compact Hausdorff space. We then express K (resp. $\overline{K}$) as a direct limit of finite free Λ-modules and invoke the fact that cohomology on a compact Hausdorff space commutes with direct limits [**10**, II, 4.12.1].

For ordinary higher direct images, the situation is once again much harder. Fortunately, the adaptation of Berkovich's technique in the proof of Theorem 1.4.8.1 for higher direct images does carry over to the case of constructible K-sheaves and constructible $\overline{K}$-sheaves, provided that we use Behrend's formalism of derived categories of K-sheaves and $\overline{K}$-sheaves. Thus, we have a very satisfactory theory of comparison isomorphisms over $\mathbf{C}$.

1.5. Finite fields and Deligne's theorem

Schemes over finite fields admit natural Frobenius operators on their ℓ-adic cohomology. These operators can be constructed geometrically and algebraically. The algebraic approach is closely related to the role of Frbobenius elements as they arise in the study of Galois representations, whereas the geometric approach is sometimes easier to use in geometric arguments. We address both points of view in §1.5.1, and explain their agreement. The Frobenius-actions on the cohomology of constructible ℓ-adic sheaves satisfy remarkable properties with respect to compactly-supported cohomology functors. These properties are discussed in §1.5.2, where we define L-functions and formulate the Lefschetz trace formula. We conclude in §1.5.3 by defining purity and mixedness for ℓ-adic sheaves, and stating Deligne's generalized purity theorem concerning weights of Frobenius-eigenvalues in the cohomology of constructible $\overline{\mathbf{Q}}_\ell$-sheaves.

1.5.1. Frobenius actions on sheaves. For any $\mathbf{F}_p$-scheme X, the *absolute Frobenius morphism* $F_X : X \to X$ is the identity map on the underlying space and is the pth-power map on the structure sheaf. Observe that for any morphism $f : X \to Y$ of $\mathbf{F}_p$-schemes, the maps $F_Y \circ f$ and $f \circ F_X$ coincide. On the level of sheaves of rings, this is simply the fact that the p-th power map on $\mathbf{F}_p$-algebras commutes with all ring homomorphisms. The r-fold iterate of F_X is the *absolute q-Frobenius morphism*, with $q = p^r$.

Let X be an arbitrary scheme over a finite field κ of size $q = p^r$ and let $\phi_r : X \to X$ be the absolute q-Frobenius morphism. Let $\overline{\kappa}$ be an algebraic closure of κ and $\overline{X} = X \times_\kappa \overline{\kappa}$. Let $\overline{\phi}_r = \phi_r \times 1$; this is a $\overline{\kappa}$-endomorphism of $\overline{X}$ and is not the intrinsic r-fold absolute Frobenius endomorphism of $\overline{X}$ as a κ-scheme. The map $\overline{\phi}_r$ is geometric in the sense that it corresponds to raising coordinates to the qth power if X is given to us inside of some projective space over κ.

Let Frob_κ denote the κ-automorphism $a \mapsto a^q$ of $\overline{\kappa}$, so $1 \times \mathrm{Frob}_\kappa$ is a κ-endomorphism $\overline{X}$. This endomorphism commutes with $\overline{\phi}_r$ and its composite with $\overline{\phi}_r$ is the r-fold absolute Frobenius endomorphism of the κ-scheme $\overline{X}$. The map $\mathrm{Frob}_\kappa^{-1}$ is called a *geometric Frobenius element*; let us explain the origin of this terminology. Let M be an abelian group. The endomorphism ring of the scheme $\overline{X}$ acts contravariantly on the cohomology group $\mathrm{H}^i_{\text{ét}}(\overline{X}, M)$ by pullback, so the operations of pullback by $\overline{\phi}_r = \phi_r \times 1$ and pullback by $1 \times \mathrm{Frob}_\kappa$ commute with each other. The composite of these commuting endomorphisms is pullback along the r-fold absolute Frobenius endomorphism of $\overline{X}$.

As we shall see in Example 1.5.1.1, for an arbitrary $\mathbf{F}_p$-scheme Z and an arbitrary constant sheaf M on Z, the pullback-action of absolute Frobenius on $\mathrm{H}^i_{\text{ét}}(Z, M)$ is the identity map. Thus, for $Z = \overline{X}$ we conclude that the geometric endomorphism $\overline{\phi}_r$ of $\overline{X}$ induces a pullback action on $\mathrm{H}^i_{\text{ét}}(\overline{X}, M)$ that is equal to the pullback action of $1 \times \mathrm{Frob}_\kappa^{-1}$. That is $\mathrm{Frob}_\kappa^{-1} \in \mathrm{Gal}(\overline{\kappa}/\kappa)$ induces a pullback action on cohomology that agrees with the action of the map $\overline{\phi}_r$. More generally, for κ-scheme $f : X \to \operatorname{Spec} \kappa$, the action of $\mathrm{Frob}_\kappa^{-1}$ on the Galois module attached to the étale sheaf $\mathrm{R}^i f_*(M)$ is the same as the pullback-action by the geometric map ϕ_r.

For technical flexibility in our later considerations, we now generalize these constructions to define Frobenius-actions on the cohomology of étale sheaves that are not necessarily constant. For any étale sheaf of sets $\mathscr{F}$ on an $\mathbf{F}_p$-scheme X, there is a natural isomorphism

$$\mathscr{F} \simeq (F_X)_* \mathscr{F} \tag{1.5.1.1}$$

in $\text{Ét}(X)$ that is defined as follows. For any X-scheme U, the (relative Frobenius) map $F_{U/X} : U \to F_X^{-1}(U)$ over X sits in the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[rr, bend left, "F_U"] \arrow[dr] & F_X^{-1}(U) \arrow[r] \arrow[d] & U \arrow[d] \\
& X \arrow[r, "F_X"'] & X
\end{tikzcd}$$

so $F_{U/X}$ is a radicial surjection (since F_U and F_X are). When U is étale over X then the radicial X-map $F_{U/X}$ is a map between étale X-schemes and thus it is an étale radicial surjection. By [**17**, 17.9.1], $F_{U/X}$ is therefore an isomorphism for U in $X_{\text{ét}}$. In the special case $X = \operatorname{Spec} k$, this is the fact that if K/k is a finite separable extension of fields of positive characteristic p then $K = k \cdot K^p$. The isomorphism $F_{U/X}$ yields an isomorphism $\mathscr{F}(F_{U/X}) : \mathscr{F}(F_X^{-1}(U)) \simeq \mathscr{F}(U)$, and the inverse of this defines (1.5.1.1).

The sheaf map that is adjoint to (1.5.1.1) is denoted

$$\mathrm{Fr}_{\mathscr{F}} : F_X^* \mathscr{F} \to \mathscr{F}; \tag{1.5.1.2}$$

for sheaves of modules, this is compatible with tensor constructions (*e.g.*, symmetric powers). This is an isomorphism because it is a composite

$$F_X^*\mathscr{F} \simeq F_X^*(F_X)_*\mathscr{F} \to \mathscr{F},$$

with the first step defined via (1.5.1.1) and the second step equal to the adjunction (that is an isomorphism because F_X is a universal homeomorphism; see Remark 1.1.6.4). Note that when $\mathscr{F} = \underline{\Sigma}$ is the constant sheaf on a set Σ, the map $\mathrm{Fr}_{\underline{\Sigma}}$ is an inverse to the canonical isomorphism $\underline{\Sigma} \simeq F_X^*(\underline{\Sigma})$; this is easily proved by using the fact that $\underline{\Sigma}$ is representable. The special case when Σ is an abelian group is the degree-0 case in the following calculation:

Example 1.5.1.1. For any $\mathbf{F}_p$-scheme Z and any abelian group M, the map

$$\mathrm{H}^i_{\text{ét}}(Z, M) \stackrel{F_Z^*}{\to} \mathrm{H}^i_{\text{ét}}(Z, F_Z^*M) \stackrel{\mathrm{Fr}_M}{\to} \mathrm{H}^i_{\text{ét}}(Z, M)$$

is the identity. To verify this, it suffices to prove more generally that if $\mathscr{F}$ is any abelian étale sheaf on Z, then the composite

$$\mathrm{H}^i_{\text{ét}}(Z, \mathscr{F}) \stackrel{F_Z^*}{\to} \mathrm{H}^i_{\text{ét}}(Z, F_Z^*\mathscr{F}) \stackrel{\mathrm{Fr}_{\mathscr{F}}}{\to} \mathrm{H}^i_{\text{ét}}(Z, \mathscr{F})$$

is the identity. By a universal δ-functor argument, we reduce to the case $i = 0$. Functoriality reduces us to the trivial constant case $\mathscr{F} = \underline{\mathbf{Z}}$.

The isomorphism $\mathrm{Fr}_{\mathscr{F}}$ is compatible with higher direct images in the sense that if $f : X \to S$ is a map of $\mathbf{F}_p$-schemes, so the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

commutes (but is generally not cartesian), then the composite

$$F_S^*\mathrm{R}^i f_*\mathscr{F} \longrightarrow \mathrm{R}^i f_*(F_X^*\mathscr{F}) \xrightarrow[\simeq]{\mathrm{R}^i f_*(\mathrm{Fr}_{\mathscr{F}})} \mathrm{R}^i f_*\mathscr{F} \tag{1.5.1.3}$$

is equal to the isomorphism $\mathrm{Fr}_{\mathrm{R}^i f_*\mathscr{F}}$; thus, the first step of (1.5.1.3) is also an isomorphism. Briefly, the method of proof is to use universal δ-functor arguments to reduce to the case $i = 0$, and this case is handled by adjointness and the commutativity of the diagram

$$\begin{tikzcd} f_*\mathscr{G} \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & F_{S*}f_*\mathscr{G} \arrow[d, equal] \\ & f_*F_{S*}\mathscr{G} \end{tikzcd}$$

Everything we have said goes through the same way with r-fold iterates $F_{X,r}$ if we work in the category of κ-schemes for a finite field κ of size p^r. Note also that when $S = \operatorname{Spec}\kappa$, and so $F_{S,r}$ is the identity, the operation $\mathrm{Fr}_{\mathscr{F},r} : F_{S,r}^*\mathscr{F} \to \mathscr{F}$ for an étale sheaf of sets $\mathscr{F}$ on S may be viewed as an endomorphism of $\mathscr{F}$. This endomorphism is usually not the identity, for it is the inverse of the isomorphism $\mathscr{F}(U) \to \mathscr{F}(U)$ induced by pullback along F_U. That is, in terms of discrete left $\mathrm{Gal}(\overline{\kappa}/\kappa)$-sets, $\mathrm{Fr}_{\mathscr{F},r}$ is the action of geometric Frobenius under the dictionary between étale sheaves over a field and discrete Galois-sets. A variant on this is provided by the next lemma (if we specialize to the case $S = \operatorname{Spec}\kappa$).

LEMMA 1.5.1.2. *For any κ-scheme S and finite-type separated map $f : X \to S$, and for any torsion abelian sheaf $\mathscr{F}$ on $X_{\text{ét}}$, the morphism*

$$F_{S,r}^* \mathrm{R}^i f_! \mathscr{F} \to \mathrm{R}^i f_! (F_{X,r}^* \mathscr{F}) \xrightarrow{\mathrm{Fr}_{\mathscr{F},r}} \mathrm{R}^i f_! \mathscr{F}$$

of étale sheaves on $S_{\text{ét}}$ is equal to $\mathrm{Fr}_{\mathrm{R}^i f_! \mathscr{F}, r}$.

PROOF. Working locally on the base, we may assume S is affine (or at least quasi-compact and quasi-separated). Let $j : X \to \overline{X}$ be an open immersion into a proper S-scheme with structure map $\overline{f} : \overline{X} \to S$. The diagram

$$\begin{tikzcd} j_! F_{X,r}^* \mathscr{F} \arrow[r, "j_! \mathrm{Fr}_{\mathscr{F},r}"] \arrow[d, "\simeq"'] & j_! \mathscr{F} \\ F_{\overline{X},r}^* (j_! \mathscr{F}) \arrow[ur, "\mathrm{Fr}_{j_! \mathscr{F},r}"'] & \end{tikzcd}$$

commutes because $\mathrm{Fr}_{\mathscr{F},r}$ respects Zariski-localization and the stalks are 0 away from X. Replacing f by $\overline{f}$ and $\mathscr{F}$ by $j_! \mathscr{F}$ therefore reduces us to the case of higher direct images. This case is treated by universal δ-functor arguments, as in our analysis of (1.5.1.3). $\square$

EXAMPLE 1.5.1.3. The Galois-compatibility that was checked after Theorem 1.3.5.2 implies, in the setup of the preceding lemma with $S = \operatorname{Spec} \kappa$ (so $F_{S,r}$ is the identity), that the Frobenius endomorphism of the étale sheaf $\mathrm{R}^i f_! \mathscr{F}$ corresponds to the endomorphism of $\mathrm{H}^i_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F}_{/\overline{\kappa}})$ induced by pullback along geometric Frobenius in $\mathrm{Gal}(\overline{\kappa}/\kappa)$. It is these actions that arise in Deligne's purity theorems, as we will now explain.

1.5.2. L-functions and Lefschetz trace formula. Let X be a scheme of finite type over a finite field κ with characteristic p and size q. Fix a complete discrete valuation ring $(\Lambda, \mathfrak{m})$ with finite residue field of characteristic $\ell \neq p$ and fraction field K of characteristic zero. Let $\mathscr{F}$ be a constructible $\mathfrak{m}$-adic sheaf on $X_{\text{ét}}$. Since everything we do will be relative to κ, all Frobenius operations will now be understood to rest on q-Frobenius maps rather than absolute Frobenius maps. As in §1.5.1, the q-Frobenius $F_X : X \to X$ induces a natural Frobenius morphism $\mathrm{Fr}_{\mathscr{F}} : F_X^* \mathscr{F} \to \mathscr{F}$ that is pullback-functorial in X. Strictly speaking, we only gave the construction functorially on ordinary sheaves, but applying these operations termwise on a projective system gives a similar functorial construction on the Artin-Rees category.

For each x in the set $|X|$ of closed points on X, say with $d_x \stackrel{\text{def}}{=} [\kappa(x) : \kappa]$ (so $\mathrm{N}(x) \stackrel{\text{def}}{=} |\kappa(x)| = q^{d_x}$), the d_x-fold iterate $F_X^{d_x *} \mathscr{F} \to \mathscr{F}$ of $\mathrm{Fr}_{\mathscr{F},x}$ has pullback along x that is an endomorphism $\mathrm{Fr}_{\mathscr{F}_x} : \mathscr{F}_x \to \mathscr{F}_x$ as a constructible $\mathfrak{m}$-adic sheaf on $(\operatorname{Spec} \kappa(x))_{\text{ét}}$. The intervention of a d_x-fold iteration is necessary because it is the $\mathrm{N}(x)$-Frobenius that acts trivially on $\kappa(x)$, and $\mathrm{N}(x) = q^{d_x}$.

Upon choosing a separable closure $\kappa(\overline{x})$ of $\kappa(x)$, we can identity $\mathscr{F}_x$ with a finite Λ-module endowed with a continuous linear action of the Galois group $\pi_1(x, \overline{x})$, so in particular we get an action of the geometric Frobenius element $\phi_x \in \mathrm{Gal}(\kappa(\overline{x})/\kappa(x))$ on $\mathscr{F}_x$. This action agrees with the abstract endomorphism $\mathrm{Fr}_{\mathscr{F}_x}$ that was constructed above by using $\mathrm{Fr}_{\mathscr{F}}$. The description in terms of ϕ_x has the advantage of being concrete, but the description in terms of $\mathrm{Fr}_{\mathscr{F}}$ is better-suited for some global theoretical arguments.

These considerations all adapt in an obvious way when we use a constructible K-sheaf (or constructible $\overline{\mathbf{Q}}_\ell$-sheaf) rather than a constructible $\mathfrak{m}$-adic sheaf.

DEFINITION 1.5.2.1. The *L-function* of a constructible K-sheaf $\mathscr{F}$ on $X_{\text{ét}}$ is the formal power series

$$L(\mathscr{F},t) = L(X,\mathscr{F},t) = \prod_{x\in|X|} \det(1-\phi_x t^{d_x} \mid \mathscr{F}_x)^{-1} \in 1 + t\Lambda[\![t]\!].$$

The Lefschetz trace formula expresses an invariance property of L-functions with respect to higher direct images with proper supports:

THEOREM 1.5.2.2 (Lefschetz trace formula). *Let $f : X \to S$ be a separated map between finite-type κ-schemes. For any constructible K-sheaf $\mathscr{F}$ on X,*

$$L(X,\mathscr{F},t) = \prod_{n\geq 0} L(S, \mathrm{R}^n f_! \mathscr{F}, t)^{(-1)^n}.$$

The reader is referred to [**12**, Exp. III] and [**14**, Exp. XII] for the proof by Grothendieck–Verdier, or either [**6**, Rapport] or [**9**, Ch. II, §2–§4] for another proof resting on a method of Neilsen–Wecken. We shall only explain how to put this formula in a more concrete form that resembles the Lefschetz trace formula in topology (and is the assertion that is the main focus of the proof in all approaches).

Consider the special case $S = \operatorname{Spec}\kappa$. In this case, the Lefschetz trace formula says that for any separated finite-type κ-scheme X and any constructible K-sheaf $\mathscr{F}$ on X, there is an Euler-characteristic formula

(1.5.2.1)
$$L(X,\mathscr{F},t) = \prod_{n\geq 0} L(\operatorname{Spec}\kappa, \mathrm{R}^n f_!\mathscr{F},t)^{(-1)^n} = \prod_{n\geq 0} \det(1-\phi t \mid \mathrm{H}^n_{\mathrm{c,ét}}(X_{/\overline{\kappa}},\mathscr{F}))^{(-1)^{n+1}},$$

where $\phi \in \mathrm{Gal}(\overline{\kappa}/\kappa)$ is a geometric Frobenius element and it acts on geometric cohomology via pullback along $\mathrm{Fr}_{\mathscr{F}}$. In particular, $L(X,\mathscr{F},t)$ is a rational function in t whose zeros and poles are eigenvalues for Frobenius-actions on the compactly-supported cohomology of $\mathscr{F}$.

Let us rewrite this formula in a manner that explains why it is called a trace formula. For any endomorphism F of a finite-dimensional vector space V over a field k, there is a general identity

$$\det(1-Ft \mid V)^{-1} = \exp\left(\sum_{i\geq 1} \mathrm{Tr}(F^i)\frac{t^i}{i}\right); \tag{1.5.2.2}$$

indeed, we may put F in upper-triangular form after extending scalars on k, and then the formula is obtained by multiplying the elementary formulas

$$(1-\lambda t)^{-1} = \exp(-\log(1-\lambda t)) = \exp\left(\sum_{i\geq 1}\lambda^i \cdot \frac{t^i}{i}\right)$$

for the eigenvalues. This enables us to rewrite the right side of (1.5.2.1) as

$$\exp\left(\sum_{i\geq 1}\big(\sum_n (-1)^n \cdot \mathrm{Tr}(\phi^i \mid \mathrm{H}^n_{\mathrm{c,ét}}(X_{/\overline{\kappa}},\mathscr{F}))\big)\cdot\frac{t^i}{i}\right).$$

Equivalently, the log-derivative of the L-function is

$$\frac{L'(X,\mathscr{F},t)}{L(X,\mathscr{F},t)}=\sum_{i\geq 1}\chi(\phi^i\,|\,\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}}(X_{/\overline{\kappa}},\mathscr{F}))t^{i-1},$$

where the χ-term is the alternating sum of traces on compactly-supported cohomology. However, the exponential formula (1.5.2.2) may also be inserted into the infinite-product definition of $L(X,\mathscr{F},t)$ to yield

$$\begin{aligned}L(X,\mathscr{F},t) &= \exp\left(\sum_{m\geq 1}\sum_{x\in|X|}\mathrm{Tr}(\phi_x^m\,|\,\mathscr{F}_x)\cdot\frac{t^{d_x m}}{m}\right)\\ &= \exp\left(\sum_{i\geq 1}\sum_{x\in|X|,d_x|i} d_x\cdot\mathrm{Tr}(\phi_x^i\,|\,\mathscr{F}_x)\cdot\frac{t^i}{i}\right),\end{aligned}$$

so by taking log-derivatives of this identity and comparing coefficients of t^{i-1} in the two formulas for L'/L, we arrive at the following reformulation of the Lefschetz trace formula for $S=\mathrm{Spec}\,\kappa$:

$$\chi(\phi^i\,|\,\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}}(X_{/\overline{\kappa}},\mathscr{F}))=\sum_{x\in X(\kappa_i)}\mathrm{Tr}(\phi_x\,|\,\mathscr{F}_x),$$

where $\kappa_i\subseteq\overline{\kappa}$ is the degree-i extension of κ and ϕ_x is understood to be the q^i-Frobenius for $x\in X(\kappa_i)$; the multiplier d_x (for $x\in|X|$) in the previous formula is absorbed by the fact that if $d|i$ then there are exactly d points in $X(\kappa_i)$ over each $x\in|X|$ with $d_x=d$.

Upon renaming κ_i as κ, the final formula may be written

$$\chi(\phi\,|\,\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}}(X_{/\overline{\kappa}},\mathscr{F}))=\sum_{x\in X(\kappa)}\mathrm{Tr}(\phi_x\,|\,\mathscr{F}_x).$$

This expresses the alternating sum of traces for the Frobenius-action on cohomology as a sum of local traces at fixed-points for the Frobenius-action on X, and this is the essential content of the Lefschetz trace formula. In the special case $\mathscr{F}=\mathbf{Q}_\ell$, this literally is the étale-topology version of the Lefschetz trace formula for the self-map $F_X:X\to X$ whose graph in $X\times X$ is transverse to the diagonal: it counts the number of fixed points of F_X as the alternating sum of traces of F_X on the compactly-supported cohomology of $X_{/\overline{\kappa}}$.

Let us next show that the special case $S=\mathrm{Spec}\,\kappa$ implies the general case, and so the general case is reduced to a trace-formula identity. We first reduce ourselves to the case of separated S by excision: if S contains an open subscheme U with closed complement Z then the infinite-product formula for $L(X,\mathscr{F},t)$ provides a decomposition $L(X,\mathscr{F},t)=L(X_U,\mathscr{F}_U,t)L(X_Z,\mathscr{F}_Z,t)$, and determinantal Euler-characteristics in the relative excision sequence for $\mathscr{F}$ with respect to $\{X_U,X_Z\}$ yield the identity

$$\prod_{n\geq 0}L(S,\mathrm{R}^n f_!\mathscr{F},t)^{(-1)^n}=\prod_{n\geq 0}L(U,\mathrm{R}^n f_{U!}\mathscr{F}_U,t)^{(-1)^n}\cdot\prod_{n\geq 0}L(Z,\mathrm{R}^n f_{Z!}\mathscr{F}_Z,t)^{(-1)^n}.$$

Thus, it suffices to work separately over U and Z, and so we may assume S (and hence X) is separated over κ.

If we define

$$\Delta(\phi, \mathscr{G}) = \prod_{w \geq 0} \det(1 - t\phi \,|\, \mathrm{H}^w_{\mathrm{c,\acute{e}t}}(X_{/\overline{\kappa}}, \mathscr{G}))^{(-1)^{w+1}}$$

for a constructible K-sheaf $\mathscr{G}$ on $X_{\mathrm{\acute{e}t}}$, then by using the special case of the base $\operatorname{Spec} \kappa$ we can restate the general formula (for separated S) as the identity

$$\Delta(\phi, \mathscr{F}) = \prod_{n \geq 0} \Delta(\phi, \mathrm{R}^n f_! \mathscr{F})^{(-1)^n}.$$

This latter formula is easily proved by taking determinantal Euler characteristics in the Frobenius-equivariant Leray spectral sequence

$$\mathrm{E}_2^{m,n} = \mathrm{H}^m_{\mathrm{c,\acute{e}t}}(S_{/\overline{\kappa}}, \mathrm{R}^n f_! \mathscr{F}) \Rightarrow \mathrm{H}^{n+m}_{\mathrm{c,\acute{e}t}}(X_{/\overline{\kappa}}, \mathscr{F});$$

the Frobenius-equivariance is Lemma 1.5.1.2.

1.5.3. Purity and Deligne's theorem. Weil conjectured that if X is smooth and projective over a finite field κ with size q, then the Frobenius-eigenvalues on $\mathrm{H}^w_{\mathrm{\acute{e}t}}(X_{/\overline{\kappa}}, \mathbf{Q}_\ell)$ are algebraic numbers whose embeddings into $\mathbf{C}$ have common absolute value $q^{w/2}$. It is a very strong condition on an algebraic number that it have the same absolute value under all embeddings into $\mathbf{C}$, and this class of numbers is given a special name:

Definition 1.5.3.1. Let E be a field of characteristic 0, and choose $q, w \in \mathbf{R}$ with $q > 0$. An element $\lambda \in E$ is a *q-Weil number of weight w* if λ is algebraic over $\mathbf{Q}$ and all $\mathbf{C}$-roots of its minimal polynomial over $\mathbf{Q}$ have absolute value $q^{w/2}$.

Definition 1.5.3.2. Let $\mathscr{F}$ be a constructible $\overline{\mathbf{Q}}_\ell$-sheaf on the κ-scheme X. The sheaf $\mathscr{F}$ is *pure of weight* w if, for every closed point $x \in X$, the $\overline{\mathbf{Q}}_\ell$-eigenvalues of $\mathrm{Fr}_{\mathscr{F},x}$ on $\mathscr{F}_x$ are $\mathrm{N}(x)$-Weil numbers with weight w. If $\mathscr{F}$ is admits a finite increasing filtration by constructible $\overline{\mathbf{Q}}_\ell$-subsheaves with successive quotients that are pure of weights $w_1, \ldots$, then $\mathscr{F}$ is *mixed*, and the w_j's are the *weights* of $\mathscr{F}$.

If $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ is a choice of isomorphism, then $\mathscr{F}$ is *ι-pure of weight* w if, for every $x \in |X|$, we have $|\iota(\lambda)| = \mathrm{N}(x)^{w/2}$ for each $\overline{\mathbf{Q}}_\ell$-eigenvalue λ of $\mathrm{Fr}_{\mathscr{F},x}$ on $\mathscr{F}_x$. Likewise, $\mathscr{F}$ is *ι-mixed* if it admits an increasing filtration by constructible $\overline{\mathbf{Q}}_\ell$-sheaves with successive quotients that are ι-pure of weights $w_1, \ldots$.

Example 1.5.3.3. The sheaf $\overline{\mathbf{Q}}_\ell(r)$ on X is pure of weight $-2r$. Indeed, on the stalk at $x \in |X|$ with $q_x = |\kappa(x)|$, the action of geometric q_x-Frobenius on ℓ-power roots of unity in $\kappa(\overline{x})$ is given by raising to the power q_x^{-1}, so the action on $\overline{\mathbf{Q}}_\ell(1)_{\overline{x}}$ is given by multiplication by $q_x^{-2/2}$. Now pass to rth tensor powers.

Example 1.5.3.4. Weil's Riemann hypothesis asserts that if $f : X \to S$ is smooth and projective, then $\mathrm{R}^w f_* \mathbf{Q}_\ell$ is pure with weight w.

It is trivial to follow purity through operations such as tensor products, symmetric powers, and linear duals. For example, if $\mathscr{F}$ is lisse and pure of weight w, then the twisted linear dual $\mathscr{F}^\vee(r)$ is pure of weight $-w - 2r$ since the natural geometric Frobenius action on $\mathscr{F}_x^\vee$ goes via the linear dual of the action of the inverse of geometric Frobenius on $\mathscr{F}_x$. Likewise, passing to the dual carries mixed sheaves to mixed sheaves, and it converts upper bounds on weights into lower bounds on weights (through negation).

EXAMPLE 1.5.3.5. If $\iota_1, \iota_2 : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ are two isomorphisms, it can happen that a constructible $\overline{\mathbf{Q}}_\ell$-sheaf is ι_1-pure but not ι_2-pure. For example, consider the scheme $\operatorname{Spec} \mathbf{F}_p$ and the rank-2 lisse sheaf such that geometric Frobenius acts by a matrix in $\mathrm{GL}_2(\mathbf{Z}_\ell)$ with distinct algebraically independent non-algebraic eigenvalues in $1 + \ell\mathbf{Z}_\ell$. Choose ι_1 to send these eigenvalues to distinct points on the unit circle and ι_2 to send them to elements of $\mathbf{C}$ with distinct absolute values.

It is obvious that if $\mathscr{F}$ is ι-pure of weight w for all isomorphisms $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$, then $\mathscr{F}$ is pure of weight w. Indeed, the hypothesis forces each eigenvalue on a stalk to have all of its $\mathrm{Aut}(\mathbf{C})$-conjugates with the same absolute value; note that this condition immediately rules out the possibility of transcendental eigenvalues.

Here is the main result in the theory [**7**, 3.3.1]:

THEOREM 1.5.3.6 (Deligne's purity theorem). *Let $f : X \to S$ be a separated map between finite-type κ-schemes. Let $\mathscr{F}$ be a constructible $\overline{\mathbf{Q}}_\ell$-sheaf on X. Let w be a real number. Fix an isomorphism $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. If $\mathscr{F}$ is ι-mixed (resp. mixed) of weights $\leq w$, then $\mathrm{R}^j f_! \mathscr{F}$ is ι-mixed (resp. mixed) of weights $\leq w + j$ for all j, with each weight congruent mod $\mathbf{Z}$ to a weight of $\mathscr{F}$.*

For our purposes, Deligne's theorem is relevant due to:

COROLLARY 1.5.3.7. *Let X be a smooth separated scheme of finite type over κ, and let $\mathscr{F}$ be a lisse $\overline{\mathbf{Q}}_\ell$-sheaf that is pure of weight $w \in \mathbf{Z}$. The image $\widetilde{\mathrm{H}}^i_{\text{ét}}(X_{/\overline{\kappa}}, \mathscr{F})$ of $\mathrm{H}^i_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F})$ in $\mathrm{H}^i_{\text{ét}}(X_{/\overline{\kappa}}, \mathscr{F})$ is pure of weight $w + i$.*

In the special case when X is proper and smooth, and $\mathscr{F} = \overline{\mathbf{Q}}_\ell$, the hypotheses are satisfied for $w = 0$. The conclusion in this case is exactly Weil's Riemann hypothesis, with projectivity relaxed to properness. From this perspective, Deligne's purity theorem generalizes Weil's original conjecture in two directions: it allows non-properness and non-constant sheaves.

PROOF. (of Corollary). We may assume that the κ-smooth X has some pure dimension d. By Deligne's purity theorem, $\mathrm{H}^i_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F})$ is mixed of weights $\leq w + i$. Since $\mathscr{F}$ is lisse and X is smooth and separated of pure dimension d, there is a perfect Poincaré-duality pairing

$$\mathrm{H}^i_{\text{ét}}(X_{/\overline{\kappa}}, \mathscr{F}) \otimes \mathrm{H}^{2d-i}_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F}^\vee(d)) \to \mathrm{H}^{2d}_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \overline{\mathbf{Q}}_\ell(d)) \to \overline{\mathbf{Q}}_\ell.$$

This is Galois equivariant, and in particular we can identify $\mathrm{H}^i_{\text{ét}}(X_{/\overline{\kappa}}, \mathscr{F})$ with the linear dual of $\mathrm{H}^{2d-i}_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F}^\vee(d))$, where the action of geometric q-Frobenius ϕ on $\mathrm{H}^i_{\text{ét}}$ goes over to the linear dual of ϕ^{-1} acting on $\mathrm{H}^{2d-i}_{\mathrm{c,ét}}$.

Since $\mathscr{F}^\vee(d)$ is pure of weight $-2d - w$, Deligne's theorem implies that the action of ϕ on $\mathrm{H}^{2d-i}_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F}^\vee(d))$ is mixed of weights $\leq -w - i$. Hence, passing to the linear dual endowed with the dual action of the inverse of geometric q-Frobenius, we conclude that $\mathrm{H}^i_{\text{ét}}(X_{/\overline{\kappa}}, \mathscr{F})$ is mixed of weights $\geq w + i$. The map

$$\mathrm{H}^i_{\mathrm{c,ét}}(X_{/\overline{\kappa}}, \mathscr{F}) \to \mathrm{H}^i_{\text{ét}}(X_{/\overline{\kappa}}, \mathscr{F})$$

is equivariant for the action of geometric q-Frobenius, so the image is mixed of weights $\geq w + i$ (due to the mixedness with lower bound on the target) and is mixed of weights $\leq w + i$ (due to mixedness with upper bound on the source). Hence, this image is pure of weight $w + i$. $\square$

Bibliography

[1] K. Berhend. *Derived ℓ-adic categories for algebraic stacks.* American Math. Soc., Providence, 2003.

[2] V. Berkovich. Étale cohomology for non-archimedean analytic spaces. *Publ. Math. IHES*, 78:5–161, 1993.

[3] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models.* Springer-Verlag, Berlin, 1990.

[4] G. Cornell and J. Silverman, editors. *Arithmetic geometry* (*Storrs, Connecticut,* 1984). Springer-Verlag, 1986.

[5] A.J. de Jong. Smoothness, semi-stability, and alterations. *Publ. Math. IHES*, 83:51–93, 1996.

[6] P. Deligne. *Cohomologie étale* . Number 569 in Lecture Notes in Mathematics. Springer-Verlag, 1977.

[7] P. Deligne. La conjecture de Weil II. *Publ. Math. IHES*, 52:137–252, 1980.

[8] P. Deligne and W. Kuyk, editors. *Modular functions of one variable II* (*Antwerp* 1972), number 349 in Lecture Notes in Mathematics. Springer-Verlag, 1973.

[9] E. Freitag and R. Kiehl. *Étale cohomology and the Weil conjectures.* Springer-Verlag, Berlin, 1988.

[10] R. Godement. *Topologie algébrique et théorie des faisceaux.* Hermann, Paris, 1973.

[11] A. Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math J.*, 9:119–221, 1957.

[12] A. Grothendieck. *Dix esposes sur la cohomologie des schemas.* North-Holland, Amsterdam, 1968.

[13] A. Grothendieck. *Revêtements étales et groupe fondamental* (1960-1961). Number 224 in Lecture Notes in Mathematics. Springer-Verlag, 1971.

[14] A. Grothendieck. *Cohomologie ℓ-adique et fonctions L* (1965-1966). Number 589 in Lecture Notes in Mathematics. Springer-Verlag, 1977.

[15] A. Grothendieck, M. Artin, and J-L. Verdier. *Théorie des topos et cohomologie étale des schémas* (1963-1964). Number 269, 270, 305 in Lecture Notes in Mathematics. Springer-Verlag, 1972-73.

[16] A. Grothendieck and J. Dieudonné. Étude cohomologique des faisceaux cohérents. Publ. Math. IHES. vol. 11 (1961), vol. 17 (1963).

[17] A. Grothendieck and J. Dieudonné. Étude locale des schémas et des morphismes de schémas. Publ. Math. IHES. vol. 20 (1964), vol. 24 (1965), vol. 28 (1966), vol. 32 (1967).

[18] A. Grothendieck and J. Dieudonné. Le langage des schémas. Publ. Math. IHES. vol. 4 (1960).

[19] A. Grothendieck, P.Deligne, N. Katz, M. Raynaud, and D. S. Rim. *Groupes de monodromie en géométrie algébrique.* Number 288, 340 in Lecture Notes in Mathematics. Springer-Verlag, 1972-73.

[20] W. Lütkebohmert. On compactification of schemes. *Manuscripta Mathematica*, 80:95–111, 1993.

[21] H. Matsumura. *Commutative ring theory.* Cambridge Univ. Press, Cambridge, 1989.

[22] D. Mumford. *Abelian varieties.* Oxford Univ. Press, Bombay, 1970.

[23] M. Raynaud. *Anneaux locaux Henséliens,* Lecture Notes in Mathematics, no. 169. Springer-Verlag, Berlin, 1970.

[24] J-P. Serre. *Local fields.* Springer-Verlag, New York, 1979.

[25] R. Thomason and T. Trobaugh. "Higher algebraic K-theory of schemes and of derived categories", in *The Grothendieck Festschrift III.* Birkhäuser, Boston, 1990.

[26] J-L. Verdier. "A duality theorem in the étale cohomology of schemes", in *Proc. Conf. Local Fields* (Dreibergen, 1966). Springer-Verlag, New York, 1967.