ABELIAN VARIETIES

Homework assignment, 13 November 2014

Hand in your work by Thursday 27 November 2014 at the latest. It is most convenient if you write your solutions in TeX or LaTeX and send them to us in pdf format by email at J.Commelin@math.ru.nl and B.Moonen@math.ru.nl (use both).

Your solutions to these exercises should convince us that you have understood what has been discussed in the course thus far. You may use everything you know, including results that can be found in HAG. If you find an exercise or result in the literature that is similar to one of the exercises below, it is of course not acceptable to cite the literature by way of solution.

The main thing we care about is that you understand what you are doing. In cases of doubt we will certainly ask you to provide further details. *What you hand in must be your own work, not the result of a collaboration with another student.*

Give full details for your arguments, with precise references for the results you use. If you have questions about the exercises, feel free to contact us.

Throughout, by a *variety* over a field k, we mean a separated k-scheme of finite type that is geometrically integral, i.e., such that $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ is reduced and irreducible. As discussed in HAG, at the end of section II.4, if k algebraically closed then this notion of a variety is "the same" as the classical concept.

Exercise 1. A *ring variety* over a field k is a commutative group variety $(A, +, 0)$ over k equipped with a k-morphism $A \times_{\mathrm{Spec}(k)} A \to A$ (the ring multiplication, denoted on points by $(x, y) \mapsto xy$) and a k-valued point $1 \in A(k)$, such that the ring multiplication is associative (on points: $(xy)z = x(yz)$), the distributive laws hold ($x(y + z) = xy + xz$ and $(x + y)z = xz + yz$), and such that 1 is a two-sided neutral element for the multiplication ($1x = x = x1$).

(i) Give an explicit example of an affine non-commutative ring variety. Specify the coordinate ring, and decribe the addition, the additive inverse, the multiplication, and the points 0 and 1 on rings. Also say what the A in your example is as a contravariant functor from k-schemes to the category of rings.

(ii) Let k be an algebraically closed field of arbitrary characteristic. Prove that there does not exist a complete ring variety over k other than the trivial example $A = \mathrm{Spec}(k)$.

Exercise 2. Let X_1 and X_2 be varieties over a field k. Suppose $Y = X_1 \times X_2$ carries the structure of an abelian variety. Show that X_1 and X_2 each have a unique structure of an abelian variety such that $Y = X_1 \times X_2$ *as abelian varieties.*

Exercise 3. Let X be an abelian variety over a field k. Let $X^t = \mathrm{Pic}^0_{X/k}$ be the dual abelian variety and $\mathscr{P}$ the Poincaré line bundle on $X \times X^t$. To a line bundle L on X we have associated a homomorphism $\varphi_L \colon X \to X^t$, given on points by $x \mapsto [t_x^*(L) \otimes L^{-1}]$. Consider the homomorphism $(\mathrm{id}_X, \varphi_L)\colon X \to X \times X^t$. Prove that

$$(\mathrm{id}_X, \varphi_L)^* \mathscr{P} \cong L \otimes [-1]^* L\,.$$

Exercise 4. Let R be a (commutative unitary) ring, and let a, $b \in R$ be elements with $ab = -2$. If A is a commutative R-algebra, write

$$G_{a,b}(A) = \bigl\{ y \in A \bigm| y^2 = ay \bigr\}\,.$$

(i) Prove that for y_1, $y_2 \in G_{a,b}(A)$ the element $y_1 \oplus y_2 := y_1 + y_2 + by_1y_2$ again lies in $G_{a,b}(A)$, and that $(y_1, y_2) \mapsto y_1 \oplus y_2$ gives $G_{a,b}(A)$ the structure of an abelian group with neutral element $0 \in G_{a,b}(A)$, and with inverse given by $y \mapsto y$.

(ii) Prove that there is a unique commutative group scheme $G_{a,b}$ over $\mathrm{Spec}(R)$ that is locally free of rank 2 (in fact it is even free), such that, for $T = \mathrm{Spec}(A)$ with A a commutative R-algebra, $G_{a,b}(T)$ is the group $G_{a,b}(A)$ as in (i).

(iii) Prove that $G_{a,b}$ and $G_{b,a}$ are Cartier dual.