§ 1. *Formation of quotients and the descent of coherent sheaves.*

**(7.1) Definition.** Let $S$ be a base scheme. Let $\rho\colon G \times_S X \to X$ be an action (from the left) of an $S$-group scheme $G$ on an $S$-scheme $X$. Let $F$ be a coherent sheaf of $O_X$-modules. Then an *action of $G$ on $F$, compatible with the action $\rho$*, is an isomorphism $\lambda\colon \mathrm{pr}_2^*F \xrightarrow{\sim} \rho^*F$ of sheaves on $G \times_S X$, such that on $G \times_S G \times_S X$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{pr}_3^*F & \xrightarrow{\mathrm{pr}_{23}^*(\lambda)} & \mathrm{pr}_{23}^*\rho^*F \\ {\scriptstyle (m\times \mathrm{id}_X)^*(\lambda)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_G\times\rho)^*(\lambda)} \\ (m \times \mathrm{id}_X)^*\rho^*F & = & (\mathrm{id}_G \times \rho)^*\rho^*F \end{array} .$$

Here is a more concrete explanation of what this means. If $T$ is an $S$-scheme and $g \in G(T)$, write $\rho_g\colon X_T \to X_T$ for the action of the element $g$. Then to have an action of $G$ on $F$ that is compatible with $\rho$ means that for every $g \in G(T)$ we have an isomorphism of sheaves $\lambda_g\colon F_T \xrightarrow{\sim} \rho_g^*F_T$ such that $\lambda_{gh} = \rho_h^*(\lambda_g) \circ \lambda_h$ for all $g$, $h \in G(T)$.

If $F$ is a locally free $O_X$-module we can take a more geometric point of view. First recall that a locally free $O_X$-module is "the same" as a geometric vector bundle over $X$. Namely, $V := \mathbb{V}(F^\vee)$ is a geometric vector bundle over $X$, and $F$ is the sheaf of sections of the structure morphism $\pi\colon V \to X$. Then a $\rho$-compatible $G$-action on $F$ corresponds to an action $\tilde\rho\colon G\times_S V \to V$ such that (i) the structure morphism $\pi\colon V \to X$ is $G$-equivariant, and (ii) the action $\tilde\rho$ is "fibrewise linear", meaning that for every $S$-scheme $T$ and every $g \in G(T)$, $x \in X(T)$, the isomorphism $\tilde\rho(g)\colon V_x \to V_{gx}$ is $O_T$-linear. We refer to such an action $\tilde\rho$ as a lifting of $\rho$.

With this notion of a $G$-action on a sheaf, we can formulate a useful result on the descent of modules.

**(7.2) Proposition.** *Let $\rho\colon G \times_S X \to X$ be an action of an $S$-group scheme $G$ on an $S$-scheme $X$. Suppose there exists an fppf quotient $p\colon X \to Y$ of $X$ by $G$. If $F$ is a coherent sheaf of $O_Y$-modules then the canonical isomorphism $\lambda_{\mathrm{can}}\colon \mathrm{pr}_2^*(p^*F) \xrightarrow{\sim} \rho^*(p^*F)$ defines a $\rho$-compatible $G$-action on $p^*F$. The functor $F \mapsto (p^*F, \lambda_{\mathrm{can}})$ gives an equivalence between the category of coherent $O_Y$-modules and the category of coherent $O_X$-modules with ($\rho$-compatible) $G$-action. This restricts to an equivalence between the category of finite locally free $O_Y$-modules and the category of finite locally free $O_X$-modules with $G$-action.*

This proposition should be seen as a statement in (faithfully flat) descent theory; it follows for instance from the results of SGA 1, Exp. VIII, § 1. (See also [BLR], § 6.1, Thm. 4.) Given such results in descent theory, the only point here is that a $\rho$-compatible $G$-action on a coherent $O_X$-module is *the same* as a descent datum on this module. (Recall that we have an isomorphism $(\rho, \mathrm{pr}_2)\colon G \times_S X \xrightarrow{\sim} X \times_Y X$.) The assertion that finite locally free $O_X$-modules with $G$-action give rise to finite locally free $O_Y$-modules follows from EGA IV, Prop. 2.5.2.

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**(7.3) Example.** We consider the situation of the proposition. The geometric vector bundle corresponding to the structure sheaf $O_X$ is just the affine line $\mathbb{A}^1_X$ over $X$.

On $O_X$ (geometrically: on $\mathbb{A}^1_X$) we have a "trivial" action $\tilde\rho_{\mathrm{triv}}$, given by

$$\tilde\rho_{\mathrm{triv}} = \rho \times \mathrm{id}_{\mathbb{A}^1_S}\colon G \times_S \mathbb{A}^1_X = G \times_S X \times_S \mathbb{A}^1_S \longrightarrow X \times_S \mathbb{A}^1_S = \mathbb{A}^1_X\,.$$

The $O_Y$-module corresponding to $(O_X, \tilde\rho_{\mathrm{triv}})$ is just $O_Y$ itself.

Let $\tilde\rho$ be some other lifting of $\rho$ to a $G$-action on $\mathbb{A}^1_X$. Let $T$ be an $S$-scheme and $g \in G(T)$. The automorphism $\tilde\rho(g) \cdot \tilde\rho_{\mathrm{triv}}(g)^{-1}$ of $\mathbb{A}^1_X \times_S T = \mathbb{A}^1_{X_T}$ is given on every fibre $\mathbb{A}^1_x$ by some (invertible) scalar multiplication. This means that $\tilde\rho(g) \cdot \tilde\rho_{\mathrm{triv}}(g)^{-1}$ is given by an element $\nu(g) \in \Gamma(X_T, O^*_{X_T})$. We find that an action $\tilde\rho$ gives rise to a morphism of functors $\nu\colon G \to \mathrm{Res}_{X/S}\mathbb{G}_{m,X}$ on the category $\mathsf{Sch}_{/S}$. The condition that $\tilde\rho$ is a group action means that $\nu$ satisfies a cocycle condition $\nu(g_1g_2)(x) = \nu(g_1)\bigl(g_2x\bigr) \cdot \nu(g_2)\bigl(x\bigr)$, where we simply write $g_2x$ for $\rho(g_2)\bigl(x\bigr)$. Conversely, given a morphism $\nu\colon G \to \mathrm{Res}_{X/S}\mathbb{G}_{m,X}$ that satisfies this condition, one finds back a $G$-action $\tilde\rho$ by $\tilde\rho(g) = \nu(g) \cdot \tilde\rho_{\mathrm{triv}}(g)$.

Now suppose that the structure morphism $f\colon X \to S$ satisfies $f_*(O_{X_T}) = O_T$ for all $S$-schemes $T$. This holds for instance if $X$ is a proper variety over a field. Then $\mathrm{Res}_{X/S}\mathbb{G}_{m,X} \cong \mathbb{G}_{m,S}$ as functors on $\mathsf{Sch}_{/S}$. In particular, any morphism $\nu\colon G \to \mathrm{Res}_{X/S}\mathbb{G}_{m,X}$ is $G$-invariant, in the sense that for all $g_1$, $g_2 \in G(T)$ and $x \in X(T)$ we have $\nu(g_1)\bigl(g_2x\bigr) = \nu(g_1)\bigl(x\bigr)$. Hence the cocycle condition in this case just says that $\nu$ is a homomorphism. So the conclusion is that the liftings $\tilde\rho$ of $\rho$ to a $G$-action on $\mathbb{A}^1_X$ are in bijective correspondence with $\mathrm{Hom}_{\mathsf{GSch}_{/S}}(G, \mathbb{G}_m)$. In case $G$ is a commutative, finite locally free $S$-group scheme this is just the Cartier dual $G^D(S)$.

Via Proposition (7.2), we can use this to obtain a description of the line bundles $L$ on $Y$ such that $p^*L \cong O_X$. The result is as follows.

**(7.4) Proposition.** *Let $G$ be a commutative, finite locally free $S$-group scheme. Let $\rho\colon G \times_S X \to X$ be a free action of $G$ on an $S$-scheme $X$. Let $p\colon X \to Y$ be the quotient of $X$ by $G$. Suppose that $f_*(O_{X_T}) = O_T$ for all $S$-schemes $T$. Then for any $S$-scheme $T$ there is a canonical isomorphism of groups*

$$\delta_T\colon \begin{pmatrix} \text{isomorphism classes of line bundles} \\ L \text{ on } Y_T \text{ with } p^*L \cong O_{X_T} \end{pmatrix} \xrightarrow{\sim} G^D(T)\,,$$

*and this isomorphism is compatible with base change $T' \to T$.*

*Proof.* To define $\delta_T$ for arbitrary $S$-schemes $T$ we may replace $S$ by $T$ and $p\colon X \to Y$ by $p_T\colon X_T \to Y_T$. Note that by Theorem (4.16) and what was explained in Example (4.29), $p_T$ is again the quotient morphism of $X_T$ by the action of $G_T$, and of course also the assumption that $f_*(O_{X_T}) = O_T$ for all $S$-schemes $T$ is preserved under base change. Hence it suffices to define the isomorphism $\delta_S$.

Let $L$ be a line bundle on $Y$ with $p^*L \cong O_X$. Via the choice of an isomorphism $\alpha\colon p^*L \xrightarrow{\sim} O_X$ (or, more geometrically, the isomorphism $\alpha\colon p^*\mathbb{V}(L^{-1}) \xrightarrow{\sim} \mathbb{A}^1_X$ over $X$) the canonical $G$-action on $p^*L$ translates into a $G$-action $\tilde\rho$ on $\mathbb{A}^1_X$, and as explained above this gives us a character $\nu\colon G \to \mathbb{G}_{m,S}$. We claim that this character is independent of the choice of $\alpha$. In general, any other isomorphism $p^*L \xrightarrow{\sim} O_X$ is of the form $\alpha' = \gamma \circ \alpha$ for some $\gamma \in \Gamma(X, O^*_X)$. Write $\tilde\rho$ and $\tilde\rho'$ for the $G$-actions on $\mathbb{A}^1_X$ obtained using $\alpha$ and $\alpha'$, respectively, and let $\nu$ and $\nu'$ be the associated characters. If $g \in G(T)$ and $y$ is a $T$-valued point of $p^*\mathbb{V}(L^{-1})$ lying over $x \in X(T)$ then we have the relations

$$\tilde\rho_{\mathrm{triv}}\bigl(g, \alpha'(y)\bigr) = \gamma(x) \cdot \tilde\rho_{\mathrm{triv}}\bigl(g, \alpha(y)\bigr) \quad \text{and} \quad \tilde\rho'\bigl(g, \alpha'(y)\bigr) = \gamma(gx) \cdot \tilde\rho\bigl(g, \alpha(y)\bigr)\,,$$

where $\gamma(x)$ is the image of $\gamma$ under the homomorphism $\Gamma(X, O_X^*) \to \Gamma(T, O_T^*)$ induced by $x: T \to X$, and similarly for $\gamma(gx)$. (Note that elements such as $\tilde{\rho}\bigl(g, \alpha(y)\bigr)$ are $T$-valued points of $\mathbb{A}^1_X$ lying over the point $gx \in X(T)$, and on such elements we have the "fibrewise" multiplication by functions on $T$.) But now our assumption that $f_*(O_X) = O_S$ implies that $\gamma$ is the pull-back of an element in $\Gamma(S, O_S^*)$, so $\gamma(x) = \gamma(gx)$. Hence $\nu = \nu'$, as claimed.

Now we can simply apply the conclusion from (7.3), and define $\delta_S$ as the map that sends the isomorphism class of $L$ to the character $\nu: G \to \mathbb{G}_{m,S}$ given on points by $\nu(g) = \tilde{\rho}(g) \cdot \tilde{\rho}_{\mathrm{triv}}(g)^{-1}$. By Proposition (7.2), together with what was explained in Example (7.3), the map $\delta_S$ thus obtained is indeed an isomorphism.

Finally we note that the maps $\delta_T$ are indeed compatible with base change, as is immediate from the construction. $\square$

§ 2. *Two duality theorems.*

**(7.5) Theorem.** *Let $f: X \to Y$ be an isogeny of abelian varieties. Then $f^t: Y^t \to X^t$ is again an isogeny and there is a canonical isomorphism of group schemes*

$$\mathrm{Ker}(f)^D \xrightarrow{\sim} \mathrm{Ker}(f^t)\,.$$

*Proof.* If $T$ is a $k$-scheme, any class in $\mathrm{Ker}(f^t)(T)$ is uniquely represented by a line bundle $L$ on $Y_T$ such that $f^*L \cong O_{X_T}$. Indeed, if $L'$ represents a class in $\mathrm{Ker}(f^t)(T)$ then there is a line bundle $M$ on $T$ such that $f^*L' \cong \mathrm{pr}_T^*M$. Then $L := L' \otimes \mathrm{pr}_T^*M^{-1}$ represents the same class as $L'$ and satisfies $f^*L \cong O_{X_T}$. Conversely, if $L_1$ and $L_2$ represent the same class then they differ by a line bundle of the form $\mathrm{pr}_T^*M$; hence $f^*L_1 \cong f^*L_2$ implies $L_1 \cong L_2$.

Applying Proposition (7.4) we obtain the desired isomorphism $\mathrm{Ker}(f^t) \xrightarrow{\sim} \mathrm{Ker}(f)^D$. In particular this shows that $f^t$ has a finite kernel and therefore is again an isogeny. $\square$

**(7.6) Proposition.** *Let $f: X \to Y$ be a homomorphism. Let $M$ be a line bundle on $Y$ and write $L = f^*M$. Then $\varphi_L: X \to X^t$ equals the composition*

$$X \xrightarrow{f} Y \xrightarrow{\varphi_M} Y^t \xrightarrow{f^t} X^t\,.$$

*If $f$ is an isogeny and $M$ is non-degenerate then $L$ is non-degenerate too, and* $\mathrm{rank}(K(L)) = \deg(f)^2 \cdot \mathrm{rank}(K(M))$.

*Proof.* That $\varphi_L = f^t \circ \varphi_M \circ f$ is clear from the formula $t_x^* f^* M = f^* t_{f(x)}^* M$. For the second assertion recall that a line bundle $L$ is non-degenerate precisely if $\varphi_L$ is an isogeny, in which case $\mathrm{rank}(K(L)) = \deg(\varphi_L)$. Now use (7.5). $\square$

**(7.7)** The Poincaré bundle on $X \times X^t$ comes equipped with a rigidification along $\{0\} \times X^t$. As $\mathscr{P}_{|X \times \{0\}} \cong O_X$ we can also choose a rigidification of $\mathscr{P}$ along $X \times \{0\}$. Such a rigidification is unique up to an element of $\Gamma(X, O_X^*) = k^*$. Hence there is a unique rigidification along $X \times \{0\}$ such that the two rigidifications agree at the origin $(0, 0)$.

Now we view $\mathscr{P}$ as a family of line bundles on $X^t$ parametrised by $X$. This gives a morphism

$$\kappa_X: X \longrightarrow X^{tt}\,.$$

As $\kappa_X(0) = 0$ it follows from Prop. (1.13) that $\kappa_X$ is a homomorphism.

**(7.8) Lemma.** *Let $L$ be a line bundle on $X$. Then $\varphi_L = \varphi_L^t \circ \kappa_X\colon X \to X^t$.*

*Proof.* Let $s\colon X \times X \to X \times X$ and $s\colon X \times X^t \to X^t \times X$ be the morphisms switching the two factors; on points: $s(x,y) = (y,x)$. We have a canonical isomorphism $s^*\Lambda(L) \cong \Lambda(L)$. Let $T$ be a $k$-scheme and $x \in X(T)$. Writing $[M]$ for the class of a bundle $M$ on $X \times T$ in $\mathrm{Pic}^0_{X/k}(T)$ we have

$$\begin{aligned}\varphi_L(x) &= \Big[(X \times T \xrightarrow{\mathrm{id}\times x} X \times X)^*\Lambda(L)\Big]\\ &= \Big[(X \times T \xrightarrow{\mathrm{id}\times x} X \times X \xrightarrow{s} X \times X)^*\Lambda(L)\Big]\\ &= \Big[(X \times T \xrightarrow{\mathrm{id}\times x} X \times X \xrightarrow{s} X \times X \xrightarrow{\mathrm{id}\times\varphi_L} X \times X^t)^*\mathscr{P}\Big]\\ &= \Big[(X \times T \xrightarrow{\varphi_L\times\mathrm{id}} X^t \times T \xrightarrow{\mathrm{id}\times x} X^t \times X \xrightarrow{s} X \times X^t)^*\mathscr{P}\Big] = \varphi_L^t \circ \kappa_X(x)\,.\end{aligned}$$

As this holds for all $T$ and $x$ the lemma is proven. □

**(7.9) Theorem.** *Let $X$ be an abelian variety over a field. Then the homomorphism $\kappa_X\colon X \longrightarrow X^{tt}$ is an isomorphism.*

*Proof.* Choose an ample line bundle $L$ on $X$. The formula $\varphi_L = \varphi_L^t \circ \kappa_X$ shows that $\mathrm{Ker}(\kappa_X)$ is finite; hence $\kappa_X$ is an isogeny. Furthermore,

$$\mathrm{rank}\big(K(L)\big) = \deg(\varphi_L) = \deg(\varphi_L^t) \cdot \deg(\kappa_X) = \mathrm{rank}\big(K(L)^D\big) \cdot \deg(\kappa_X)\,,$$

using (7.5). But $\mathrm{rank}\big(K(L)^D\big) = \mathrm{rank}\big(K(L)\big)$, so $\kappa_X$ has degree 1. □

**(7.10) Corollary.** *If $L$ is a non-degenerate line bundle on $X$ then $K(L) \cong K(L)^D$.*

*Proof.* Apply (7.5) to $\varphi_L$ and use (7.8) and (7.9). □

## § 3. *Further properties of* $\mathrm{Pic}^0_{X/k}$.

Let $X$ be an abelian variety over a field $k$. A line bundle $L$ on $X$ gives rise to a homomorphism $\varphi_L\colon X \to X^t$. We are going to extend this construction to a more general situation. Namely, let $T$ be a $k$-scheme, and suppose $L$ is a line bundle on $X_T := X \times_k T$. We are going to associate to $L$ a homomorphism $\varphi_L\colon X_T \to X_T^t$.

As usual we write $\Lambda(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ for the Mumford bundle on $X_T \times_T X_T$ associated to $L$. (Note that we are working in the relative setting, viewing $T$ as the base scheme. If we rewrite $X_T \times_T X_T$ as $X \times_k X \times_k T$ then $\Lambda(L)$ becomes $(m \times \mathrm{id}_T)^*L \otimes p_{13}^*L^{-1} \otimes p_{23}^*L^{-1}$.) In order to distinguish the two factors $X_T$, let us write $X_T^{(1)} = X_T \times_T e(T)$ and $X_T^{(2)} = e(T) \times_T X_T$ where $e(T) \subset X_T$ is the image of the zero section $e\colon T \to X_T$. Viewing $\Lambda(L)$ as a family of line bundles on $X_T^{(1)}$ parametrized by $X_T^{(2)}$ we obtain a morphism

$$\varphi_L\colon X_T = X_T^{(2)} \longrightarrow \mathrm{Pic}_{X_T/T} = \mathrm{Pic}_{X/k} \times_k T\,.$$

As $\varphi_L(0) = 0$ and the fibres $X_t$ are connected, $\varphi_L$ factors through $X_T^t = \mathrm{Pic}^0_{X/k} \times_k T$.

**(7.11) Lemma.** (i) *The morphism $\varphi_L$ only depends on the class of $L$ in* $\mathrm{Pic}_{X/k}(T)$.

(ii) *Let $f\colon T \to S$ be a morphism of $k$-schemes. If $M$ is a line bundle on $X_S$ and $L = (\mathrm{id}_X \times f)^*M$ on $X_T$, then $\varphi_L\colon X_T \to X_T^t$ is the morphism obtained from $\varphi_M\colon X_S \to X_S^t$ by pulling back via $f$ on the basis.*

(iii) *The morphism $\varphi_L\colon X_T \to X_T^t$ is a homomorphism.*

Part (i) of the lemma will be sharpened in (7.15) below. As a particular case of (ii), note that the fibre of $\varphi_L$ above a point $t \in T$ is just $\varphi_{L_t}$, where we write $L_t$ for the restriction of $L$ to $X \times \{t\}$.

*Proof.* (i) If $L_1$ and $L_2$ have the same class then they differ by a factor $\mathrm{pr}_T^*M$. But then $\Lambda(L_1)$ and $\Lambda(L_2)$ differ by a factor $\pi^*M^{-1}$, where $\pi\colon X_T \times_T X_T \to T$ is the structural morphism. This implies that $\varphi_{L_1} = \varphi_{L_2}$, as claimed.

(ii) This readily follows from the definitions.

(iii) The assertion that $\varphi_L$ is a homomorphism means that we have an equality of two morphisms

$$\varphi_L \circ m = m \circ (\varphi_L \times \varphi_L)\colon X_T \times_T X_T \longrightarrow X_T^t\,.$$

For every $t \in T$ we already know that the two morphisms agree on the fibres above $t$. Hence the lemma is true if $T$ is reduced. In particular, the lemma is true in the "universal" case that $T = \mathrm{Pic}_{X/k}$ and $L$ is the Poincaré bundle on $X \times_k \mathrm{Pic}_{X/k}$. In the general case, consider the morphism $f\colon T \to \mathrm{Pic}_{X/k}$ associated to the line bundle $L$. This morphism is characterized by the property that $L$ and $(\mathrm{id} \times f)^*\mathscr{P}$ have the same class in $\mathrm{Pic}_{X/k}(T)$. Now apply (i) and (ii). $\square$

In the above we allow $L$—to be thought of as a family of line bundles on $X$ parametrized by $T$—to be non-constant. But the abelian variety we work on is a constant one. We can go one step further by also letting the abelian varieties $X_t$ "vary with $t$". This generalization will be discussed in Chapter ??; see in particular (?.?).

We write $K(L) := \mathrm{Ker}(\varphi_L) \subset X_T$. It is the maximal subscheme of $X_T$ over which $\Lambda(L)$ is trivial, viewing $X_T \times_T X_T$ as a scheme over $X_T$ via the second projection. In particular, $\varphi_L = 0$ if and only if $\Lambda(L)$ is trivial over $X_T$, meaning that $\Lambda(L) = \mathrm{pr}_2^*M$ for some line bundle $M$ on $X_T$. Using (2.17) we can make this a little more precise.

**(7.12) Lemma.** *Let $T$ be a locally noetherian $k$-scheme. Write $\pi\colon X_T \times_T X_T \to T$ for the structural morphism. For a line bundle $L$ on $X_T$, consider the following conditions.*

(a) $\varphi_L = 0$.

(b) $\Lambda(L) \cong \mathrm{pr}_2^*M$ *for some line bundle $M$ on $X_T$.*

(c) $\Lambda(L) \cong \pi^*N$ *for some line bundle $N$ on $T$.*

(d) $\varphi_{L_t} = 0$ *for some $t \in T$.*

*Then* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d), *and if $T$ is connected then all four conditions are equivalent. If these equivalent conditions are satisfied then $N \cong e^*L^{-1}$ and $M = \mathrm{pr}_T^*N$.*

*Proof.* The implications (d) $\Leftarrow$ (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c) are clear. Let us write $X_T \times_T X_T$ as $X \times_k X \times_k T$. In this notation we have $\Lambda(L) = (m \times \mathrm{id}_T)^*L \otimes p_{13}^*L^{-1} \otimes p_{23}^*L^{-1}$ and $\pi$ becomes the projection onto the third factor. Set $N := e^*L^{-1}$. We find that

$$\Lambda(L)_{|\{0\}\times X\times T} \cong \mathrm{pr}_T^*N \cong \Lambda(L)_{|X\times\{0\}\times T}$$

as line bundles on $X \times T$.

Suppose $T$ is connected and $\varphi_{L_t} = 0$ for some $t \in T$. Then

$$\Lambda(L)_{|X\times X\times\{t\}} \cong O_{X\times X\times\{t\}}$$

by (iii) of (2.17). By Thm. (2.5) the line bundle $\Lambda(L) \otimes p_3^* N^{-1}$ on $X \times X \times T$ is trivial, i.e., $\Lambda(L) \cong \pi^* N$. This shows that (d) $\Rightarrow$ (a) for connected $T$. For arbitrary $T$ we get the implication (a) $\Rightarrow$ (c) by applying the previous to each of its connected components.

The last assertion of the lemma is obtained by restricting $\Lambda(L)$ to $\{0\} \times \{0\} \times T$ and to $\{0\} \times X \times T$. $\square$

**(7.13) Fact.** *Let $X$ and $Y$ be two projective varieties over a field $k$. Then the contravariant functor*

$$\mathit{Hom}_{\mathsf{Sch}}(X, Y)\colon (\mathsf{Sch}_{/k}) \to \mathsf{Sets} \quad \textit{given by} \quad T \mapsto \mathrm{Hom}_{\mathsf{Sch}_{/T}}(X_T, Y_T)$$

*is representable by a $k$-scheme, locally of finite type.*

This fact is a consequence of the theory of Hilbert schemes. A reference is ??. Note that in this proof the projectivity of $X$ and $Y$ is used in an essential way. See also Matsumura-Oort [1] for related results for non-projective varieties.

**(7.14) Proposition.** *Let $X$ and $Y$ be two abelian varieties over a field $k$. Then the functor*

$$\mathit{Hom}_{\mathsf{AV}}(X, Y)\colon (\mathsf{Sch}_{/k}) \to \mathsf{Ab} \quad \textit{given by} \quad T \mapsto \mathrm{Hom}_{\mathsf{GSch}_{/T}}(X_T, Y_T)$$

*is representable by an étale commutative $k$-group scheme.*

*Proof.* Let $H = \mathit{Hom}_{\mathsf{Sch}}(X, Y)$ and $H' = \mathit{Hom}_{\mathsf{Sch}}(X \times X, Y)$. Let $f\colon X_H \to Y_H$ be the universal morphism. Consider the morphism $g\colon (X \times X)_H \to Y_H$ given on points by $g(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2)$. Consider also the "trivial" morphism $e\colon (X \times X)_H \to Y_H$ given on points by $e(x_1, x_2) = e_Y$. Then $g$ and $e$ are $H$-valued points of $H'$; in other words, they correspond to morphisms $\psi_g$, $\psi_e\colon H \to H'$. The functor $\mathit{Hom}_{\mathsf{AV}}(X, Y)$ is represented by the subscheme of $H$ given by the condition that $\psi_g = \psi_e$; in other words, it is given by the cartesian diagram

$$\begin{tikzcd}
\mathit{Hom}_{\mathsf{AV}}(X, Y) \arrow[r] \arrow[d] & H' \arrow[d, "\Delta_{H'/k}"] \\
H \arrow[r, "(\psi_g, \psi_e)"] & H' \times_k H'
\end{tikzcd} .$$

To get a group scheme structure on $\mathit{Hom}_{\mathsf{AV}}(X, Y)$ we just note that $\mathit{Hom}_{\mathsf{AV}}(X, Y)$ is naturally a group functor; now apply (3.6).

It remains to be shown that $\mathit{Hom}_{\mathsf{AV}}(X, Y)$ is an étale group scheme. We already know it is locally of finite type over $k$, so it suffices to show that its tangent space at the origin is trivial. It suffices to prove this in the special case that $Y = X$, for $\mathit{Hom}_{\mathsf{AV}}(X, Y)$ embeds as a closed subgroup scheme of $\mathscr{E}\mathit{nd}_{\mathsf{AV}}(X \times Y) := \mathit{Hom}_{\mathsf{AV}}(X \times Y, X \times Y)$ by sending $f\colon X \to Y$ to the endomorphism $(x, y) \mapsto \big(0, f(x)\big)$ of $X \times Y$.

A tangent vector of $\mathscr{E}\mathit{nd}_{\mathsf{AV}}(X)$ at the point $\mathrm{id}_X$ is the same as a homomorphism $\xi\colon X_{k[\varepsilon]} \to X_{k[\varepsilon]}$ over $\mathrm{Spec}\big(k[\varepsilon]\big)$ that reduces to the identity modulo $\varepsilon$. Note that $\xi$ is necessarily an automorphism. (It is the identity on underlying topological spaces, and it is an easy exercise to show that $\xi$ gives an automorphism of the structure sheaf.) Hence by the results in Exercise (1.3), $\xi$ corresponds to a global vector field $\Xi$ on $X$. As we know, the global vector fields on $X$ are precisely the translation-invariant vector fields. On the other hand, a necessary condition for $\xi$ to be an endomorphism is that it maps the identity section of $X_{k[\varepsilon]}$ to itself. This just means that $\Xi(e_X) = 0$. Hence $\Xi$ is the trivial vector field. This shows that $\mathrm{id}_X$ has non non-trivial first order deformations. $\square$

In line with the notational conventions introduced in (1.16), we shall usually simply write $Hom(X, Y)$ for the group scheme of homomorphisms from $X$ to $Y$. If we wish to refer to the bigger scheme of arbitrary scheme morphisms from $X$ to $Y$, or if there is a risk of confusion, we shall use a subscript "AV" or "Sch" to indicate which of the two we mean.

By (i) and (ii) of Lemma (7.11), $L \mapsto \varphi_L$ gives rise to a morphism of functors $\varphi\colon \mathrm{Pic}_{X/k} \to Hom(X, X^t)$. If $L$ and $M$ are line bundles on $X_T$ then $\Lambda(L \otimes M) \cong \Lambda(L) \otimes \Lambda(M)$ and we find that $\varphi_{L\otimes M} = \varphi_L + \varphi_M$. Summing up, we obtain a homomorphism of $k$-group schemes

$$\varphi\colon \mathrm{Pic}_{X/k} \to Hom(X, X^t)\,.$$

**(7.15) Lemma.** *Let $T$ be a connected $k$-scheme. Let $L$ be a line bundle on $X_T$. Write $L_t$ for $L_{|X\times\{t\}}$. Then for any two $k$-valued points $s$, $t \in T(k)$ we have $\varphi_{L_s} = \varphi_{L_t}$. In particular,* $\mathrm{Pic}^0_{X/k} \subset \mathrm{Ker}(\varphi)$.

*Proof.* By (d) $\Rightarrow$ (a) of (7.12), applied with $T = X^t$ and with $L = \mathscr{P}$ the Poincaré bundle, we find that $X^t = \mathrm{Pic}^0_{X/k} \subset \mathrm{Ker}(\varphi)$. As $\varphi$ is a homomorphism, it is constant on the connected components of $\mathrm{Pic}_{X/k}$.

Let $f\colon T \to \mathrm{Pic}_{X/k}$ be the morphism corresponding to $L$; it factors through some connected component $C \subset \mathrm{Pic}_{X/k}$. Let $M := \mathscr{P}_{|X\times C}$ be the restriction of the Poincaré bundle to $X \times C$. Using (i) and (ii) of (7.11) we find that $\varphi_L\colon X_T \to X^t_T$ is obtained from $\varphi_M\colon X_C \to X^t_C$ by pulling back via $f$ on the basis. But by the above, $\varphi_{M_{f(s)}} = \varphi_{M_{f(t)}}$. $\square$

**(7.16) Lemma.** *Let $X$ be an abelian variety over $k$. Let $T$ be a $k$-scheme and let $L$ be a line bundle on $X_T$ such that $\varphi_L = 0$.*

(i) *If $Y$ is a $T$-scheme then for any two morphisms $f$, $g\colon Y \to X_T$ of schemes over $T$ we have $\big[(f+g)^*L\big] = \big[f^*L \otimes g^*L\big]$ in $\mathrm{Pic}_{Y/T}(T)$.*

(ii) *For $n \in \mathbb{Z}$ we have $[n^*L] = [L^n]$ in $\mathrm{Pic}_{X/k}(T)$.*

*Proof.* If $\varphi_L = 0$ then $\Lambda(L) = \pi^*N$ for some line bundle $N$ on $T$. Pulling back via $(f, g)\colon Y \to X_T \times_T X_T$ gives $(f+g)^*L = f^*L \otimes g^*L \otimes \pi^*N$, where $\pi\colon Y \to T$ is the structural morphism. But $\pi^*N$ is trivial in $\mathrm{Pic}_{Y/T}(T)$, so we get (i). Applying this with $f = \mathrm{id}_{X_T}$ and $g = n_{X_T}$ gives the relation $[(n+1)^*L] = [L \otimes n^*L]$. By double induction on $n$, starting with the cases $n = 0$ and $n = 1$, we obtain (ii). $\square$

Using that $\mathrm{Pic}^0_{X/k} \subset \mathrm{Ker}(\varphi)$ we obtain a positive answer to the questions posed in (6.20).

**(7.17) Corollary.** *Let $X$ and $Y$ be abelian varieties over $k$. Then the map $Hom(X, Y) \to Hom(Y^t, X^t)$ given on points by $f \mapsto f^t$ is a homomorphism of $k$-group schemes. For all $n \in \mathbb{Z}$ we have $(n_X)^t = n_{X^t}$.*

Combining this last result with (7.5) we find that $X^t[n]$ is canonically isomorphic to the Cartier dual of $X[n]$, for every $n \in \mathbb{Z}_{>0}$.

**(7.18)** Let $X$ be an abelian variety. We call a homomorphism $f\colon X \to X^t$ symmetric if $f = f^t$, taking the isomorphism $\kappa_X\colon X \xrightarrow{\sim} X^{tt}$ of (7.9) as an identification. It follows from the previous corollary that the functor of symmetric homomorphisms $X \to X^t$ is represented by a closed subgroup scheme

$$Hom^{\mathrm{symm}}(X, X^t) \subset Hom(X, X^t)\,.$$

In fact, $Hom^{\mathrm{symm}}(X, X^t)$ is just the kernel of the endomorphism of $Hom(X, X^t)$ given by $f \mapsto f - f^t$.

By Lemma (7.8), the homomorphism $\varphi\colon \mathrm{Pic}_{X/k} \to Hom(X, X^t)$ factors through the subgroup $Hom^{\mathrm{symm}}(X, X^t)$. (Because $Hom(X, X^t)$ is étale, it suffices to know that $\varphi$ maps into $Hom^{\mathrm{symm}}$ for points with values in a field.)

Our next goal is to show that not only $\mathrm{Pic}^0_{X/k} \subset \mathrm{Ker}(\varphi)$ but that the two are in fact equal. First we prove a lemma about the cohomology of line bundles $L$ with $\varphi_L = 0$. Note that we are here again working over a field; this lemma has no straightforward generalization to the relative setting.

**(7.19) Lemma.** *Let $L$ be a line bundle on $X$ with $\varphi_L = 0$. If $L \not\cong O_X$ then $H^i(X, L) = 0$ for all $i$.*

*Proof.* First we treat the group $H^0(X, L)$. If there is a non-trivial section $s$ then $(-1)^*s$ is a non-trivial section of $(-1)^*L \cong L^{-1}$; so both $L$ and $L^{-1}$ have a non-trivial section, and this implies that $L$ is trivial. Since we have assumed this is not the case, $H^0(X, L) = \{0\}$.

Let now $i \geqslant 1$ be the smallest positive integer such that $H^i(X, L) \neq 0$. Consider the composition

$$X \to X \times X \xrightarrow{m} X\,, \quad \text{given by } x \mapsto (x, 0) \mapsto x\,.$$

On cohomology this induces the maps

$$H^i(X, L) \to H^i(X \times X, m^*L) \to H^i(X, L)\,,$$

the composition of which is the identity. But since $m^*L \cong p_1^*L \otimes p_2^*L$, the Künneth formula gives

$$H^i(X \times X, m^*L) \cong H^i(X \times X, p_1^*L \otimes p_2^*L) \cong \sum_{a+b=i} H^a(X, L) \otimes H^b(X, L)\,.$$

Since $H^0(X, L) = \{0\}$ we may consider only those terms in the RHS where $a \geqslant 1$ and $b \geqslant 1$. But then $a < i$ which by our choice of $i$ implies that $H^a(X, L) = 0$. This shows that the identity map on $H^i(X, L)$ factors via zero. □

In the proof of the next proposition we need some facts about cohomology and base change. Here is what we need.

**(7.20) Fact.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes, with $Y$ reduced and connected. Let $F$ be a coherent sheaf of $O_X$-modules on $X$.*

*(i) If $y \mapsto \dim_{k(y)} H^q(X_y, F_y)$ is a constant function on $Y$ then $R^q f_*(F)$ is a locally free sheaf on $Y$, and for all $y \in Y$ the natural map $R^q f_*(F) \otimes_{O_Y} k(y) \to H^q(X_y, F_y)$ is an isomorphism.*

*(ii) If $R^q f_*(F) = 0$ for all $q \geqslant q_0$ then $H^q(X_y, F_y) = 0$ for all $y \in Y$ and $q \geqslant q_0$.*

A proof of this result can be found in [MAV], § 5.

**(7.21) Proposition.** *Let $X$ be an abelian variety over an algebraically closed field $k$. Let $L$ be an ample line bundle on $X$ and $M$ a line bundle with $\varphi_M = 0$. Then there exists a point $x \in X(k)$ with $M \cong t_x^*L \otimes L^{-1}$.*

*Proof.* We follow Mumford's beautiful proof. The idea is to look at the cohomology on $X \times X$ of the line bundle

$$K := \Lambda(L) \otimes p_2^* M^{-1} .$$

The projections $p_1, p_2 \colon X \times X \to X$ give rise to two Leray spectral sequences

$$E_2^{p,q} = H^p(X, R^q p_{1,*}(K)) \Rightarrow H^{p+q}(X \times X, K)$$

and

$$E_2'^{p,q} = H^p(X, R^q p_{2,*}(K)) \Rightarrow H^{p+q}(X \times X, K) .$$

The restrictions of $K$ to the horizontal and vertical fibres are given by

$$\begin{aligned} K_{|\{x\}\times X} &\cong t_x^* L \otimes L^{-1} \otimes M^{-1} , \\ K_{|X \times \{x\}} &\cong t_x^* L \otimes L^{-1} . \end{aligned}$$

Assume that there is no $x \in X(k)$ such that $t_x^* L \otimes L^{-1} \cong M$. It then follows that $K_{|\{x\}\times X}$ is a non-trivial bundle in $\mathrm{Ker}(\varphi)$ for every $x$. (Note that $[t_x^* L \otimes L^{-1}] = \varphi_L(x) \in \mathrm{Pic}^0_{X/k} \subset \mathrm{Ker}(\varphi)$.) By Lemma (7.19) and (7.20) this gives $R^q p_{1,*}(K) = (0)$ for all $q$, and from the first spectral sequence we find that $H^n(X \times X, K) = 0$ for all $n$.

Now use the second spectral sequence. For $x \notin K(L)$ the bundle $t_x^* L \otimes L^{-1}$ is a non-trivial bundle in $\mathrm{Ker}(\varphi)$. Again by Lemma (7.15) we find that $\mathrm{supp}(R^q p_{2,*} K) \subset K(L)$. Since $K(L)$ is a finite subscheme of $X$ (the bundle $L$ being ample) we find

$$E_2'^{p,q} = \begin{cases} \bigoplus\limits_{x \in K(L)} R^q p_{2,*}(K)_x & \text{if } p = 0; \\ \qquad 0 & \text{otherwise.} \end{cases}$$

As we only have non-zero terms for $p = 0$, the spectral sequence degenerates at level $E_2'$. This gives $H^n(X \times X, K) = \oplus_{x \in K(L)} R^n p_{2,*}(K)_x$.

Comparing the two answers for $H^n(X \times X, K)$ we find that $R^n p_{2,*}(K) = 0$ for all $n$. By (7.20) this implies that $H^n(X, K_{|X\times\{x\}}) = 0$ for all $x$. But $K_{|X\times\{0\}}$ is the trivial bundle, so taking $n = 0$ and $x = 0$ gives a contradiction. □

**(7.22) Corollary.** *Let $X$ be an abelian variety over a field $k$. Then* $\mathrm{Pic}^0_{X/k} = \mathrm{Ker}\big(\varphi \colon \mathrm{Pic}_{X/k} \to \mathit{Hom}(X, X^t)\big)$.

*Proof.* We already know that $\mathrm{Ker}(\varphi)$ is a subgroup scheme of $\mathrm{Pic}_{X/k}$ that contains $\mathrm{Pic}^0_{X/k}$. Hence $\mathrm{Ker}(\varphi)$ is the union of a number of connected components of $\mathrm{Pic}_{X/k}$. By the proposition, every $\bar{k}$-valued point of $\mathrm{Ker}(\varphi)$ lies in $\mathrm{Pic}^0$. The claim follows. □

**(7.23) Corollary.** *Let $X$ be an abelian variety over a field $k$. Let $L$ be a line bundle on $X$.*

(i) *If $[L^n] \in \mathrm{Pic}^0_{X/k}$ for some $n \neq 0$ then $[L] \in \mathrm{Pic}^0_{X/k}$. In particular, if $L$ has finite order, i.e., $L^n \cong O_X$ for some $n \in \mathbb{Z}_{\geqslant 1}$, then $[L] \in \mathrm{Pic}^0_{X/k}$.*

(ii) *We have $[L \otimes (-1)^* L^{-1}] \in \mathrm{Pic}^0_{X/k}$.*

(iii) *We have*

$$\begin{aligned} [L] \in \mathrm{Pic}^0_{X/k} &\Longleftrightarrow n^* L \cong L^n \quad \text{for all } n \in \mathbb{Z} \\ &\Longleftrightarrow n^* L \cong L^n \quad \text{for some } n \in \mathbb{Z} \setminus \{0, 1\}. \end{aligned}$$

*Proof.* (i) Since $\varphi$ is a homomorphism we have $\varphi_{L^n}(x) = n \cdot \varphi_L(x) = \varphi_L(n \cdot x)$. Hence if $[L^n] \in \mathrm{Pic}^0(X)$ then $\varphi_L$ is trivial on all points in the image of $n_X$. But $n_X$ is surjective, so $\varphi_L$ is trivial.

(ii) Direct computation shows that $\varphi_{(-1)^*L}(x) = -\varphi_L(x)$ for all $L$ and $x$. Since also $\varphi_{L^{-1}}(x) = -\varphi_L(x)$, we find that $[L \otimes (-1)^*L^{-1}] \in \mathrm{Ker}(\varphi)$.

(iii) The first implication "$\Rightarrow$" was proven in (7.16) above; the second is trivial. Suppose that $n^*L \cong L^n$ for some $n \notin \{0, 1\}$. Since $n^*L \cong L^n \otimes [L \otimes (-1)^*L]^{(n^2-n)/2}$ it follows that $L \otimes (-1)^*L$ has finite order, hence its class lies in $\mathrm{Pic}^0_{X/k}$. By (ii) we also have $[L \otimes (-1)^*L^{-1}] \in \mathrm{Pic}^0_{X/k}$. Hence $[L^2] \in \mathrm{Pic}^0_{X/k}$ and by (i) then also $[L] \in \mathrm{Pic}^0_{X/k}$. □

**(7.24)** In (3.29) we have associated to any group scheme $G$ locally of finite type over a field $k$ an étale group scheme of connected components, denoted by $\varpi_0(G)$. We now apply this with $G = \mathrm{Pic}_{X/k}$ for $X/k$ an abelian variety. The associated component group scheme

$$\mathrm{NS}_{X/k} := \varpi_0(\mathrm{Pic}_{X/k})$$

is called the Néron-Severi group scheme of $X$ over $k$. The natural homomorphism $q\colon \mathrm{Pic}_{X/k} \to \mathrm{NS}_{X/k}$ realizes $\mathrm{NS}_{X/k}$ as the fppf quotient of $\mathrm{Pic}_{X/k}$ modulo $\mathrm{Pic}^0_{X/k}$; hence we could also write

$$\mathrm{NS}_{X/k} = \mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k}\,.$$

We refer to the group

$$\mathrm{NS}(X) := \mathrm{NS}_{X/k}(k)$$

as the Néron-Severi group of $X$. Note that $\mathrm{NS}(X)$ equals the subgroup of $\mathrm{Gal}(k_s/k)$-invariants in $\mathrm{NS}(X_{k_s})$.

We say that two line bundles $L$ and $M$ are algebraically equivalent, notation $L \sim_{\mathrm{alg}} M$, if $[L]$ and $[M]$ have the same image in $\mathrm{NS}(X)$. As $\mathrm{NS}(X)$ naturally injects into $\mathrm{NS}(X_{\bar{k}})$, algebraic equivalence of line bundles (or divisors) can be tested over $\bar{k}$, and there it coincides with the notion defined in Remark (6.9). Hence we can think of the Néron-Severi group scheme as being given by the classical, geometric Néron-Severi group $\mathrm{NS}(X_{k_s}) = \mathrm{NS}(X_{\bar{k}})$ of line bundles (or divisors) modulo algebraic equivalence, together with its natural action of $\mathrm{Gal}(k_s/k)$. Note, however, that a $k$-rational class $\xi \in \mathrm{NS}(X)$ may not always be representable by a line bundle on $X$ over the ground field $k$.

Let us rephrase some of the results that we have obtained in terms of the Néron-Severi group.

**(7.25) Corollary.** *The Néron-Severi group* $\mathrm{NS}(X)$ *is torsion-free. If* $n \in \mathbb{Z}$ *and* $L$ *is a line bundle on* $X$ *then* $n^*L$ *is algebraically equivalent to* $L^{n^2}$*; in other words,* $n^*\colon \mathrm{NS}(X) \to \mathrm{NS}(X)$ *is multiplication by* $n^2$.

*Proof.* The first assertion is just (i) of Corollary (7.23). The second assertion follows from (ii) of that Corollary together with Corollary (2.12). □

**(7.26)** Corollary (7.22) can be restated by saying that the natural homomorphism $\varphi\colon \mathrm{Pic}_{X/k} \to \mathit{Hom}^{\mathrm{symm}}(X, X^t) \subset \mathit{Hom}(X, X^t)$ factors as

$$\mathrm{Pic}_{X/k} \xrightarrow{q} \mathrm{NS}_{X/k} \xhookrightarrow{\psi} \mathit{Hom}^{\mathrm{symm}}(X, X^t)$$

for some injective homomorphism $\psi\colon \mathrm{NS}_{X/k} \hookrightarrow \mathit{Hom}^{\mathrm{symm}}(X, X^t)$. This says that the homomorphism $\varphi_L$ associated to a line bundle $L$ only depends on the algebraic equivalence class of $L$, and that $\varphi_L = \varphi_M$ only if $L \sim_{\mathrm{alg}} M$. We shall later show that $\psi$ is actually an isomorphism; see Corollary (11.3).

## § 4. *Applications to cohomology.*

**(7.27) Proposition.** *Let $X$ be an abelian variety with $\dim(X) = g$. Cup-product gives an isomorphism $\wedge^\bullet H^1(X, O_X) \xrightarrow{\sim} H^\bullet(X, O_X)$. For every $p$ and $q$ we have a natural isomorphism $H^q(X, \Omega^p_{X/k}) \cong (\wedge^q T_{X^t,0}) \otimes (\wedge^p T^\vee_{X,0})$. The Hodge numbers $h^{p,q} = \dim H^q(X, \Omega^p_{X/k})$ are given by $h^{p,q} = \binom{g}{p}\binom{g}{q}$.*

*Proof.* Use (6.13) and the isomorphisms $\Omega^p_{X/k} \cong (\wedge^p T^\vee_{X,0}) \otimes_k O_X$. □

**(7.28) Corollary.** *Multiplication by an integer $n$ on $X$ induces multiplication by $n^{p+q}$ on $H^q(X, \Omega^p_X)$.*

*Proof.* Immediate from the fact that $n_X$ induces multiplication by $n$ on $T_{X,0}$, applied to both $X$ and $X^t$. □

Before we state the next corollary, let us recall that the algebraic de Rham cohomology of a smooth proper algebraic variety $X$ over a field $k$ is defined to be the hypercohomology of the de Rham complex

$$\Omega^\bullet_{X/k} = (O_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots),$$

with $O_X$ in degree zero. We have the so-called "stupid filtration" of this complex, by the subcomplexes $\sigma_{\geqslant p}\Omega^\bullet_{X/k}$ given by

$$[\sigma_{\geqslant p}\Omega^\bullet_{X/k}]^i = \begin{cases} 0 \text{ for } i < p \\ \Omega^i_{X/k} \text{ for } i \geqslant p. \end{cases}$$

This gives rise to a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{\mathrm{dR}}(X/k)$$

called the "Hodge–de Rham" spectral sequence.

If $k$ has characteristic zero then it follows from Hodge theory that this spectral sequence degenerates at the $E_1$-level, see Deligne [1], section 5. If $k$ has characteristic $p > 0$ then this is no longer true in general. For examples and further results we refer to Deligne-Illusie [1] and Oesterlé [1].

As we shall now show, for abelian varieties the degeneration of the Hodge-de Rham spectral sequence at level $E_1$ follows from (6.12) without any restrictions on the field $k$.

**(7.29) Corollary.** *Let $X$ be an abelian variety over a field $k$. Then the "Hodge-de Rham" spectral sequence of $X$ degenerates at level $E_1$.*

*Proof.* We follow the proof given by Oda [1]. We have to show that the differentials $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ are zero for all $r \geqslant 1$. By induction we may assume that this holds at all levels $< r$. (The empty assumption if $r = 1$.)

Write $E_r^*(X) = \oplus E_r^{p,q}$, graded by total degree. Cup-product makes $E_r^*(X)$ into a connected, graded-commutative $k$-algebra. By our induction assumption and the Künneth formula there is a canonical isomorphism

$$E_r^*(X \times X) \cong E_r^*(X) \otimes_k E_r^*(X)\,.$$

Write $\mu$: $E_r^*(X) \to E_r^*(X) \otimes_k E_r^*(X)$ for the map induced by the multiplication law on $X$, and write $\varepsilon$: $E_r^*(X) \to E_k^0(X) = k$ for the projection onto the degree zero component. One checks that $\mu$ and $\varepsilon$ give $E_r^*(X)$ the structure of a graded bialgebra over $k$.

Let $g = \dim(X)$. By what was shown above, $E_r^1(X) = H^1(X, O_X) \oplus H^0(X, \Omega^1_{X/k})$ has dimension $2g$. Also, $E_r^i(X) = 0$ for $i > 2g$. The Borel-Hopf structure theorem (6.12) then gives

$$E_r^*(X) \cong \wedge^* E_r^1(X)\,.$$

Since $d_r$ is compatible with the product structure (cup-product) on $E_r^*(X)$, it suffices to show that $d_r$ is zero on $E_r^1(X)$, which is just the space of primitive elements of $E_r^*(X)$. (See 6.17.) By functoriality of the Hodge–de Rham spectral sequence we have $\mu \circ d_r = (d_r \otimes d_r) \circ \mu$. Therefore, for $\xi \in E_r^1(X)$ we have $\mu(d_r(\xi)) = d_r(\xi) \otimes 1 + 1 \otimes d_r(\xi)$. This shows that $d_r(\xi)$ is again a primitive element. But $d_r(\xi) \in E_r^2(X)$ which, by (6.17), contains no non-zero primitive elements. This shows that $d_r = 0$. $\square$

**(7.30) Corollary.** *There is an exact sequence*

$$0 \longrightarrow \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/k) \longrightarrow H^1_{\mathrm{dR}}(X/k) \longrightarrow H^1(X, O_X) \longrightarrow 0\,,$$

*where* $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/k) := H^0(X, \Omega^1_{X/k}) \cong T^\vee_{X,0}$.

To close this section let us fulfil an earlier promise and give an example of a smooth projective variety with non-reduced Picard scheme. We refer to Katsura-Ueno [1] for similar examples.

**(7.31) Example.** Let $k$ be an algebraically closed field of characteristic 3. Let $E_1$ be the elliptic curve over $k$ given by the Weierstrass equation $y^2 = x^3 - x$. From (5.27) we know that $E_1$ is supersingular. Let $\sigma$ be the automorphism of $E_1$ given by $(x, y) \mapsto (x + 1, y)$. Then $\sigma$ has order 3, so that we get an action of $G := \mathbb{Z}/3\mathbb{Z}$ on $E_1$. The quotient of $E_1$ by $G$ is isomorphic to $\mathbb{P}^1_k$; in affine coordinates te quotient map is just $(x, y) \mapsto y$.

Let $E_2$ be an ordinary elliptic curve over $k$. Let $\tau$ be the translation over a point of (exact) order 3 on $E_2$. Then $(\sigma, \tau)$ is an automorphism of order 3 of the abelian surface $X := E_1 \times E_2$; this gives a strictly free action of $G := \mathbb{Z}/3\mathbb{Z}$ on $X$, and we can form the quotient $\pi$: $X \to Y := G\backslash X$. By (??) $\pi$ is an étale morphism, so $Y$ is again a non-singular algebraic surface. We have a natural morphism $Y \to (G\backslash E_1) \cong \mathbb{P}^1$; this exhibits $Y$ as an elliptic surface over $\mathbb{P}^1$. In fact, for all $P \in \mathbb{P}^1(k)$ with $P \neq \infty$ the fibre $Y_P$ above $P$ is isomorphic to $E_2$.

We compute $h^1(Y, O_Y)$ using Hirzebruch-Riemann-Roch and Chern numbers for algebraic surfaces. (A reference is ??.) The Euler number $c_2$ of $Y$ is a multiple of the Euler number of $X$, and this is 0. By the Hirzebruch-Riemann-Roch formula we have

$$1 - h^1(Y, O_Y) + h^2(Y, O_Y) = (c_1^2 + c_2)/12 = 0\,,$$

since $c_1^2 = 0$ for every elliptic surface. By Serre duality, $h^2(Y, O_Y) = h^0(Y, \Omega^2_{Y/k})$. Now we use that $H^0(Y, \Omega^2_{Y/k})$ is isomorphic to the space of $G$-invariants in $H^0(X, \Omega^2_{X/k})$. If $\omega_i$ is a basis for $H^0(E_i, \Omega^1_{E_i/k})$ then $\omega_1 \wedge \omega_2$ is a basis for $H^0(X, \Omega^2_{X/k})$. But $\omega_1$ is a multiple of $dy$, which is invariant under $\sigma$, and $\omega_2$ is translation invariant, in particular invariant under $\tau$. In sum, we find that $h^2(Y, O_Y) = 1$ and $h^1(Y, O_Y) = 2$.

On the other hand, $\pi\colon X \to Y$ induces a homomorphism $\pi^*\colon \mathrm{Pic}^0_{Y/k} \to X^t = \mathrm{Pic}^0_{X/k}$. The same arguments as in the proof of Theorem (7.5) show that $\mathrm{Ker}(\pi^*) \cong \mu_3$. On the other hand, $\pi^*$ factors via the subscheme of $G$-invariants in $X^t$. (See Exercise ?? for the existence of such a subscheme of $G$-invariants.) The point here is that we are describing line bundles on $Y$ as coming from line bundles $L$ on $X$ together with an action of $G$. But such an action is given by an isomorphism $\rho^*L \xrightarrow{\sim} \mathrm{pr}_X^*L$ of line bundles on $G \times_k X$. The existence of such an isomorphism says precisely that $L$ corresponds to a $G$-invariant point of $X^t$.

By Exercises ?? and ??, $X^t \xrightarrow{\sim} X$. The induced action of $G$ on $X^t$ is given by the automorphism $(\sigma, \mathrm{id})$. (Cf. Exercise ??) Therefore, the subscheme of $G$-invariants in $X^t$ is $E_1^{\langle\sigma\rangle} \times E_2$. The only geometric point of $E_1$ fixed under $\sigma$ is the origin. A computation in local coordinates reveals that $E_1^{\langle\sigma\rangle}$ is in fact the Frobenius kernel $E_1[F] \subset E_1$ which can be shown to be isomorphic to $\alpha_3$. In any case, we find that $\mathrm{Pic}^0_{Y/k}$ is 1-dimensional, whereas we have shown its tangent space at the identity, isomorphic to $H^1(Y, O_Y)$, to be 2-dimensional. Hence $\mathrm{Pic}^0_{Y/k}$ is non-reduced.

§ 5. *The duality between Frobenius and Verschiebung.*

**(7.32)** Let $S$ be a scheme of characteristic $p$. Recall that for any $S$-scheme $a_X\colon X \to S$ we have a commutative diagram with Cartesian square

$$X$$

$$\begin{array}{ccc} X^{(p/S)} & \xrightarrow{W_{X/S}} & X \\ \big\downarrow a_X^{(p)} & & \big\downarrow a_X \\ S & \xrightarrow{\mathrm{Frob}_S} & S \end{array}$$

If there is no risk of confusion we simply write $X^{(p)}$ for $X^{(p/S)}$. Note that if $a_T\colon T \to S$ is an $S$-scheme then we have $a_T \circ \mathrm{Frob}_T = \mathrm{Frob}_S \circ a_T$ and this gives a natural identification $(X_T)^{(p/T)} = (X^{(p/S)})_T$. We denote this scheme simply by $X_T^{(p)}$.

Let us write $T_{(p)}$ for the scheme $T$ viewed as an $S$-scheme via the morphism $a_{T_{(p)}} := \mathrm{Frob}_S \circ a_T = a_T \circ \mathrm{Frob}_T\colon T \to S$. The morphism $\mathrm{Frob}_T\colon T \to T$ is not, in general, a morphism of $S$-schemes, but if we view it as a morphism $T_{(p)} \to T$ then it is a morphism over $S$. To avoid confusion, let us write $\mathrm{Fr}_T\colon T_{(p)} \to T$ for the morphism of $S$-schemes given by $\mathrm{Frob}_T$.

Let $Y$ be an $S$-scheme. Recall that we write $Y(T)$ for the $T$-valued points of $Y$. It is understood here (though not expressed in the notation) that all schemes and morphisms of schemes are over a fixed base scheme $S$; so $Y(T)$ is the set of morphisms $T \to Y$ over $S$. There is a natural bijection

$$w_{Y,T}\colon Y^{(p)}(T) \xrightarrow{\sim} Y(T_{(p)}),$$

sending a point $\eta\colon T \to Y^{(p)}$ to $W_{Y/S} \circ \eta$, which is a $T_{(p)}$-valued point of $Y$. The composition

$$w_{Y,T} \circ F_{Y/S}(T)\colon Y(T) \to Y(T_{(p)})$$

is the map that sends $y \in Y(T)$ to $y \circ \mathrm{Fr}_T\colon T_{(p)} \to Y$, which is the same as $y \circ \mathrm{Frob}_T\colon T \to Y$ viewed as a morphism $T_{(p)} \to Y$.

**(7.33)** Consider an abelian variety $X$ over a field $k$ of characteristic $p$. Take $S := \mathrm{Spec}(k)$. If $T$ is any $S$-scheme then $X \times_S T_{(p)}$ is the same as $X^{(p)} \times_S T$, and we find that

$$\begin{aligned}\mathrm{Pic}^{(p)}_{X/k}(T) \xrightarrow[w_{\mathrm{Pic}_{X/k},T}]{\sim} \mathrm{Pic}_{X/k}(T_{(p)}) &= \left\{\begin{matrix}\text{isomorphism classes of rigidified}\\ \text{line bundles } (L,\alpha) \text{ on } X \times_S T_{(p)}\end{matrix}\right\}\\ &= \left\{\begin{matrix}\text{isomorphism classes of rigidified}\\ \text{line bundles } (L,\alpha) \text{ on } X^{(p)} \times_S T\end{matrix}\right\} = \mathrm{Pic}_{X^{(p)}/S}(T)\,.\end{aligned}$$

In this way we obtain an isomorphism $\mathrm{Pic}^{(p)}_{X/S} \xrightarrow{\sim} \mathrm{Pic}_{X^{(p)}/S}$, which we take as an identification. Applying (7.32) with $Y = \mathrm{Pic}_{X/k}$ we find that the relative Frobenius of $\mathrm{Pic}_{X/k}$ over $k$ is the homomorphism that sends a point $y \in \mathrm{Pic}_{X/k}(T)$ to $y \circ \mathrm{Frob}_T$, viewed as a morphism $T_{(p)} \to \mathrm{Pic}_{X/k}$. Because the diagram

$$\begin{tikzcd} X_T^{(p)} \arrow[r, "W_{X_T/T}"] \arrow[d, "a_X^{(p)}"'] & X_T \arrow[d, "a_X"] \\ T \arrow[r, "\mathrm{Frob}_T"] & T \end{tikzcd}$$

is Cartesian this just means that $F_{\mathrm{Pic}/k}\colon \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X^{(p)}/k}$ sends the class of a rigidified line bundle $(L,\alpha)$ on $X_T$ to the class of $\bigl(L^{(p)},\alpha^{(p)}\bigr)$ on $X_T^{(p)}$, where $L^{(p)} := W^*_{X_T/T}L$, and where $\alpha^{(p)}\colon O_T \xrightarrow{\sim} e^*L^{(p)} = \mathrm{Frob}_T^*(e^*L)$ is the rigidification of $L^{(p)}$ along the zero section obtained by pulling back $\alpha$ via $\mathrm{Frob}_T$.

**(7.34) Proposition.** *Let $X$ be an abelian variety over a field $k$ of characteristic $p$. We identify $(X^t)^{(p)} = (X^{(p)})^t$ as in (7.33), and we denote this abelian variety by $X^{t,(p)}$. Then we have the identities*

$$F^t_{X/k} = V_{X^t/k}\colon X^{t,(p)} \to X^t \qquad \text{and} \qquad V^t_{X/k} = F_{X^t/k}\colon X^t \to X^{t,(p)}\,.$$

*Proof.* It suffices to prove that $F^t_{X/k} \circ F_{X^t/k}\colon X^t \to X^t$ equals $[p]_{X^t}$, because if this holds then together with Proposition (5.20) and the fact that $F_{X^t/k}$ is an isogeny it follows that $F^t_{X/k} = V_{X^t/k}$. The other assertion follows by duality.

Let $T$ be a $k$-scheme. Consider a rigidified line bundle $(L,\alpha)$ on $X_T$ that gives a point of $X^t(T)$. As explained in (7.33) $F_{X^t/k}$ sends $(L,\alpha)$ to $\bigl(L^{(p)},\alpha^{(p)}\bigr)$ with $L^{(p)} = W^*_{X_T/T}L$. Because $W_{X_T/T} \circ F_{X_T/T} = \mathrm{Frob}_{X_T}$, pull-back via $F_{X_T/T}$ gives the line bundle $\mathrm{Frob}^*_{X_T}L$ on $X_T$. But if $Y$ is any scheme of characteristic $p$ and $M$ is a line bundle on $Y$ then $\mathrm{Frob}_Y^*(M) \cong M^p$; this follows for instance by taking a trivialization of $M$ and remarking that $\mathrm{Frob}_Y$ raises all transition functions to the power $p$. The rigidification we have on $F^*_{X_T/T}W^*_{X_T/T}L = \mathrm{Frob}^*_{X_T}L = L^p$ is the isomorphism

$$O_T = \mathrm{Frob}_T^* O_T \xrightarrow{\sim} e^*_{X_T}F^*_{X_T/T}W^*_{X_T/T}L = e^*_{X_T^{(p)}}W^*_{X_T/T}L = \mathrm{Frob}_T^* e^*_{X_T}L = (e^*_{X_T}L)^p$$

that is obtained from $\alpha$ by pulling back via $\mathrm{Frob}_T$, which just means it is $\alpha^p$. In sum, $F^t_{X/k} \circ F_{X^t/k}$ sends $(L,\alpha)$ to $(L^p,\alpha^p)$, which is what we wanted to prove. $\square$

**Exercises.**

**(7.1)** Let $X$ be an abelian variety. Let $m_X\colon X \times X \to X$ be the group law, and let $\Delta_X\colon X \to X \times X$ be the diagonal morphism. Show that $(m_X)^t = \Delta_{X^t}\colon X^t \times X^t \to X^t$, and that $(\Delta_X)^t = m_{X^t}\colon X^t \times X^t \to X^t$.

**(7.2)** Let $L$ be a line bundle on an abelian variety $X$.

(i) Show that, for $n \in \mathbb{Z}$,

$$n^*L \cong O_X \quad \Longleftrightarrow \quad L^n \cong O_X\,.$$

(ii) Show that, for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$,

$$n^*L \cong L \quad \Longleftrightarrow \quad L^{n-1} \cong O_X\,.$$

**(7.3)** Let $X$ be an abelian variety over an algebraically closed field $k$. Show that every line bundle $L$ on $X$ can be written as $L = L_1 \otimes L_2$, where $L_1$ is symmetric and $[L_2] \in \mathrm{Pic}^0_{X/k}$. [*Hint:* By (7.23), the class of the line bundle $(-1)^*L \otimes L^{-1}$ is in $\mathrm{Pic}^0_{X/k}$. As $\mathrm{Pic}^0$ is an abelian variety and $k = \overline{k}$, there exists a line bundle $M$ on $X$ with $[M] \in \mathrm{Pic}^0$ and $M^2 \cong (-1)^*L \otimes L^{-1}$. Now show that $L \otimes M$ is symmetric.]

**(7.4)** Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$. For $m$, $n \in \mathbb{Z}$, consider the endomorphism $(m, n)$ of $X \times X^t$. Show that $(m, n)^*\mathscr{P} \cong \mathscr{P}^{mn}$.

**(7.5)** Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$. Show that the associated homomorphism $\varphi_{\mathscr{P}}\colon X \times X^t \to X^t \times X^{tt}$ is the homomorphism given by $\varphi_{\mathscr{P}}(x, \xi) = \bigl(\xi, \kappa_X(x)\bigr)$. [*Hint:* Compute the restrictions of $t_(x, \xi)^*\mathscr{P} \otimes \mathscr{P}^{-1}$ to $X \times \{0\}$ and $\{0\} \times X^t$.]

**(7.6)** If $\tau$ is a translation on an abelian variety, then what is the induced automorphism $\tau^t$ of the dual abelian variety?

**(7.7)** Let $X$ be an abelian variety over a field $k$. Let $i\colon Y \hookrightarrow X$ be an abelian subvariety. Write $q\colon X \to Z := X/Y$ for the fppf quotient morphism, which exists by Thm. (4.38). Note that $Z$ is an abelian variety; see Example (4.40).

(i) Show that for any $k$-scheme $T$ we have $q_*(O_{X_T}) = O_{Z_T}$.

(ii) Prove that $q^t\colon Z^t \to X^t$ is injective and gives an isomorphism between $Z^t$ and $\mathrm{Ker}(i^t\colon X^t \to Y^t)^0_{\mathrm{red}}$.

**(7.8)** Let $L$ be a line bundle on an abelian variety $X$. For a symmetric $m \times m$-matrix $S$ with integer coefficients $s_{ij}$ we define a line bundle $L^{\boxtimes S}$ on $X^m$ by

$$L^{\boxtimes S} := \Bigl(\bigotimes_{i=1}^{m} p_i^* L^{s_{ii}}\Bigr) \otimes \Bigl(\bigotimes_{1 \leqslant i < j \leqslant m} p_{ij}^* \Lambda(L)^{s_{ij}}\Bigr)\,.$$

If $\alpha = \bigl(a_{ij}\bigr)$ is an integer valued matrix of size $m \times n$ we define a homomorphism of abelian varieties $[\alpha]_X\colon X^n \to X^m$ by $\alpha(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$ with $y_i = \sum_{j=1}^{n} a_{ij} x_j$.

(i) Prove that $[\alpha]_X^*\bigl(L^{\boxtimes S}\bigr)$ is algebraically equivalent to $L^{\boxtimes({}^{\mathrm{t}}\alpha S \alpha)}$.

(ii) Assume that $L$ is a symmetric line bundle. Prove that $[\alpha]_X^*\bigl(L^{\boxtimes S}\bigr) \cong L^{\boxtimes({}^{\mathrm{t}}\alpha S \alpha)}$.

**Notes.** (nog aanvullen)