Chapter VII. Duality.

§ 1. Formation of quotients and the descent of coherent sheaves.

(7.1) Definition. Let $S$ be a base scheme. Let $\rho: G \times_S X \to X$ be an action (from the left) of an $S$-group scheme $G$ on an $S$-scheme $X$. Let $F$ be a coherent sheaf of $O_X$-modules. Then an action of $G$ on $F$, compatible with the action $\rho$, is an isomorphism $\lambda: \text{pr}_2^*F \sim \rho^*F$ of sheaves on $G \times_S X$, such that on $G \times_S G \times_S X$ we have a commutative diagram

\[
\begin{array}{ccc}
\text{pr}_2^*F & \xrightarrow{\text{pr}_2^*(\lambda)} & \text{pr}_2^*\rho^*F \\
(m \times \text{id}_X)^*(\lambda) & \downarrow & \downarrow (\text{id}_G \times \rho)^*(\lambda) \\
(m \times \text{id}_X)^*\rho^*F & = & (\text{id}_G \times \rho)^*\rho^*F.
\end{array}
\]

Here is a more concrete explanation of what this means. If $T$ is an $S$-scheme and $g \in G(T)$, write $\rho_g: X_T \to X_T$ for the action of the element $g$. Then to have an action of $G$ on $F$ that is compatible with $\rho$ means that for every $g \in G(T)$ we have an isomorphism of sheaves $\lambda_g: F_T \sim \rho_g^*F_T$ such that $\lambda_{gh} = \rho_h^*(\lambda_g) \circ \lambda_h$ for all $g, h \in G(T)$.

If $F$ is a locally free $O_X$-module we can take a more geometric point of view. First recall that a locally free $O_X$-module is “the same” as a geometric vector bundle over $X$. Namely, $V := \mathcal{V}(F^\vee)$ is a geometric vector bundle over $X$, and $F$ is the sheaf of sections of the structure morphism $\pi: V \to X$. Then a $\rho$-compatible $G$-action on $F$ corresponds to an action $\tilde{\rho}: G \times_S V \to V$ such that (i) the structure morphism $\pi: V \to X$ is $G$-equivariant, and (ii) the action $\tilde{\rho}$ is “fibrewise linear”, meaning that for every $S$-scheme $T$ and every $g \in G(T)$, $x \in X(T)$, the isomorphism $\tilde{\rho}(g): V_x \to V_{gx}$ is $O_T$-linear. We refer to such an action $\tilde{\rho}$ as a lifting of $\rho$.

With this notion of a $G$-action on a sheaf, we can formulate a useful result on the descent of modules.

(7.2) Proposition. Let $\rho: G \times_S X \to X$ be an action of an $S$-group scheme $G$ on an $S$-scheme $X$. Suppose there exists an fppf quotient $p: X \to Y$ of $X$ by $G$. If $F$ is a coherent sheaf of $O_Y$-modules then the canonical isomorphism $\lambda_{\text{can}}: \text{pr}_2^*(p^*F) \sim \rho^*(p^*F)$ defines a $\rho$-compatible $G$-action on $p^*F$. The functor $F \mapsto (p^*F, \lambda_{\text{can}})$ gives an equivalence between the category of coherent $O_Y$-modules and the category of coherent $O_X$-modules with ($\rho$-compatible) $G$-action. This restricts to an equivalence between the category of finite locally free $O_Y$-modules and the category of finite locally free $O_X$-modules with $G$-action.

This proposition should be seen as a statement in (faithfully flat) descent theory; it follows for instance from the results of SGA 1, Exp. VIII, § 1. (See also [BLR], § 6.1, Thm. 4.) Given such results in descent theory, the only point here is that a $\rho$-compatible $G$-action on a coherent $O_X$-module is the same as a descent datum on this module. (Recall that we have an isomorphism $(\rho, \text{pr}_2): G \times_S X \sim \to X \times_Y X$.) The assertion that finite locally free $O_X$-modules with $G$-action give rise to finite locally free $O_Y$-modules follows from EGA IV, Prop. 2.5.2.

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(7.3) Example. We consider the situation of the proposition. The geometric vector bundle corresponding to the structure sheaf $O_X$ is just the affine line $\mathbb{A}^1_X$ over $X$.

On $O_X$ (geometrically: on $\mathbb{A}^1_X$) we have a “trivial” action $\tilde{\rho}_{\text{triv}}$, given by

$$\tilde{\rho}_{\text{triv}} = \rho \times \text{id}_{\mathbb{A}^1_X} : G \times S \mathbb{A}^1_X = G \times S X \times S \mathbb{A}^1_S \longrightarrow X \times S \mathbb{A}^1_S = \mathbb{A}^1_X.$$ 

The $O_Y$-module corresponding to $(O_X, \tilde{\rho}_{\text{triv}})$ is just $O_Y$ itself.

Let $\tilde{\rho}$ be some other lifting of $\rho$ to a $G$-action on $\mathbb{A}^1_X$. Let $T$ be an $S$-scheme and $g \in G(T)$. The automorphism $\tilde{\rho}(g) \cdot \tilde{\rho}_{\text{triv}}^{-1}(g)$ of $\mathbb{A}^1_X \times S T = \mathbb{A}^1_X$ is given on every fibre $\mathbb{A}^1_Z$ by some (invertible) scalar multiplication. This means that $\tilde{\rho}(g) \cdot \tilde{\rho}_{\text{triv}}^{-1}(g)$ is given by an element $\nu(g) \in \Gamma(X_T, O_{X_T})$. We find that an action $\tilde{\rho}$ gives rise to a morphism of functors $\nu : G \rightarrow \text{Res}_{X/S} \mathbb{G}_{m,X}$ on the category $\mathbf{Sch}_{/S}$. The condition that $\tilde{\rho}$ is a group action means that $\nu$ satisfies a cocycle condition $\nu(g_1 g_2)(x) = \nu(g_1)(g_2 x) \cdot \nu(g_2)(x)$, where we simply write $g_2 x$ for $\rho(g_2)(x)$. Conversely, given a morphism $\nu : G \rightarrow \text{Res}_{X/S} \mathbb{G}_{m,X}$ that satisfies this condition, one finds back a $G$-action $\tilde{\rho}$ by $\tilde{\rho}(g) = \nu(g) \cdot \tilde{\rho}_{\text{triv}}(g)$.

Now suppose that the structure morphism $f : X \rightarrow S$ satisfies $f_*(O_X) = O_T$ for all $S$-schemes $T$. This holds for instance if $X$ is a proper variety over a field. Then $\text{Res}_{X/S} \mathbb{G}_{m,X} \cong \mathbb{G}_{m,S}$ as functors on $\mathbf{Sch}_{/S}$. In particular, any morphism $\nu : G \rightarrow \text{Res}_{X/S} \mathbb{G}_{m,X}$ is $G$-invariant, in the sense that for all $g_1, g_2 \in G(T)$ and $x \in X(T)$ we have $\nu(g_1)(g_2 x) = \nu(g_1)(x)$. Hence the cocycle condition in this case just says that $\nu$ is a homomorphism. So the conclusion is that the liftings $\tilde{\rho}$ of $\rho$ to a $G$-action on $\mathbb{A}^1_X$ are in bijective correspondence with $\text{Hom}_{\mathbf{Grpg}_{/S}}(G, \mathbb{G}_m)$. In case $G$ is a commutative, finite locally free group scheme this is just the Cartier dual $G^D(S)$.

Via Proposition (7.2), we can use this to obtain a description of the line bundles $L$ on $Y$ such that $p^* L \cong O_X$. The result is as follows.

(7.4) Proposition. Let $G$ be a commutative, finite locally free group scheme. Let $\rho : G \times S X \rightarrow X$ be a free action of $G$ on an $S$-scheme $X$. Let $p : X \rightarrow Y$ be the quotient of $X$ by $G$. Suppose that $f_*(O_{X_T}) = O_T$ for all $S$-schemes $T$. Then for any $S$-scheme $T$ there is a canonical isomorphism of groups

$$\delta_T : \left( \text{isomorphism classes of line bundles } L \text{ on } Y_T \text{ with } p^* L \cong O_{X_T} \right) \sim \sim \rightarrow G^D(T),$$

and this isomorphism is compatible with base change $T' \rightarrow T$.

Proof. To define $\delta_T$ for arbitrary $S$-schemes $T$ we may replace $S$ by $T$ and $p : X \rightarrow Y$ by $p_T : X_T \rightarrow Y_T$. Note that by Theorem (4.16) and what was explained in Example (4.29), $p_T$ is again the quotient morphism of $X_T$ by the action of $G_T$, and of course also the assumption that $f_*(O_{X_T}) = O_T$ for all $S$-schemes $T$ is preserved under base change. Hence it suffices to define the isomorphism $\delta_S$.

Let $L$ be a line bundle on $Y$ with $p^* L \cong O_X$. Via the choice of an isomorphism $\alpha : p^* L \sim \sim \rightarrow O_X$ (or, more geometrically, the isomorphism $\alpha : p^* \mathcal{V}(L^{-1}) \sim \sim \rightarrow \mathbb{A}^1_X$ over $X$) the canonical $G$-action on $p^* L$ translates into a $G$-action $\tilde{\rho}$ on $\mathbb{A}^1_X$, and as explained above this gives us a character $\nu : G \rightarrow \mathbb{G}_{m,S}$. We claim that this character is independent of the choice of $\alpha$. In general, any other isomorphism $p^* L \sim \sim \rightarrow O_X$ is of the form $\alpha' = \gamma \circ \alpha$ for some $\gamma \in \Gamma(X, O_X^*)$. Write $\tilde{\rho}$ and $\tilde{\rho}'$ for the $G$-actions on $\mathbb{A}^1_X$ obtained using $\alpha$ and $\alpha'$, respectively, and let $\nu$ and $\nu'$ be the associated characters. If $g \in G(T)$ and $y$ is a $T$-valued point of $p^* \mathcal{V}(L^{-1})$ lying over $x \in X(T)$ then we have the relations

$$\tilde{\rho}_{\text{triv}}(g, \alpha'(y)) = \gamma(x) \cdot \tilde{\rho}_{\text{triv}}(g, \alpha(y)) \quad \text{and} \quad \tilde{\rho}'(g, \alpha'(y)) = \gamma(gx) \cdot \tilde{\rho}(g, \alpha(y)),$$
where $\gamma(x)$ is the image of $\gamma$ under the homomorphism $\Gamma(X, O_X) \to \Gamma(T, O_T^*)$ induced by $x: T \to X$, and similarly for $\gamma(gx)$. (Note that elements such as $\tilde{\rho}(g, \alpha(y))$ are $T$-valued points of $\mathbb{A}_X^*$ lying over the point $gx \in X(T)$, and on such elements we have the “fibrewise” multiplication by functions on $T$.) But now our assumption that $f_* (O_X) = O_S$ implies that $\gamma$ is the pull-back of an element in $\Gamma(S, O_S^*)$, so $\gamma(x) = \gamma(gx)$. Hence $\nu = \nu'$, as claimed.

Now we can simply apply the conclusion from (7.3), and define $\delta_S$ as the map that sends the isomorphism class of $L$ to the character $\nu: G \to \mathbb{G}_{m, S}$ given on points by $\nu(g) = \tilde{\rho}(g) \cdot \tilde{\rho}_{triv}(g)^{-1}$. By Proposition (7.2), together with what was explained in Example (7.3), the map $\delta_S$ thus obtained is indeed an isomorphism.

Finally we note that the maps $\delta_T$ are indeed compatible with base change, as is immediate from the construction. 

\[\Box\]

§ 2. Two duality theorems.

(7.5) Theorem. Let $f: X \to Y$ be an isogeny of abelian varieties. Then $f^t: Y^t \to X^t$ is again an isogeny and there is a canonical isomorphism of group schemes

$$\text{Ker}(f)^D \sim \text{Ker}(f^t).$$

Proof. If $T$ is a $k$-scheme, any class in $\text{Ker}(f^t)(T)$ is uniquely represented by a line bundle $L$ on $Y_T$ such that $f^*L \cong O_{X_T}$. Indeed, if $L'$ represents a class in $\text{Ker}(f^t)(T)$ then there is a line bundle $M$ on $T$ such that $f^*L' \cong \text{pr}_T^*M$. Then $L := L' \otimes \text{pr}_T^*M^{-1}$ represents the same class as $L'$ and satisfies $f^*L \cong O_{X_T}$. Conversely, if $L_1$ and $L_2$ represent the same class then they differ by a line bundle of the form $\text{pr}_T^*M$; hence $f^*L_1 \cong f^*L_2$ implies $L_1 \cong L_2$.

Applying Proposition (7.4) we obtain the desired isomorphism $\text{Ker}(f^t) \sim \text{Ker}(f)^D$. In particular this shows that $f^t$ has a finite kernel and therefore is again an isogeny. 

\[\Box\]

(7.6) Proposition. Let $f: X \to Y$ be a homomorphism. Let $M$ be a line bundle on $Y$ and write $L = f^*M$. Then $\varphi_L: X \to X^t$ equals the composition

$$X \xrightarrow{f} Y \xrightarrow{\varphi_M} Y^t \xrightarrow{f^t} X^t.$$ 

If $f$ is an isogeny and $M$ is non-degenerate then $L$ is non-degenerate too, and $\text{rank}(K(L)) = \deg(f)^2 \cdot \text{rank}(K(M))$.

Proof. That $\varphi_L = f^t \circ \varphi_M \circ f$ is clear from the formula $t_x^*f^*M = f^*t_{f(x)}^*M$. For the second assertion recall that a line bundle $L$ is non-degenerate precisely if $\varphi_L$ is an isogeny, in which case $\text{rank}(K(L)) = \deg(\varphi_L)$. Now use (7.5). 

\[\Box\]

(7.7) The Poincaré bundle on $X \times X^t$ comes equipped with a rigidification along $\{0\} \times X^t$. As $\mathcal{P}_{|X \times \{0\}} \cong O_X$ we can also choose a rigidification of $\mathcal{P}$ along $X \times \{0\}$. Such a rigidification is unique up to an element of $\Gamma(X, O_X^*) = k^*$. Hence there is a unique rigidification along $X \times \{0\}$ such that the two rigidifications agree at the origin $(0, 0)$.

Now we view $\mathcal{P}$ as a family of line bundles on $X^t$ parametrised by $X$. This gives a morphism

$$\kappa_X: X \to X^{tt}.$$ 

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As $\kappa_X(0) = 0$ it follows from Prop. (1.13) that $\kappa_X$ is a homomorphism.

(7.8) Lemma. Let $L$ be a line bundle on $X$. Then $\varphi_L = \varphi_L^\prime \circ \kappa_X: X \to X^t$.

Proof. Let $s: X \times X \to X \times X$ and $s: X \times X^t \to X^t \times X$ be the morphisms switching the two factors; on points: $s(x, y) = (y, x)$. We have a canonical isomorphism $s^\ast \Lambda(L) \cong \Lambda(L)$. Let $T$ be a $k$-scheme and $x \in X(T)$. Writing $[M]$ for the class of a bundle $M$ on $X \times T$ in $\text{Pic}^0_{X/k}(T)$ we have

$$\varphi_L(x) = \left( (X \times T \xrightarrow{\text{id} \times x} X \times X)^\ast \Lambda(L) \right) = \left( (X \times T \xrightarrow{\text{id} \times x} X \times X \xrightarrow{\varphi_L} X \times X)^\ast \Lambda(L) \right)$$

$$= \left( (X \times T \xrightarrow{\varphi_L \times \text{id}} X^t \times T \xrightarrow{\text{id} \times x} X^t \times X \xrightarrow{\varphi_L} X \times X)^\ast \mathcal{O} \right) = \varphi_L^\prime \circ \kappa_X(x).$$

As this holds for all $T$ and $x$ the lemma is proven. \qed

(7.9) Theorem. Let $X$ be an abelian variety over a field. Then the homomorphism $\kappa_X: X \to X^t$ is an isomorphism.

Proof. Choose an ample line bundle $L$ on $X$. The formula $\varphi_L = \varphi_L^\prime \circ \kappa_X$ shows that $\text{Ker}(\kappa_X)$ is finite; hence $\kappa_X$ is an isogeny. Furthermore,

$$\text{rank}(K(L)) = \text{deg}(\varphi_L) = \text{deg}(\varphi_L^\prime) \cdot \text{deg}(\kappa_X) = \text{rank}(K(L)^D) \cdot \text{deg}(\kappa_X),$$

using (7.5). But $\text{rank}(K(L)^D) = \text{rank}(K(L))$, so $\kappa_X$ has degree 1. \qed

(7.10) Corollary. If $L$ is a non-degenerate line bundle on $X$ then $K(L) \cong K(L)^D$.

Proof. Apply (7.5) to $\varphi_L$ and use (7.8) and (7.9). \qed

§ 3. Further properties of $\text{Pic}^0_{X/k}$.

Let $X$ be an abelian variety over a field $k$. A line bundle $L$ on $X$ gives rise to a homomorphism $\varphi_L: X \to X^t$. We are going to extend this construction to a more general situation. Namely, let $T$ be a $k$-scheme, and suppose $L$ is a line bundle on $X_T := X \times_k T$. We are going to associate to $L$ a homomorphism $\varphi_L: X_T \to X_T^t$.

As usual we write $\Lambda(L) := m^\ast L \otimes p_1^T L^{-1} \otimes p_2^T L^{-1}$ for the Mumford bundle on $X_T \times T X_T$ associated to $L$. (Note that we are working in the relative setting, viewing $T$ as the base scheme. If we rewrite $X_T \times T X_T$ as $X_k X \times_k T$ then $\Lambda(L)$ becomes $(m \times \text{id}_T)^\ast L \otimes p_1^T L^{-1} \otimes p_2^T L^{-1}$.) In order to distinguish the two factors $X_T$, let us write $X_T^{(1)} = X_T \times_T \epsilon(T)$ and $X_T^{(2)} = \epsilon(T) \times_T X_T$ where $\epsilon(T) \subset X_T$ is the image of the zero section $\epsilon: T \to X_T$. Viewing $\Lambda(L)$ as a family of line bundles on $X_T^{(1)}$ parametrized by $X_T^{(2)}$ we obtain a morphism

$$\varphi_L: X_T = X_T^{(2)} \to \text{Pic}_{X_T/T} = \text{Pic}_{X/k} \times_k T.$$

As $\varphi_L(0) = 0$ and the fibres $X_t$ are connected, $\varphi_L$ factors through $X_T = \text{Pic}^0_{X/k} \times_k T$. 

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(7.11) Lemma. (i) The morphism \( \varphi_L \) only depends on the class of \( L \) in \( \text{Pic}_{X/k}(T) \).

(ii) Let \( f: T \to S \) be a morphism of \( k \)-schemes. If \( M \) is a line bundle on \( X_S \) and \( L = (\text{id}_X \times f)^*M \) on \( X_T \), then \( \varphi_L: X_T \to X_T^1 \) is the morphism obtained from \( \varphi_M: X_S \to X_S^1 \) by pulling back via \( f \) on the basis.

(iii) The morphism \( \varphi_L: X_T \to X_T^1 \) is a homomorphism.

Part (i) of the lemma will be sharpened in (7.15) below. As a particular case of (ii), note that the fibre of \( \varphi_L \) above a point \( t \in T \) is just \( \varphi_{L_t} \), where we write \( L_t \) for the restriction of \( L \) to \( X \times \{t\} \).

Proof. (i) If \( L_1 \) and \( L_2 \) have the same class then they differ by a factor \( \text{pr}_T^*M \). But then \( \Lambda(L_1) \) and \( \Lambda(L_2) \) differ by a factor \( \pi^*M^{-1} \), where \( \pi: X_T \times_T X_T \to T \) is the structural morphism. This implies that \( \varphi_{L_1} = \varphi_{L_2} \), as claimed.

(ii) This readily follows from the definitions.

(iii) The assertion that \( \varphi_L \) is a homomorphism means that we have an equality of two morphisms

\[ \varphi_L \circ m = m \circ (\varphi_L \times \varphi_L): X_T \times_T X_T \to X_T^1. \]

For every \( t \in T \) we already know that the two morphisms agree on the fibres above \( t \). Hence the lemma is true if \( T \) is reduced. In particular, the lemma is true in the “universal” case that \( T = \text{Pic}_{X/k} \) and \( L \) is the Poincaré bundle on \( X \times_k \text{Pic}_{X/k} \). In the general case, consider the morphism \( f: T \to \text{Pic}_{X/k} \) associated to the line bundle \( L \). This morphism is characterized by the property that \( L \) and \( (\text{id} \times f)^*\mathcal{O} \) have the same class in \( \text{Pic}_{X/k}(T) \). Now apply (i) and (ii). \( \square \)

In the above we allow \( L \)—to be thought of as a family of line bundles on \( X \) parametrized by \( T \)—to be non-constant. But the abelian variety we work on is a constant one. We can go one step further by also letting the abelian varieties \( X_t \) “vary with \( t \”. This generalization will be discussed in Chapter ??; see in particular (??).

We write \( K(L) := \text{Ker}(\varphi_L) \subset X_T \). It is the maximal subscheme of \( X_T \) over which \( \Lambda(L) \) is trivial, viewing \( X_T \times_T X_T \) as a scheme over \( X_T \) via the second projection. In particular, \( \varphi_L = 0 \) if and only if \( \Lambda(L) \) is trivial over \( X_T \), meaning that \( \Lambda(L) = \text{pr}_2^*M \) for some line bundle \( M \) on \( X_T \).

Using (2.17) we can make this a little more precise.

(7.12) Lemma. Let \( T \) be a locally noetherian \( k \)-scheme. Write \( \pi: X_T \times_T X_T \to T \) for the structural morphism. For a line bundle \( L \) on \( X_T \), consider the following conditions.

(a) \( \varphi_L = 0 \).

(b) \( \Lambda(L) \cong \text{pr}_2^*M \) for some line bundle \( M \) on \( X_T \).

(c) \( \Lambda(L) \cong \pi^*N \) for some line bundle \( N \) on \( T \).

(d) \( \varphi_{L_t} = 0 \) for some \( t \in T \).

Then \( (a) \iff (b) \iff (c) \implies (d) \), and if \( T \) is connected then all four conditions are equivalent. If these equivalent conditions are satisfied then \( N \cong \pi^*L^{-1} \) and \( M = \text{pr}_2^*N \).

Proof. The implications \( (d) \iff (a) \iff (b) \iff (c) \) are clear. Let us write \( X_T \times_T X_T \) as \( X \times_k X \times_k T \). In this notation we have \( \Lambda(L) = (m \times \text{id}_T)^*L \otimes p_{13}^*L^{-1} \otimes p_{23}^*L^{-1} \) and \( \pi \) becomes the projection onto the third factor. Set \( N := \pi^*L^{-1} \). We find that

\[ \Lambda(L)|_{\{0\} \times X \times T} \cong \text{pr}_2^*N \cong \Lambda(L)|_{X \times \{0\} \times T} \]

as line bundles on \( X \times T \).
Suppose \( T \) is connected and \( \varphi_{L_t} = 0 \) for some \( t \in T \). Then
\[
\Lambda(L)|_{X \times X \times \{t\}} \cong O_{X \times X \times \{t\}}
\]
by (iii) of (2.17). By Thm. (2.5) the line bundle \( \Lambda(L) \otimes p_t^*N^{-1} \) on \( X \times X \times T \) is trivial, i.e., \( \Lambda(L) \cong \pi^*N \). This shows that (d) \( \Rightarrow \) (a) for connected \( T \). For arbitrary \( T \) we get the implication (a) \( \Rightarrow \) (c) by applying the previous to each of its connected components.

The last assertion of the lemma is obtained by restricting \( \Lambda(L) \) to \( \{0\} \times \{0\} \times T \) and to \( \{0\} \times X \times T \).

(7.13) Fact. Let \( X \) and \( Y \) be two projective varieties over a field \( k \). Then the contravariant functor
\[
\text{Hom}_{\text{Sch}}(X, Y) : (\text{Sch}/k) \to \text{Sets}
\]
given by \( T \mapsto \text{Hom}_{\text{Sch}/T}(X_T, Y_T) \) is representable by a \( k \)-scheme, locally of finite type.

This fact is a consequence of the theory of Hilbert schemes. A reference is ???. Note that in this proof the projectivity of \( X \) and \( Y \) is used in an essential way. See also Matsumura-Oort [1] for related results for non-projective varieties.

(7.14) Proposition. Let \( X \) and \( Y \) be two abelian varieties over a field \( k \). Then the functor
\[
\text{Hom}_{\text{AV}}(X, Y) : (\text{Sch}/k) \to \text{Ab}
\]
given by \( T \mapsto \text{Hom}_{\text{Sch}/T}(X_T, Y_T) \) is representable by an étale commutative \( k \)-group scheme.

Proof. Let \( H = \text{Hom}_{\text{Sch}}(X, Y) \) and \( H' = \text{Hom}_{\text{Sch}}(X \times X, Y) \). Let \( f : X_H \to Y_H \) be the universal morphism. Consider the morphism \( g : (X \times X)_H \to Y_H \) given on points by \( g(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2) \). Consider also the “trivial” morphism \( e : (X \times X)_H \to Y_H \) given on points by \( e(x_1, x_2) = e_Y \). Then \( g \) and \( e \) are \( H \)-valued points of \( H' \); in other words, they correspond to morphisms \( \psi_y, \psi_e : H \to H' \). The functor \( \text{Hom}_{\text{AV}}(X, Y) \) is represented by the subscheme of \( H \) given by the condition that \( \psi_y = \psi_e \); in other words, it is given by the cartesian diagram
\[
\begin{array}{ccc}
\text{Hom}_{\text{AV}}(X, Y) & \longrightarrow & H' \\
\downarrow & & \downarrow \Delta_{H'/k} \\
H & \xrightarrow{(\psi_y, \psi_e)} & H' \times_k H'.
\end{array}
\]

To get a group scheme structure on \( \text{Hom}_{\text{AV}}(X, Y) \) we just note that \( \text{Hom}_{\text{AV}}(X, Y) \) is naturally a group functor; now apply (3.6).

It remains to be shown that \( \text{Hom}_{\text{AV}}(X, Y) \) is an étale group scheme. We already know it is locally of finite type over \( k \), so it suffices to show that its tangent space at the origin is trivial. It suffices to prove this in the special case that \( Y = X \), for \( \text{Hom}_{\text{AV}}(X, Y) \) embeds as a closed subgroup scheme of \( \text{End}_{\text{AV}}(X \times X) := \text{Hom}_{\text{AV}}(X \times Y, X \times Y) \) by sending \( f : X \to Y \) to the endomorphism \( (x, y) \mapsto (0, f(x)) \) of \( X \times Y \).

A tangent vector of \( \text{End}_{\text{AV}}(X) \) at the point \( \text{id}_X \) is the same as a homomorphism \( \xi : X_k[\varepsilon] \to X_k[\varepsilon] \) over \( \text{Spec}(k[\varepsilon]) \) that reduces to the identity modulo \( \varepsilon \). Note that \( \xi \) is necessarily an automorphism. (It is the identity on underlying topological spaces, and it is an easy exercise to show that \( \xi \) gives an automorphism of the structure sheaf.) Hence by the results in Exercise (1.3), \( \xi \) corresponds to a global vector field \( \Xi \) on \( X \). As we know, the global vector fields on \( X \) are
precisely the translation-invariant vector fields. On the other hand, a necessary condition for $\xi$ to be an endomorphism is that it maps the identity section of $X_{k[t]}$ to itself. This just means that $\Xi(\varepsilon_X) = 0$. Hence $\Xi$ is the trivial vector field. This shows that $\text{id}_X$ has non-trivial first order deformations. \hfill $\Box$

In line with the notational conventions introduced in (1.16), we shall usually simply write $\text{Hom}(X, Y)$ for the group scheme of homomorphisms from $X$ to $Y$. If we wish to refer to the bigger scheme of arbitrary scheme morphisms from $X$ to $Y$, or if there is a risk of confusion, we shall use a subscript “$\text{Sch}$” or “$\text{AV}$” to indicate which of the two we mean.

By (i) and (ii) of Lemma (7.11), $L \mapsto \varphi_L$ gives rise to a morphism of functors $\varphi: \text{Pic}_X/k \rightarrow \text{Hom}(X, X^t)$. If $L$ and $M$ are line bundles on $X_T$ then $\Lambda(L \otimes M) \cong \Lambda(L) \otimes \Lambda(M)$ and we find that $\varphi_{L \otimes M} = \varphi_L + \varphi_M$. Summing up, we obtain a homomorphism of $k$-group schemes

$$\varphi: \text{Pic}_X/k \rightarrow \text{Hom}(X, X^t).$$

(7.15) Lemma. Let $T$ be a connected $k$-scheme. Let $L$ be a line bundle on $X_T$. Write $L_i$ for $L_{|X \times T}$ Then for any two $k$-valued points $s, t \in T(k)$ we have $\varphi_{L_s} = \varphi_{L_t}$. In particular, $\text{Pic}^0_X/k \subset \text{Ker}(\varphi)$.

Proof. By (d) $\Rightarrow$ (a) of (7.12), applied with $T = X^t$ and with $L = \mathcal{P}$ the Poincaré bundle, we find that $X^t = \text{Pic}^0_X/k \subset \text{Ker}(\varphi)$. As $\varphi$ is a homomorphism, it is constant on the connected components of $\text{Pic}_X/k$.

Let $f: T \rightarrow \text{Pic}_X/k$ be the morphism corresponding to $L$; it factors through some connected component $C \subset \text{Pic}_X/k$. Let $M := \mathcal{P}_{|X \times C}$ be the restriction of the Poincaré bundle to $X \times C$. Using (i) and (ii) of (7.11) we find that $\varphi_M: X_C \rightarrow X^t_C$ is obtained from $\varphi_M: X_C \rightarrow X^t_C$ by pulling back via $f$ on the basis. But by the above, $\varphi_{M_{f(t)}} = \varphi_{M_{f(s)}}$. \hfill $\Box$

(7.16) Lemma. Let $X$ be an abelian variety over $k$. Let $T$ be a $k$-scheme and let $L$ be a line bundle on $X_T$ such that $\varphi_L = 0$.

(i) If $Y$ is a $T$-scheme then for any two morphisms $f, g: Y \rightarrow X_T$ of schemes over $T$ we have $[(f + g)^*L] = [f^*L \otimes g^*L]$ in $\text{Pic}_{Y/T}(T)$.

(ii) For $n \in \mathbb{Z}$ we have $[n^*L] = [L^n]$ in $\text{Pic}_X/k(T)$.

Proof. If $\varphi_L = 0$ then $\Lambda(L) = \pi^*N$ for some line bundle $N$ on $T$. Pulling back via $(f, g): Y \rightarrow X_T \times_T X_T$ gives $(f + g)^*L = f^*L \otimes g^*L \otimes \pi^*N$, where $\pi: Y \rightarrow T$ is the structural morphism. But $\pi^*N$ is trivial in $\text{Pic}_{Y/T}(T)$, so we get (i). Applying this with $f = \text{id}_{X_T}$ and $g = n_{X_T}$ gives the relation $[(n + 1)^*L] = [L \otimes n^*L]$. By double induction on $n$, starting with the cases $n = 0$ and $n = 1$, we obtain (ii). \hfill $\Box$

Using that $\text{Pic}^0_X/k \subset \text{Ker}(\varphi)$ we obtain a positive answer to the questions posed in (6.20).

(7.17) Corollary. Let $X$ and $Y$ be abelian varieties over $k$. Then the map $\text{Hom}(X, Y) \rightarrow \text{Hom}(Y^t, X^t)$ given on points by $f \mapsto f^t$ is a homomorphism of $k$-group schemes. For all $n \in \mathbb{Z}$ we have $(n_X)^t = n_X^t$.

Combining this last result with (7.5) we find that $X^t[n]$ is canonically isomorphic to the Cartier dual of $X[n]$, for every $n \in \mathbb{Z}_{\geq 0}$.

(7.18) Let $X$ be an abelian variety. We call a homomorphism $f: X \rightarrow X^t$ symmetric if $f = f^t$, taking the isomorphism $\kappa_X: X \iso X^t$ of (7.9) as an identification. It follows from the previous
corollary that the functor of symmetric homomorphisms $X \to X^t$ is represented by a closed subgroup scheme

$$\text{Hom}^{\text{symm}}(X, X^t) \subset \text{Hom}(X, X^t).$$

In fact, $\text{Hom}^{\text{symm}}(X, X^t)$ is just the kernel of the endomorphism of $\text{Hom}(X, X^t)$ given by $f \mapsto f - f^t$.

By Lemma (7.8), the homomorphism $\varphi : \text{Pic}_{X/k} \to \text{Hom}(X, X^t)$ factors through the subgroup $\text{Hom}^{\text{symm}}(X, X^t)$. (Because $\text{Hom}(X, X^t)$ is étale, it suffices to know that $\varphi$ maps into $\text{Hom}^{\text{symm}}$ for points with values in a field.)

Our next goal is to show that not only $\text{Pic}_{X/k}^0 \subset \text{Ker}(\varphi)$ but that the two are in fact equal.

First we prove a lemma about the cohomology of line bundles $L$ with $\varphi_L = 0$. Note that we are here again working over a field; this lemma has no straightforward generalization to the relative setting.

(7.19) Lemma. Let $L$ be a line bundle on $X$ with $\varphi_L = 0$. If $L \not\cong O_X$ then $H^i(X, L) = 0$ for all $i$.

Proof. First we treat the group $H^0(X, L)$. If there is a non-trivial section $s$ then $(-1)^*s$ is a non-trivial section of $(-1)^*L \cong L^{-1}$; so both $L$ and $L^{-1}$ have a non-trivial section, and this implies that $L$ is trivial. Since we have assumed this is not the case, $H^0(X, L) = \{0\}$.

Let now $i \geq 1$ be the smallest positive integer such that $H^i(X, L) \neq 0$. Consider the composition

$$X \to X \times X \xrightarrow{m} X,$$

given by $x \mapsto (x, 0) \mapsto x$.

On cohomology this induces the maps

$$H^i(X, L) \to H^i(X \times X, m^*L) \to H^i(X, L),$$

the composition of which is the identity. But since $m^*L \cong p_1^*L \oplus p_2^*L$, the Künneth formula gives

$$H^i(X \times X, m^*L) \cong H^i(X \times X, p_1^*L \otimes p_2^*L) \cong \sum_{a+b=i} H^a(X, L) \otimes H^b(X, L).$$

Since $H^0(X, L) = \{0\}$ we may consider only those terms in the RHS where $a \geq 1$ and $b \geq 1$. But then $a < i$ which by our choice of $i$ implies that $H^a(X, L) = 0$. This shows that the identity map on $H^i(X, L)$ factors via zero. \hfill \Box

In the proof of the next proposition we need some facts about cohomology and base change. Here is what we need.

(7.20) Fact. Let $f : X \to Y$ be a proper morphism of noetherian schemes, with $Y$ reduced and connected. Let $F$ be a coherent sheaf of $O_X$-modules on $X$.

(i) If $y \mapsto \dim_{k(y)} H^q(X_y, F_y)$ is a constant function on $Y$ then $R^q f_*(F)$ is a locally free sheaf on $Y$, and for all $y \in Y$ the natural map $R^q f_*(F) \otimes_{O_Y} k(y) \to H^q(X_y, F_y)$ is an isomorphism.

(ii) If $R^q f_*(F) = 0$ for all $q \geq q_0$ then $H^q(X_y, F_y) = 0$ for all $y \in Y$ and $q \geq q_0$.

A proof of this result can be found in [MAV], § 5.

(7.21) Proposition. Let $X$ be an abelian variety over an algebraically closed field $k$. Let $L$ be an ample line bundle on $X$ and $M$ a line bundle with $\varphi_M = 0$. Then there exists a point $x \in X(k)$ with $M \cong t_x^*L \otimes L^{-1}$. 

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Proof. We follow Mumford’s beautiful proof. The idea is to look at the cohomology on $X \times X$ of the line bundle

$$K := \Lambda(L) \otimes p_2^*M^{-1}.$$  

The projections $p_1, p_2: X \times X \to X$ give rise to two Leray spectral sequences

$$E_2^{p,q} = H^p(X, R^q p_1_* (K)) \Rightarrow H^{p+q}(X \times X, K)$$  

and

$$E_2^{p,q} = H^p(X, R^q p_2_* (K)) \Rightarrow H^{p+q}(X \times X, K).$$

The restrictions of $K$ to the horizontal and vertical fibres are given by

$$K_{\{x\} \times X} \cong t_x^*L \otimes L^{-1} \otimes M^{-1},$$  

$$K_{[X \times \{x\}] \cong t_x^*L \otimes L^{-1}.$$  

Assume that there is no $x \in X(k)$ such that $t_x^*L \otimes L^{-1} \cong M$. It then follows that $K_{\{x\} \times X}$ is a non-trivial bundle in Ker$(\varphi)$ for every $x$. (Note that $[t_x^*L \otimes L^{-1}] = \varphi(L)(x) \in \text{Pic}^0_{X/k} \subset \text{Ker}(\varphi)$.) By Lemma (7.19) and (7.20) this gives $R^q p_1_* (K) = (0)$ for all $q$, and from the first spectral sequence we find that $H^n(X \times X, K) = 0$ for all $n$.

Now use the second spectral sequence. For $x \notin K(L)$ the bundle $t_x^*L \otimes L^{-1}$ is a non-trivial bundle in Ker$(\varphi)$. Again by Lemma (7.15) we find that supp$(R^q p_2_* K) \subset K(L)$. Since $K(L)$ is a finite subscheme of $X$ (the bundle $L$ being ample) we find

$$E_2^{p,q} = \left\{ \begin{array}{ll} \bigoplus_{x \in K(L)} R^q p_2_* (K)_x & \text{if } p = 0; \\ 0 & \text{otherwise.} \end{array} \right.$$  

As we only have non-zero terms for $p = 0$, the spectral sequence degenerates at level $E_2$. This gives $H^n(X \times X, K) = \bigoplus_{x \in K(L)} R^n p_2_* (K)_x$.

Comparing the two answers for $H^n(X \times X, K)$ we find that $R^n p_2_* (K) = (0)$ for all $n$. By (7.20) this implies that $H^n(X, K_{[X \times \{x\}]}) = 0$ for all $x$. But $K_{[X \times \{0\}] is the trivial bundle, so taking $n = 0$ and $x = 0$ gives a contradiction.  

(7.22) Corollary. Let $X$ be an abelian variety over a field $k$. Then $\text{Pic}^0_{X/k} = \text{Ker}(\varphi: \text{Pic}^0_{X/k} \to \text{Hom}(X, X^*))$.  

Proof. We already know that Ker$(\varphi)$ is a subgroup scheme of $\text{Pic}^0_{X/k}$ that contains $\text{Pic}^0_{X/k}$. Hence Ker$(\varphi)$ is the union of a number of connected components of $\text{Pic}^0_{X/k}$. By the proposition, every $k$-valued point of Ker$(\varphi)$ lies in Pic$^0$. The claim follows.  

(7.23) Corollary. Let $X$ be an abelian variety over a field $k$. Let $L$ be a line bundle on $X$.

(i) If $[L^n] \in \text{Pic}^0_{X/k}$ for some $n \neq 0$ then $[L] \in \text{Pic}^0_{X/k}$. In particular, if $L$ has finite order, i.e., $L^n \cong O_X$ for some $n \in \mathbb{Z}_{\geq 1}$, then $[L] \in \text{Pic}^0_{X/k}$.

(ii) We have $[L \otimes (-1)^s L^{-1}] \in \text{Pic}^0_{X/k}$.

(iii) We have

$$[L] \in \text{Pic}^0_{X/k} \iff n^*L \cong L^n \quad \text{for all } n \in \mathbb{Z}$$

$$\iff n^*L \cong L^n \quad \text{for some } n \in \mathbb{Z} \setminus \{0, 1\}.$$
Proof. (i) Since \( \varphi \) is a homomorphism we have \( \varphi_{L^n}(x) = n \cdot \varphi_L(x) = \varphi_L(n \cdot x) \). Hence if \( [L^n] \in \text{Pic}^0(X) \) then \( \varphi_L \) is trivial on all points in the image of \( n_X \). But \( n_X \) is surjective, so \( \varphi_L \) is trivial.

(ii) Direct computation shows that \( \varphi_{L^{-1}}(x) = -\varphi_L(x) \) for all \( L \) and \( x \). Since also \( \varphi_{L^{-1}}(x) = -\varphi_L(x) \), we find that \( [L \otimes (-1)^*L^{-1}] \in \text{Ker}(\varphi) \).

(iii) The first implication \( \Rightarrow \) was proven in (7.16) above; the second is trivial. Suppose that \( n^*L \cong L^n \) for some \( n \notin \{0,1\} \). Since \( n^*L \cong L^n \otimes [L \otimes (-1)^*L]^{(n^*-n)/2} \) it follows that \( L \otimes (-1)^*L \) has finite order, hence its class lies in \( \text{Pic}^0_{X/k} \). By (ii) we also have \( [L \otimes (-1)^*L^{-1}] \in \text{Pic}^0_{X/k} \).

Hence \( [L^2] \in \text{Pic}^0_{X/k} \) and by (i) then also \( [L] \in \text{Pic}^0_{X/k} \).

\((7.24)\) In (3.29) we have associated to any group scheme \( G \) locally of finite type over a field \( k \) an étale group scheme of connected components, denoted by \( \varpi_0(G) \). We now apply this with \( G = \text{Pic}^0_{X/k} \) for \( X/k \) an abelian variety. The associated component group scheme

\[ \text{NS}_{X/k} := \varpi_0(\text{Pic}^0_{X/k}) \]

is called the Néron-Severi group scheme of \( X \) over \( k \). The natural homomorphism \( q: \text{Pic}^0_{X/k} \rightarrow \text{NS}_{X/k} \) realizes \( \text{NS}_{X/k} \) as the fppf quotient of \( \text{Pic}^0_{X/k} \) modulo \( \text{Pic}^0_{X/k} \); hence we could also write

\[ \text{NS}_{X/k} = \text{Pic}^0_{X/k}/\text{Pic}^0_{X/k} \cdot \]

We refer to the group

\[ \text{NS}(X) := \text{NS}_{X/k}(k) \]

as the Néron-Severi group of \( X \). Note that \( \text{NS}(X) \) equals the subgroup of \( \text{Gal}(k_s/k) \)-invariants in \( \text{NS}(X_{k_s}) \).

We say that two line bundles \( L \) and \( M \) are algebraically equivalent, notation \( L \sim_{\text{alg}} M \), if \( [L] \) and \( [M] \) have the same image in \( \text{NS}(X) \). As \( \text{NS}(X) \) naturally injects into \( \text{NS}(X_{\overline{K}}) \), algebraic equivalence of line bundles (or divisors) can be tested over \( \overline{K} \), and there it coincides with the notion defined in Remark (6.9). Hence we can think of the Néron-Severi group scheme as being given by the classical, geometric Néron-Severi group \( \text{NS}(X_{k_s}) = \text{NS}(X_{\overline{K}}) \) of line bundles (or divisors) modulo algebraic equivalence, together with its natural action of \( \text{Gal}(k_s/k) \). Note, however, that a \( k \)-rational class \( \xi \in \text{NS}(X) \) may not always be representable by a line bundle on \( X \) over the ground field \( k \).

Let us rephrase some of the results that we have obtained in terms of the Néron-Severi group.

\((7.25)\) Corollary. The Néron-Severi group \( \text{NS}(X) \) is torsion-free. If \( n \in \mathbb{Z} \) and \( L \) is a line bundle on \( X \) then \( n^*L \) is algebraically equivalent to \( L^{n^2} \); in other words, \( n^*: \text{NS}(X) \rightarrow \text{NS}(X) \) is multiplication by \( n^2 \).

Proof. The first assertion is just (i) of Corollary (7.23). The second assertion follows from (ii) of that Corollary together with Corollary (2.12). \( \square \)

\((7.26)\) Corollary (7.22) can be restated by saying that the natural homomorphism \( \varphi: \text{Pic}^0_{X/k} \rightarrow \text{Hom}^\text{symm}(X, X^t) \subset \text{Hom}(X, X^t) \) factors as

\[ \text{Pic}^0_{X/k} \xrightarrow{q} \text{NS}^0_{X/k} \xleftarrow{\psi} \text{Hom}^\text{symm}(X, X^t) \]
for some injective homomorphism $\psi: \text{NS}_{X/k} \to \text{Hom}^{\text{symm}}(X, X^t)$. This says that the homomorphism $\varphi_L$ associated to a line bundle $L$ only depends on the algebraic equivalence class of $L$, and that $\varphi_L = \varphi_M$ only if $L \sim_{\text{alg}} M$. We shall later show that $\psi$ is actually an isomorphism; see Corollary (11.3).

§ 4. Applications to cohomology.

(7.27) Proposition. Let $X$ be an abelian variety with $\dim(X) = g$. Cup-product gives an isomorphism $\wedge^\bullet H^1(X, O_X) \sim H^\bullet(X, O_X)$. For every $p$ and $q$ we have a natural isomorphism $H^q(X, \Omega^p_{X/k}) \cong (\wedge^p T_{X/k}^\vee) \otimes (\wedge^q T_{X/k}^\vee)$. The Hodge numbers $h^{p,q} = \dim H^q(X, \Omega^p_{X/k})$ are given by $h^{p,q} = (g^p)(g^q)$.

Proof. Use (6.13) and the isomorphisms $\Omega^p_{X/k} \cong (\wedge^p T_{X/k}^\vee) \otimes_k O_X$. \hfill $\square$

(7.28) Corollary. Multiplication by an integer $n$ on $X$ induces multiplication by $n^{p+q}$ on $H^q(X, \Omega^p_{X})$.

Proof. Immediate from the fact that $n_X$ induces multiplication by $n$ on $T_{X,0}$, applied to both $X$ and $X^t$. \hfill $\square$

Before we state the next corollary, let us recall that the algebraic de Rham cohomology of a smooth proper algebraic variety $X$ over a field $k$ is defined to be the hypercohomology of the de Rham complex $\Omega^\bullet_{X/k} = (O_X \to \Omega_1^1 \to \Omega_2^1 \to \cdots)$, with $O_X$ in degree zero. We have the so-called “stupid filtration” of this complex, by the subcomplexes $\sigma \geq p \Omega^\bullet_{X/k}$ given by

$$[\sigma \geq p \Omega^\bullet_{X/k}]^i = \begin{cases} 0 & \text{for } i < p \\ \Omega^i_{X/k} & \text{for } i \geq p. \end{cases}$$

This gives rise to a spectral sequence

$$E_{1}^{p,q} = H^q(X, \Omega^p_{X}) \Rightarrow H^{p+q}_{\text{dR}}(X/k)$$

called the “Hodge-de Rham” spectral sequence.

If $k$ has characteristic zero then it follows from Hodge theory that this spectral sequence degenerates at the $E_1$-level, see Deligne [1], section 5. If $k$ has characteristic $p > 0$ then this is no longer true in general. For examples and further results we refer to Deligne-Illusie [1] and Oesterlé [1].

As we shall now show, for abelian varieties the degeneration of the Hodge-de Rham spectral sequence at level $E_1$ follows from (6.12) without any restrictions on the field $k$.

(7.29) Corollary. Let $X$ be an abelian variety over a field $k$. Then the “Hodge-de Rham” spectral sequence of $X$ degenerates at level $E_1$.

Proof. We follow the proof given by Oda [1]. We have to show that the differentials $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ are zero for all $r \geq 1$. By induction we may assume that this holds at all levels $< r$. (The empty assumption if $r = 1$.)
Write \( E^r_*(X) = \oplus E^p_q \), graded by total degree. Cup-product makes \( E^r_*(X) \) into a connected, graded-commutative \( k \)-algebra. By our induction assumption and the K"unneth formula there is a canonical isomorphism
\[
E^r_*(X \times X) \cong E^r_*(X) \otimes_k E^r_*(X).
\]
Write \( \mu: E^r_*(X) \to E^r_*(X) \otimes_k E^r_*(X) \) for the map induced by the multiplication law on \( X \), and write \( \varepsilon: E^r_*(X) \to E^0_*(X) = k \) for the projection onto the degree zero component. One checks that \( \mu \) and \( \varepsilon \) give \( E^r_*(X) \) the structure of a graded bialgebra over \( k \).

Let \( g = \dim(X) \). By what was shown above, \( E^1_*(X) = H^1(X, O_X) \oplus H^0(X, \Omega^1_{X/k}) \) has dimension \( 2g \). Also, \( E^i_*(X) = 0 \) for \( i > 2g \). The Borel-Hopf structure theorem \( (6.12) \) then gives
\[
E^r_*(X) \cong \wedge^r E^1_*(X).
\]

Since \( d_r \) is compatible with the product structure (cup-product) on \( E^r_*(X) \), it suffices to show that \( d_r \) is zero on \( E^1_*(X) \), which is just the space of primitive elements of \( E^r_*(X) \). (See \( 6.17 \).) By functoriality of the Hodge–de Rham spectral sequence we have \( \mu = d_r = (d_r \otimes d_r) + \mu \). Therefore, for \( \xi \in E^1_*(X) \) we have \( \mu(d_r(\xi)) = d_r(\xi) \otimes 1 + 1 \otimes d_r(\xi) \). This shows that \( d_r(\xi) \) is again a primitive element. But \( d_r(\xi) \in E^2_*(X) \) which, by \( (6.17) \), contains no non-zero primitive elements. This shows that \( d_r = 0 \). \( \square \)

(7.30) Corollary. There is an exact sequence
\[
0 \longrightarrow \Fil^1 H^1_{\text{dR}}(X/k) \longrightarrow H^1_{\text{dR}}(X/k) \longrightarrow H^1(X, O_X) \longrightarrow 0,
\]
where \( \Fil^1 H^1_{\text{dR}}(X/k) := H^0(X, \Omega^1_{X/k}) \cong T^\vee_{X,0} \).

To close this section let us fulfil an earlier promise and give an example of a smooth projective variety with non-reduced Picard scheme. We refer to Katsura-Ueno [1] for similar examples.

(7.31) Example. Let \( k \) be an algebraically closed field of characteristic 3. Let \( E_1 \) be the elliptic curve over \( k \) given by the Weierstrass equation \( y^2 = x^3 - x \). From \( (5.27) \) we know that \( E_1 \) is supersingular. Let \( \sigma \) be the automorphism of \( E_1 \) given by \( (x, y) \mapsto (x + 1, y) \). Then \( \sigma \) has order 3, so that we get an action of \( G := \mathbb{Z}/3\mathbb{Z} \) on \( E_1 \). The quotient of \( E_1 \) by \( G \) is isomorphic to \( \mathbb{P}^1_k \); in affine coordinates the quotient map is just \( (x, y) \mapsto y \).

Let \( E_2 \) be an ordinary elliptic curve over \( k \). Let \( \tau \) be the translation over a point of (exact) order 3 on \( E_2 \). Then \( (\sigma, \tau) \) is an automorphism of order 3 of the abelian surface \( X := E_1 \times E_2 \); this gives a strictly free action of \( G := \mathbb{Z}/3\mathbb{Z} \) on \( X \), and we can form the quotient \( X \to Y := G\backslash X \). By (??) \( \pi \) is an étale morphism, so \( Y \) is again a non-singular algebraic surface. We have a natural morphism \( Y \to (G\backslash E_1) \cong \mathbb{P}^1 \); this exhibits \( Y \) as an elliptic surface over \( \mathbb{P}^1 \). In fact, for all \( P \in \mathbb{P}^1(k) \) with \( P \not= \infty \) the fibre \( Y_P \) above \( P \) is isomorphic to \( E_2 \).

We compute \( h^1(Y, O_Y) \) using Hirzebruch-Riemann-Roch and Chern numbers for algebraic surfaces. (A reference is ??.) The Euler number \( c_2 \) of \( Y \) is a multiple of the Euler number of \( X \), and this is 0. By the Hirzebruch-Riemann-Roch formula we have
\[
1 - h^1(Y, O_Y) + h^2(Y, O_Y) = (c_1^2 + c_2)/12 = 0,
\]
since \( c_1^2 = 0 \) for every elliptic surface. By Serre duality, \( h^2(Y, O_Y) = h^0(Y, \Omega^2_Y/k) \). Now we use that \( H^0(Y, \Omega^2_Y/k) \) is isomorphic to the space of \( G \)-invariants in \( H^0(X, \Omega^2_{X/k}) \). If \( \omega_1 \) is a basis for \( H^0(E_1, \Omega^1_{E_1/k}) \) then \( \omega_1 \wedge \omega_2 \) is a basis for \( H^0(X, \Omega^2_{X/k}) \). But \( \omega_1 \) is a multiple of \( dy \), which is
invariant under $\sigma$, and $\omega_2$ is translation invariant, in particular invariant under $\tau$. In sum, we find that $h^2(Y, O_Y) = 1$ and $h^1(Y, O_Y) = 2$.

On the other hand, $\pi: X \to Y$ induces a homomorphism $\pi^*: \text{Pic}^0_Y \to X^t = \text{Pic}^0_{X/k}$. The same arguments as in the proof of Theorem (7.5) show that $\text{Ker}(\pi^*) \cong \mu_3$. On the other hand, $\pi^*$ factors via the subscheme of $G$-invariants in $X^t$. (See Exercise ?? for the existence of such a subscheme of $G$-invariants.) The point here is that we are describing line bundles on $Y$ as coming from line bundles $L$ on $X$ together with an action of $G$. But such an action is given by an isomorphism $\rho^* L \sim \text{pr}^*_X L$ of line bundles on $G \times_k X$. The existence of such an isomorphism says precisely that $L$ corresponds to a $G$-invariant point of $X^t$.

By Exercises ?? and ??, $X^t \sim X$. The induced action of $G$ on $X^t$ is given by the automorphism $(\sigma, \text{id})$. (Cf. Exercise ??) Therefore, the subscheme of $G$-invariants in $X^t$ is $E_1^{(\sigma)} \times E_2$. The only geometric point of $E_1$ fixed under $\sigma$ is the origin. A computation in local coordinates reveals that $E_1^{(\sigma)}$ is in fact the Frobenius kernel $E_1[F] \subset E_1$ which can be shown to be isomorphic to $\alpha_3$. In any case, we find that $\text{Pic}^0_Y$ is 1-dimensional, whereas we have shown its tangent space at the identity, isomorphic to $H^1(Y, O_Y)$, to be 2-dimensional. Hence $\text{Pic}^0_Y$ is non-reduced.

§ 5. The duality between Frobenius and Verschiebung.

(7.32) Let $S$ be a scheme of characteristic $p$. Recall that for any $S$-scheme $a_X: X \to S$ we have a commutative diagram with Cartesian square

$$
\begin{array}{ccc}
X^{(p/S)} & \xrightarrow{W_{X/S}} & X \\
\downarrow a_X^{(p)} & & \downarrow a_X \\
S & \xrightarrow{\text{Frob}_S} & S
\end{array}
$$

If there is no risk of confusion we simply write $X^{(p)}$ for $X^{(p/S)}$. Note that if $a_T: T \to S$ is an $S$-scheme then we have $a_T \circ \text{Frob}_T = \text{Frob}_S \circ a_T$ and this gives a natural identification $(X^{(p/T)}) = (X^{(p/S)})_T$. We denote this scheme simply by $X^{(p)}$.

Let us write $T \circ (p)$ for the scheme $T$ viewed as an $S$-scheme via the morphism $a_{T(p)} := \text{Frob}_S \circ a_T = a_T \circ \text{Frob}_T: T \to S$. The morphism $\text{Frob}_T: T \to T$ is not, in general, a morphism of $S$-schemes, but if we view it as a morphism $T(p) \to T$ then it is a morphism over $S$. To avoid confusion, let us write $\text{Fr}_T: T(p) \to T$ for the morphism of $S$-schemes given by $\text{Frob}_T$.

Let $Y$ be an $S$-scheme. Recall that we write $Y(T)$ for the $T$-valued points of $Y$. It is understood here (though not expressed in the notation) that all schemes and morphisms of schemes are over a fixed base scheme $S$; so $Y(T)$ is the set of morphisms $T \to Y$ over $S$. There is a natural bijection

$$
w_{Y,T}: Y^{(p)}(T) \sim Y(T),
$$

sending a point $\eta: T \to Y^{(p)}$ to $W_{Y/S \circ \eta}$, which is a $T(p)$-valued point of $Y$. The composition

$$
w_{Y,T} \circ F_{Y/S}(T): Y(T) \to Y(T)
$$

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is the map that sends $y \in Y(T)$ to $y \cdot \text{Fr}_T: T_{(p)} \to Y$, which is the same as $y \cdot \text{Frob}_T: T \to Y$ viewed as a morphism $T_{(p)} \to Y$.

(7.33) Consider an abelian variety $X$ over a field $k$ of characteristic $p$. Take $S := \text{Spec}(k)$. If $T$ is any $S$-scheme then $X \times_S T_{(p)}$ is the same as $X^{(p)} \times_S T$, and we find that

$$
\text{Pic}_{X/k}^{(p)}(T) \xrightarrow{\sim} \text{Pic}_{X/k}(T_{(p)}) = \left\{ \text{isomorphism classes of rigidified line bundles } (L, \alpha) \text{ on } X \times_S T_{(p)} \right\} 
$$

$$
= \left\{ \text{isomorphism classes of rigidified line bundles } (L, \alpha) \text{ on } X^{(p)} \times_S T \right\} = \text{Pic}_{X^{(p)}/S}(T).
$$

In this way we obtain an isomorphism $\text{Pic}_{X/S}^{(p)} \xrightarrow{\sim} \text{Pic}_{X^{(p)}/S}$, which we take as an identification. Applying (7.32) with $Y = \text{Pic}_{X/k}$ we find that the relative Frobenius of $\text{Pic}_{X/k}$ over $k$ is the homomorphism that sends a point $y \in \text{Pic}_{X/k}(T)$ to $y \cdot \text{Frob}_T$, viewed as a morphism $T_{(p)} \to \text{Pic}_{X/k}$. Because the diagram

$$
\begin{array}{ccc}
X^{(p)}_T & \xrightarrow{W_{X/T}} & X_T \\
\alpha^{(p)}_X & \downarrow & \downarrow \alpha_X \\
T & \xrightarrow{\text{Frob}_T} & T
\end{array}
$$

is Cartesian this just means $F_{\text{Pic}/k}: \text{Pic}_{X/k} \to \text{Pic}_{X^{(p)}/k}$ sends the class of a rigidified line bundle $(L, \alpha)$ on $X_T$ to the class of $(L^{(p)}, \alpha^{(p)})$ on $X^{(p)}_T$, where $L^{(p)} := W^*_X/T L$, and where $\alpha^{(p)}: O_T \xrightarrow{\sim} e^*L^{(p)} = \text{Frob}^T_T(e^*L)$ is the rigidification of $L^{(p)}$ along the zero section obtained by pulling back $\alpha$ via $\text{Frob}_T$.

(7.34) Proposition. Let $X$ be an abelian variety over a field $k$ of characteristic $p$. We identify $(X^t)^{(p)} = (X^{(p)})^t$ as in (7.33), and we denote this abelian variety by $X^t,(p)$. Then we have the identities

$$
F^t_{X/k} = V_{X^t/k}: X^t,(p) \to X^t \quad \text{and} \quad V^t_{X/k} = F^t_{X^t/k}: X^t \to X^t,(p).
$$

Proof. It suffices to prove that $F^t_{X/k} \circ F^t_{X^t/k}: X^t \to X^t$ equals $[p]_{X^t}$, because if this holds then together with Proposition (5.20) and the fact that $F^t_{X^t/k}$ is an isogeny it follows that $F^t_{X/k} = V_{X^t/k}$. The other assertion follows by duality.

Let $T$ be a $k$-scheme. Consider a rigidified line bundle $(L, \alpha)$ on $X_T$ that gives a point of $X^t(T)$. As explained in (7.33) $F^t_{X^t/k}$ sends $(L, \alpha)$ to $(L^{(p)}, \alpha^{(p)})$ with $L^{(p)} = W^*_X/T L$. Because $W_{X^t/T} \circ F_{X^t/T} = \text{Frob}_{X^t}$, pull-back via $F_{X^t/T}$ gives the line bundle $\text{Frob}_{X^t} L$ on $X_T$. But if $Y$ is any scheme of characteristic $p$ and $M$ is a line bundle on $Y$ then $\text{Frob}^Y Y(M) \cong M^p$; this follows for instance by taking a trivialization of $M$ and remarking that $\text{Frob}^Y$ raises all transition functions to the power $p$. The rigidification we have on $F^*_X/T, W^*_X/T L = \text{Frob}^*_X L = L^p$ is the isomorphism

$$
O_T = \text{Frob}^T_T O_T \xrightarrow{\sim} e^*_X F^*_X/T, W^*_X/T L = e^*_X L^p = (e^*_X L)^p
$$

that is obtained from $\alpha$ by pulling back via $\text{Frob}_T$, which just means it is $\alpha^p$. In sum, $F^t_{X/k} \circ F^t_{X^t/k}$ sends $(L, \alpha)$ to $(L^p, \alpha^p)$, which is what we wanted to prove. \hfill \square
Exercises.

(7.1) Let $X$ be an abelian variety. Let $m_X: X \times X \rightarrow X$ be the group law, and let $\Delta_X: X \rightarrow X \times X$ be the diagonal morphism. Show that $(m_X)^t = \Delta_X: X^t \times X^t \rightarrow X^t$, and that $(\Delta_X)^t = m_X^t: X^t \times X^t \rightarrow X^t$.

(7.2) Let $L$ be a line bundle on an abelian variety $X$.

(i) Show that, for $n \in \mathbb{Z}$,
\[ n^*L \cong O_X \iff L^n \cong O_X. \]

(ii) Show that, for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$,
\[ n^*L \cong L \iff L^{n-1} \cong O_X. \]

(7.3) Let $X$ be an abelian variety over an algebraically closed field $k$. Show that every line bundle $L$ on $X$ can be written as $L = L_1 \otimes L_2$, where $L_1$ is symmetric and $[L_2] \in \text{Pic}^0_X/k$. [Hint: By (7.23), the class of the line bundle $(-1)^*L \otimes L^{-1}$ is in $\text{Pic}^0_X/k$. As $\text{Pic}^0$ is an abelian variety and $k = \mathbb{F}$, there exists a line bundle $M$ on $X$ with $[M] \in \text{Pic}^0$ and $M^2 \cong (-1)^*L \otimes L^{-1}$. Now show that $L \otimes M$ is symmetric.]

(7.4) Let $\mathcal{P}$ be the Poincaré bundle on $X \times X^t$. For $m, n \in \mathbb{Z}$, consider the endomorphism $(m, n)$ of $X \times X^t$. Show that $(m, n)^*\mathcal{P} \cong \mathcal{P}^{mn}$.

(7.5) Let $\mathcal{P}$ be the Poincaré bundle on $X \times X^t$. Show that the associated homomorphism $\varphi_{\mathcal{P}}: X \times X^t \rightarrow X^t \times X^t$ is the homomorphism given by $\varphi_{\mathcal{P}}(x, \xi) = (\xi, \kappa_X(x))$. [Hint: Compute the restrictions of $t, x, \xi)^*\mathcal{P} \otimes \mathcal{P}^{-1}$ to $X \times \{0\}$ and $\{0\} \times X^t$.]

(7.6) If $\tau$ is a translation on an abelian variety, then what is the induced automorphism $\tau^t$ of the dual abelian variety?

(7.7) Let $X$ be an abelian variety over a field $k$. Let $i: Y \hookrightarrow X$ be an abelian subvariety. Write $q: X \rightarrow Z := X/Y$ for the fpqc quotient morphism, which exists by Thm. (4.38). Note that $Z$ is an abelian variety; see Example (4.40).

(i) Show that for any $k$-scheme $T$ we have $q_*(O_{X,T}) = O_{Z,T}$.

(ii) Prove that $q^t: Z^t \rightarrow X^t$ is injective and gives an isomorphism between $Z^t$ and $\text{Ker}(i^t: X^t \rightarrow Y^t)_{\text{red}}$.

(7.8) Let $L$ be a line bundle on an abelian variety $X$. For a symmetric $m \times m$-matrix $S$ with integer coefficients $s_{ij}$ we define a line bundle $L^\otimes S$ on $X^m$ by
\[ L^\otimes S := \left( \otimes_{i=1}^m p_i^* L_{x_{ij}} \right) \otimes \left( \otimes_{1 \leq i < j \leq m} p_{ij}^* \Lambda(L)^{s_{ij}} \right). \]

If $\alpha = (a_{ij})$ is an integer valued matrix of size $m \times n$ we define a homomorphism of abelian varieties $[\alpha]_X: X^n \rightarrow X^m$ by $\alpha(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$ with $y_i = \sum_{j=1}^n a_{ij} x_j$.

(i) Prove that $[\alpha]_X^* (L^\otimes S)$ is algebraically equivalent to $L^\otimes (\alpha S^0)$.

(ii) Assume that $L$ is a symmetric line bundle. Prove that $[\alpha]_X^* (L^\otimes S) \cong L^\otimes (\alpha S^0)$.

Notes. (nog aanvullen)