Chapter II. Line bundles and divisors on abelian varieties.

In this chapter we study divisors on abelian varieties. One of the main goals is to prove that abelian varieties are projective. The Theorem of the Square (2.9) plays a key role. Since abelian varieties are nonsingular, a Weil divisor defines a Cartier divisor and a line bundle, and we have a natural isomorphism $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$. We shall mainly work with line bundles, but sometimes (Weil) divisors are more convenient.

The following abuse of notation will prove handy: if L is a line bundle on a product variety $X \times Y$ then we shall write L_x for the restriction of L to $\{x\} \times Y$ and, similarly, L_y denotes the restriction $L_{|X\times\{y\}}$. Here of course x shall always be a point of X and y a point of Y.

In this chapter, varieties shall always be varieties over some ground field k, which in most cases shall not be mentioned.

§ 1. *The theorem of the square.*

(2.1) Theorem. *Let X and Y be varieties. Suppose X is complete. Let L and M be two line bundles on $X \times Y$. If for all closed points $y \in Y$ we have $L_y \cong M_y$ there exists a line bundle N on Y such that $L \cong M \otimes p^*N$, where $p = \mathrm{pr}_Y\colon X \times Y \to Y$ is the projection onto Y.*

Proof. This is a standard fact of algebraic geometry. A proof using cohomology runs as follows. Since $L_y \otimes M_y^{-1}$ is the trivial bundle and X_y is complete, the space of sections $H^0(X_y, L_y \otimes M_y^{-1})$ is isomorphic to $k(y)$, the residue field of y. This implies that $p_*(L \otimes M^{-1})$ is locally free of rank one, hence a line bundle (see MAV, §5 or HAG, Chap. III, § 12). We shall prove that the natural map

$$\alpha\colon p^*p_*(L \otimes M^{-1}) \to L \otimes M^{-1}$$

is an isomorphism. If we restict to a fibre we find the map

$$O_{X_y} \otimes \Gamma(X_y, O_{X_y}) \to O_{X_y}$$

which is an isomorphism. By Nakayama's Lemma, this implies that α is surjective and by comparing ranks we conclude that it is an isomorphism. $\square$

As an easy consequence we find a useful prinicple.

(2.2) See-saw Principle. *If, in addition to the assumptions of* (2.1), *we have $L_x = M_x$ for some point $x \in X$ then $L \cong M$.*

Proof. We have $L \cong M \otimes \mathrm{pr}_Y^*N$. Over $\{x\} \times Y$ this gives $L_x \cong M_x \otimes (\mathrm{pr}_Y^*N)_x$. Therefore, $(\mathrm{pr}_Y^*N)_x$ is trivial, and this implies that N is trivial. $\square$

(2.3) Lemma. *Let X and Y be varieties, with X complete. For a line bundle L on $X \times Y$, the set $\{y \in Y \mid L_y$ is trivial$\}$ is closed in Y.*

Proof. If M is a line bundle on a complete variety then M is trivial if and only if both $H^0(M)$ and $H^0(M^{-1})$ are non-zero. Hence

$$\{y \in Y \mid L_y \text{ is trivial}\} = \{y \in Y \mid h^0(L_y) > 0\} \cap \{y \in Y \mid h^0(L_y^{-1}) > 0\}\,. \qquad (1)$$

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But the functions $y \mapsto h^0(L_y)$ and $y \mapsto h^0(L^{-1})$ are upper semi-continuous on Y; see MAV, § 5 or HAG, Chap. III, Thm. 12.8. So the two sets in the right hand side of (1) are closed in Y. $\square$

Actually, there is a refinement of this which says the following.

(2.4) Lemma. *Let X be a complete variety over a field k, let Y be a k-scheme, and let L be a line bundle on $X \times Y$. Then there exists a closed subscheme $Y_0 \hookrightarrow Y$ which is* the maximal subscheme of Y over which L is trivial; *i.e.,* (i) *the restriction of L to $X \times Y_0$ is the pull back (under pr_{Y_0}) of a line bundle on Y_0, and* (ii) *if $\varphi\colon Z \to Y$ is a morphism such that* $(\mathrm{id}_X \times \varphi)^*(L)$ *is the pullback of a line bundle on Z under p_Z^* then φ factors through Y_0.*

For the proof we refer to MAV, §10. In Chapter 6 we shall discuss Picard schemes; once we know the existence and some properties of $\mathrm{Pic}_{X/k}$ the assertion of the lemma is a formal consequence. (See (6.4).)

The following theorem is again a general fact from algebraic geometry and could be accepted as a black box. As it turns out, it is of crucial importance for the theory of abelian varieties. In view of its importance we give a proof.

(2.5) Theorem. *Let X and Y be complete varieties and let Z be a connected, locally noetherian scheme. Let $x \in X(k)$, $y \in Y(k)$, and let z be a point of Z. If L is a line bundle on $X \times Y \times Z$ whose restriction to $\{x\} \times Y \times Z$, to $X \times \{y\} \times Z$ and to $X \times Y \times \{z\}$ is trivial then L is trivial.*

Proof. We follow the proof given by Mumford in MAV. We view L as a family of line bundles on $X \times Y$ parametrized by Z. Let Z' be the maximal closed subscheme of Z over which L is trivial, as discussed above. We have $z \in Z'$. We shall show that $Z' = Z$ by showing that Z' is an open subscheme and using the connectedness of Z.

Let ζ be a point of Z'. Write $\mathfrak{m}$ for the maximal ideal of the local ring $O_{Z,\zeta}$ and $I \subset O_{Z,\zeta}$ for the ideal defining (the germ of) Z'. We have to show that $I = (0)$. Suppose not. By Krull's Theorem (here we use that Z is locally noetherian) we have $\cap_n \mathfrak{m}^n = (0)$, hence there exists a positive integer n such that $I \subset \mathfrak{m}^n, I \not\subset \mathfrak{m}^{n+1}$. Put $a_1 = (I, \mathfrak{m}^{n+1})$, and choose an ideal a_2 with

$$\mathfrak{m}^{n+1} \subset a_2 \subset (I, \mathfrak{m}^{n+1}) = a_1 \qquad \text{and} \qquad \dim_{k(\zeta)}(a_1/a_2) = 1\,.$$

(Note that such ideals exist.) Let $Z_i \subset \mathrm{Spec}(O_{Z,\zeta})$ be the closed subscheme defined by the ideal a_i ($i = 1, 2$). We will show that the restriction of L to $X \times Y \times Z_2$ is trivial. This implies that Z_2 is contained in Z', which is a contradiction, since $I \not\subset a_2$.

Write L_i for the restriction of L to $X \times Y \times Z_i$. By construction, L_1 is trivial; choose a trivializing global section s. The inclusion $Z_1 \hookrightarrow Z_2$ induces a restriction map $\Gamma(L_2) \to \Gamma(L_1)$. We claim: L_2 is trivial if and only if s can be lifted to a global section of L_2. To see this, suppose first that we have a lift s'. The schemes $X \times Y \times Z_1$ and $X \times Y \times Z_2$ have the same underlying point sets. If $s'(P) = 0$ for some point P then also $s(P) = 0$, but this contradicts the assumption that s is a trivialization of L_1. Hence s' is nowhere zero, and since L_2 is locally free of rank 1 this implies that s' trivializes L_2. Conversely, if L_2 is trivial then the restriction map $\Gamma(L_2) \to \Gamma(L_1)$ is just $\Gamma(O_{Z_2}) \to \Gamma(O_{Z_1})$ and this is surjective.

The obstruction for lifting s to a global section of L_2 is an element $\xi \in H^1(X \times Y, O_{X\times Y})$. We know that the restrictions of L_2 to $\{x\} \times Y \times Z_2$ and to $X \times \{y\} \times Z_2$ are trivial. Writing $i_1 = (\mathrm{id}_X, y)\colon X \hookrightarrow X \times Y$ and $i_2 = (x, \mathrm{id}_Y)\colon Y \hookrightarrow X \times Y$, this means that ξ has trivial image under $i_1^*\colon H^1(X \times Y, O_{X\times Y}) \to H^1(X, O_X)$ and under $i_2^*\colon H^1(X \times Y, O_{X\times Y}) \to H^1(Y, O_Y)$. But the map (i_1^*, i_2^*) gives a (Künneth) isomorphism

$$H^1(X \times Y, O_{X\times Y}) \xrightarrow{\sim} H^1(X, O_X) \oplus H^1(Y, O_Y)\,,$$

hence $\xi = 0$ and s can be lifted. $\square$

(2.6) Remarks. (i) In the theorem as stated we require x and y to be k-rational points of X, resp. Y. We use this when we define the maps i_1 and i_2. But in fact the theorem holds without these assumptions. The point is that if $k \subset K$ is a field extension then a line bundle M on a k-variety V is trivial if and only if the line bundle M_K on V_K is trivial. (See Exercise (2.1).) Hence we may first pass to a bigger field K over which the points x and y (and even the point $z \in Z$) become rational.

(ii) The previous theorem gives a strong general result about line bundles on a product of three complete varieties. Note that the analogous statement for line bundles on a product of two complete varieties is false in general. More precisely, suppose X and Y are complete k-varieties and L is a line bundle on $X \times Y$. If there exist points $x \in X$ and $y \in Y$ such that $L_x \cong O_Y$ and $L_y \cong O_X$ then it is not true in general that $L \cong O_{X\times Y}$. For instance, take $X = Y$ to be an elliptic curve, and consider the divisor

$$D = \Delta_X - \bigl(\{0\} \times X\bigr) - \bigl(X \times \{0\}\bigr)$$

where $\Delta_X \subset X \times X$ is the diagonal. Note that $L = O_{X\times X}(D)$ restricts to the trivial bundle on $\{0\} \times X$ and on $X \times \{0\}$. (Use that the divisor $1 \cdot 0$ $(= 1 \cdot e_X)$ on X is linearly equivalent to a divisor whose support does not contain 0.) But L is certainly not the trivial bundle: if it were, $L_{|\{P\}\times X} = O_X(P - e_X) \cong O_X$ for all points $P \in X$. But then there is a function f on X with one zero and one pole and X would have to be a rational curve, which we know it is not.

Theorem (2.5), together with the previous remark, is a reflection of the quadratic character of line bundles which comes out clearer as follows. If $f(x) = ax^2 + bx + c$ is a quadratic function on the real line then

$$f(x+y+z) - f(x+y) - f(x+z) - f(y+z) + f(x) + f(y) + f(z)$$

is constant. The analogue of this for line bundles on abelian varieties is the celebrated Theorem of the Cube. First a notational convention. If X is an abelian variety and $I = \{i_1, \ldots, i_r\} \subset \{1, 2, \ldots, n\}$ then we write

$$p_I\colon X^n \to X\,, \qquad \text{or} \quad p_{i_1\cdots i_r}\colon X^n \to X\,,$$

for the morphism sending $(x_1, x_2, \ldots, x_n)$ to $x_{i_1} + \cdots + x_{i_r}$. Thus, for example, p_i is the projection onto the ith factor, $p_{12} = p_1 + p_2$, etc. With these notations we have the following important corollary to the theorem.

(2.7) Theorem of the Cube. *Let L be a line bundle on X. Then the line bundle*

$$\begin{aligned}\Theta(L) :=& \bigotimes_{I\subset\{1,2,3\}} p_I^* L^{\otimes(-1)^{1+\#I}}\\ &= p_{123}^*L \otimes p_{12}^*L^{-1} \otimes p_{13}^*L^{-1} \otimes p_{23}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L\end{aligned}$$

on $X \times X \times X$ is trivial.

Proof. Restriction of $\Theta(L)$ to $\{0\} \times X \times X$ gives the bundle

$$m^*L \otimes p_2^*L^{-1} \otimes p_3^*L^{-1} \otimes m^*L^{-1} \otimes O_{X\times X} \otimes p_2^*L \otimes p_3^*L$$

which is obviously trivial. Similarly for $X \times \{0\} \times X$ and $X \times X \times \{0\}$. By (2.5) the result follows. $\square$

We could sharpen the corollary by saying that $\Theta(L)$ is canonically trivial, see Exercise (2.2).

(2.8) Corollary. *Let Y be a scheme and let X be an abelian variety. For every triple f, g, h of morphisms $Y \to X$ and for every line bundle L on X, the bundle*

$$(f+g+h)^*L \otimes (f+g)^*L^{-1} \otimes (f+h)^*L^{-1} \otimes (g+h)^*L^{-1} \otimes f^*L \otimes g^*L \otimes h^*L$$

on Y is trivial.

Proof. Consider $(f,g,h)\colon Y \to X \times X \times X$ and use (2.7). $\square$

Another important corollary is the following.

(2.9) Theorem of the Square. *Let X be an abelian variety and let L be a line bundle on X. Then for all $x, y \in X(k)$,*

$$t_{x+y}^*L \otimes L \cong t_x^*L \otimes t_y^*L\,.$$

More generally, let T be a k-scheme and write L_T for the pull-back of L to X_T. Then

$$t_{x+y}^*L_T \otimes L_T \cong t_x^*L_T \otimes t_y^*L_T \otimes \mathrm{pr}_T^*\big((x+y)^*L \otimes x^*L^{-1} \otimes y^*L^{-1}\big)$$

for all $x, y \in X(T)$.

Proof. In the first formulation, this is immediate from (2.8) by taking for f the identity on X and for g and h the constant maps with images x and y. For the general form, take $f = \mathrm{pr}_X\colon X_T = X \times_k T \to X$, take $g = x \circ \mathrm{pr}_T$ and $h = y \circ \mathrm{pr}_T$. Then

$$f + g = \mathrm{pr}_X \circ t_x\,, \quad f + h = \mathrm{pr}_X \circ t_y\,, \quad g + h = (x+y) \circ \mathrm{pr}_T$$

and

$$f + g + h = \mathrm{pr}_X \circ t_{x+y}$$

Now again apply (2.8). $\square$

The theorem allows the following interpretation. (Compare this with what we have seen in Examples (1.7) and (1.9).)

(2.10) Corollary. *Let L be a line bundle on an abelian variety X. Let $\mathrm{Pic}(X)$ be the group of isomorphism classes of line bundles on X. Then the map $\varphi_L\colon X(k) \to \mathrm{Pic}(X)$ given by $x \mapsto [t_x^*L \otimes L^{-1}]$ is a homomorphism.*

Proof. Immediate from (2.9). $\square$

(2.11) Remark. The homomorphisms φ_L will play a very important role in the theory. In later chapters (see in particular Chapters 6 and 7) we shall introduce the dual X^t of an abelian variety X, and we shall interprete φ_L as a homomorphism $X \to X^t$. The homomorphisms $\lambda\colon X \to X^t$ that are (geometrically) of the form φ_L for an ample line bundle L are called polarizations; see Chapter 11.

At this point, let us already caution the reader that there is a sign convention in the theory that can easily lead to misunderstanding. In the theory of elliptic curves one usually describes line bundles of degree 0 (which is what the dual elliptic curve is about!) in the form $O_E(P-O)$. More precisely: if E is an elliptic curve with origin O then the map $P \mapsto O_E(P-O)$ gives an isomorphism $E \xrightarrow{\sim} E^t = \mathrm{Pic}^0_{E/k}$. This map is *not* the polarization associated to the ample line bundle $L = O_E(O)$; rather it is *minus* that map. In general, if D is a divisor on an abelian variety X then $t_x^* O_X(D)$ is $O_X\bigl((t_{-x}(D)\bigr) = O_X(D-x)$, not $O_X(D+x)$. So if $L = O_E(O)$ on an elliptic curve E, the map φ_L is given on points by $P \mapsto O_E(O-P)$.

The same remark applies to the theory of Jacobians (see in particular Chapter 14). If C is a smooth projective curve over a field k, and if $P_0 \in C(k)$ is a k-rational point then we have a natural morphism φ from C to its Jacobian variety $J = \mathrm{Jac}(C) := \mathrm{Pic}^0_{C/k}$. In most literature one considers the map $C \to J$ given on points by $P \mapsto O_C(P - P_0)$. However, we have a canonical principal polarization on J (see again Chapter 14 for further details), and in connection with this it is more natural to consider the morphism $\varphi\colon C \to J$ given by $P \mapsto O_C(P_0 - P)$.

Let X be an abelian variety. For every $n \in \mathbb{Z}$ we have a homomorphism $[n] = [n]_X\colon X \to X$ called "multiplication by n". For $n \geqslant 1$, it sends $x \in X(k)$ to $x + \cdots + x$ (n terms); for $n = -m \leqslant -1$ we have $[n]_X = i_X \circ [m]_X$. If there is no risk of confusion, we shall often simply write n for $[n]$; in particular this includes the abbreviations 1 for $[1] = \mathrm{id}_X$, 0 for $[0]$ (the constant map with value 0), and -1 for $[-1] = -\mathrm{id}_X$. The effect of n on line bundles is described by the following result.

(2.12) Corollary. *For every line bundle L on an abelian variety X we have*

$$n^*L \cong L^{n(n+1)/2} \otimes (-1)^*L^{n(n-1)/2}\,.$$

Proof. Set $f = n$, $g = 1$, and $h = -1$. Applying (2.8), one finds that

$$n^*L \otimes (n+1)^*L^{-1} \otimes (n-1)^*L^{-1} \otimes n^*L \otimes L \otimes (-1)^*L$$

is trivial, i.e.,

$$n^*L^2 \otimes (n+1)^*L^{-1} \otimes (n-1)^*L^{-1} \cong (L \otimes (-1)^*L)^{-1}\,.$$

The assertion now follows by induction, starting from the cases $n = -1$, 0, 1. □

In particular, if the line bundle L is symmetric, by which we mean that $(-1)^*L \cong L$, then we find that $n^*L \cong L^{n^2}$ for all n. For instance, if M is an arbitrary line bundle then $L_+ := M \otimes (-1)^*M$ is symmetric. Similarly, $L_- := M \otimes (-1)^*M^{-1}$ is an example of an anti-symmetric line bundle, i.e., a line bundle L for which $(-1)^*L \cong L^{-1}$; for such line bundles we have $n^*L \cong L^n$ for all n. Note the contrast between the quadratic effect of n^* in the symmetric case and the linear effect in the anti-symmetric case. Further note that with the notation just introduced we have $M^2 \cong L_+ \otimes L_-$; so we find that the square of a line bundle can be written as the product of a symmetric and an anti-symmetric part. This is a theme we shall explore in much greater detail in later chapters.

§ 2. *Projectivity of abelian varieties.*

We now turn to the question whether abelian varieties are projective. As it turns out the answer is "yes". We give two proofs of this. A fairly short proof is given in (2.26); the Theorem of the Square plays a key role in this argument. The other proof we give is longer—it takes up most of this section—but along the way we shall obtain a number of results that are interesting in their own right. We think that Proposition (2.20) is particularly remarkable.

We shall need a couple of facts about group schemes. Since these form the main objects of study of the next two chapters, we shall simply use what we need, and refer forward to the next chapter for a precise explanation. Most of what is needed in this chapter can be summarized as follows.

(2.13) Fact. *Let X be an abelian variety over a field k. Suppose $Y \hookrightarrow X$ is a closed subgroup scheme. If Y^0 is the connected component of Y containing the origin then Y^0 is an open and closed subgroup scheme of Y and Y^0 is geometrically irreducible. If furthermore k is perfect then the reduced underlying scheme $Y^0_{\mathrm{red}} \hookrightarrow X$ is an abelian subvariety of X.*

For the proof of this statement, see Prop. (3.17) and Exercise (3.2).

(2.14) Remark. The fact just stated is weaker than what is actually true. Namely, the conclusion that $Y^0_{\mathrm{red}} \hookrightarrow X$ is an abelian subvariety of X holds true without the assumption that the base field k is perfect. We shall see this in Prop. (5.31), once we have more theory at our disposal. If we already knew the stronger version of the above fact at this stage, it would simplify some of the arguments that we shall give. For instance, in the rest of this chapter we shall sometimes work over $\overline{k}$ and then later draw conclusions that are valid over an arbitrary field. The reason for this detour is that, at this stage, we can apply (2.13) only over a perfect field.

Suppose $X = A \times B$ is an abelian variety which is a product of positive dimensional abelian varieties A and B, and suppose M is a line bundle on A. If $\mathrm{pr}_A\colon A \times B \to A$ is the projection onto A then the bundle $L := \mathrm{pr}_A^* M$ is invariant under translation over the points of $\{0_A\} \times B \subset X$. Obviously, L is not ample. This suggests that if L is a line bundle on X which is invariant under many translations, then L might not be ample.

(2.15) Definition. Let L be a line bundle on an abelian variety X. On $X \times X$ we define the *Mumford line bundle* $\Lambda(L)$ by

$$\Lambda(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \,.$$

As we shall see, $\Lambda(L)$ is a very useful bundle. The restriction of $\Lambda(L)$ to a vertical fibre $\{x\} \times X$ and to a horizontal fibre $X \times \{x\}$ is $t_x^*L \otimes L^{-1}$. In particular, $\Lambda(L)$ is trivial on $\{0\} \times X$ and on $X \times \{0\}$.

(2.16) Definition. With the above notations, we define $K(L) \subseteq X$ as the maximal closed subscheme (in the sense of (2.4)) such that $\Lambda(L)_{|X \times K(L)}$ is trivial over $K(L)$, i.e., such that $\Lambda(L)_{|X\times K(L)} \cong \mathrm{pr}_2^* M$ for some line bundle M on $K(L)$.

It follows from the universal property in (2.4) that the formation of $K(L)$ is compatible with base-change. In particular, if $k \subset k'$ is a field extension, writing L' for the pull-back of L to $X \times_k k'$, we have $K(L') = K(L) \times_k k'$.

Roughly speaking, a point belongs to $K(L)$ if L is invariant under translation by this point. A more precise statement is given by the following lemma.

(2.17) Lemma. *Let T be a k-scheme and $x\colon T \to X$ a T-valued point of X.*

(i) *The morphism x factors through $K(L)$ if and only if $t_x^*L_T \otimes L_T^{-1}$ is the pull-back of a line bundle on T.*

(ii) *If $t_x^*L_T \otimes L_T^{-1} \cong \mathrm{pr}_T^*M$ then $M \cong x^*L$.*

(iii) *We have $\Lambda(L)_{|X\times K(L)} \cong O_{X\times K(L)}$.*

In (iii), note that a priori we only knew that $\Lambda(L)_{|X\times K(L)}$ is the pull-back of a line bundle on $K(L)$.

Proof. As usual, L_T denotes the pull-back of L via the projection $\mathrm{pr}_X\colon X_T \to X$. Since $\mathrm{pr}_X \circ t_x\colon X_T \to X_T \to X$ is equal to the composition $m\circ(\mathrm{id}_X\times x)\colon X_T = X\times_k T \to X\times_k X \to X$, we find

$$t_x^*L_T \cong (\mathrm{id}_X \times x)^*m^*L\,.$$

Note that we can write L_T as $L_T = (\mathrm{id}_X \times x)^*p_1^*L$. This gives

$$\begin{aligned} t_x^*L_T \otimes L_T^{-1} &\cong (\mathrm{id}_X \times x)^*\Lambda(L) \otimes (\mathrm{id}_X \times x)^*p_2^*L \\ &= (\mathrm{id}_X \times x)^*\Lambda(L) \otimes (\mathrm{pr}_T^*x^*L)\,. \end{aligned}$$

Using the defining properties of $K(L)$ as given in Lemma (2.4), the assertion of (i) readily follows from this formula.

For (ii) note that $t_x^*L_T \otimes L_T^{-1}$ restricts to x^*L on $\{0\}\times T$.

For (iii), take $T = K(L)$, and let $x\colon K(L) \to X$ be the inclusion. Then

$$\begin{aligned} t_x^*L_T \otimes L_T^{-1} &\cong \bigl(m^*L \otimes p_1^*L^{-1}\bigr)_{|X\times K(L)} \\ &= \Lambda(L)_{|X\times K(L)} \otimes \bigl(p_2^*L\bigr)_{|X\times K(L)} \\ &= p_2^*M \otimes p_2^*(L_{|K(L)}) \qquad \text{for some line bundle } M \text{ on } K(L), \end{aligned}$$

whereas $x^*L = L_{|K(L)}$. Now apply (ii) to find that $M = O_{K(L)}$. $\square$

(2.18) Proposition. *The subscheme $K(L)$ is a subgroup scheme of X.*

Proof. Strictly speaking we have not yet defined the notion of a subgroup scheme; see Definition (3.7) below. With that definition the proposition boils down to the statement that $K(L)(T) \subset X(T)$ is a subgroup, for any k-scheme T. This follows from (i) of the Lemma together with the Theorem of the Square. $\square$

The following lemma shows that an ample line bundle is invariant under only finitely many translations.

(2.19) Lemma. *If L is ample then $K(L)$ is a finite group scheme.*

Proof. Without loss of generality we may assume that k is algebraically closed. Set $Y := K(L)^0_{\mathrm{red}} \subset X$ which, as we noted in (2.13), is an abelian subvariety of X. The restriction L' of L to Y is again ample. By (iii) of Lemma (2.17) the bundle $\Lambda(L')$ on $Y\times Y$ is trivial. Pulling this bundle back to Y via $(1,-1)\colon Y \to Y\times Y$ gives that $L'\otimes(-1)^*L'$ is trivial on Y. But L' is ample, hence $(-1)^*L'$ and $L'\otimes(-1)^*L'$ are ample too. It follows that $\dim(Y) = 0$. Hence $K(L)$ is finite. $\square$

We would like to have a converse to this fact. To obtain this we first prove the following remarkable result.

(2.20) Proposition. *Let X be an abelian variety over an algebraically closed field k. Let $f\colon X \to Y$ be a morphism of k-varieties. For $x \in X$, let C_x denote the connected component of the fibre over $f(x)$ such that $x \in C_x$, and write F_x for the reduced scheme underlying C_x. Then F_0 is an abelian subvariety of X and $F_x = t_x(F_0) = x + F_0$ for all $x \in X(k)$.*

Proof. Consider the morphism $\varphi\colon X \times F_x \to Y$ obtained by restricting $f \circ m$ to $X \times F_x$. Clearly $\varphi(\{0\} \times F_x) = \{f(x)\}$. Since F_x is complete and connected, the Rigidity Lemma (1.11) implies that φ maps the fibres $\{z\} \times F_x$ to a point. In particular, we find that $f(y-x+F_x) = f(y)$ for all $x, y \in X(k)$. Putting $y = z$, $x = 0$ gives $z + F_0 \subseteq F_z$; putting $y = 0$, $x = z$ gives $-z + F_z \subseteq F_0$. This shows that $F_z = z + F_0$.

To see that F_0 is a subgroup scheme of X we take a geometric point $a \in F_0(k)$. Then obviously $F_a = F_0$ so that $a + F_0 = F_a = F_0$. Since F_0 is reduced, it follows that F_0 is a subgroup scheme of X. By (2.13) it is then an abelian subvariety. □

To illustrate the proposition, suppose X is a simple abelian variety (over $k = \overline{k}$), meaning that it does not have any non-trivial abelian subvarieties. Then the conclusion is that every morphism from X to another k-variety is either constant or finite. So the proposition puts strong restrictions on the geometry of abelian varieties.

We give another interpretation of F_0. For this, let D be an *effective* divisor on X and let $L = O_X(D)$ be the corresponding line bundle. We claim that linear system $|2D|$ has no base-points, i.e., the sections of $L^{\otimes 2}$ define a morphism of X to projective space. To see this we have to show that for every geometric point y of X there exists an element $E \in |2D|$ that does not contain y. Now the Theorem of the Square tells us that the divisors of the form

$$t_x^* D + t_{-x}^* D \tag{2}$$

belong to $|2D|$. It is easy to see that given y there exists a geometric point x such that y does not belong to the support of the divisor (2). This means that the map $\varphi\colon X \to \mathbb{P}(\Gamma(X, L^{\otimes 2})^*)$ defined by the sections of $L^{\otimes 2}$ is a morphism. Note that we also have a morphism

$$f\colon X \to \mathbb{P} = |2D|, \quad x \mapsto t_x^* D + t_{-x}^* D\,.$$

The relation between φ and f shall be discussed in ??.

We now again assume that $k = \overline{k}$. For an effective divisor D on X we define the reduced closed subscheme $H(D) \subset X$ by

$$H(D)(\overline{k}) = \bigl\{x \in X(\overline{k}) \bigm| t_x^* D = D\bigr\}\,.$$

By $t_x^* D = D$ we here mean equality of divisors, not of divisor classes. Clearly $H(D)$ is a subgroup scheme of X.

(2.21) Lemma. *Assume $k = \overline{k}$ and let L be an effective line bundle on the abelian variety X. Let $f\colon X \to \mathbb{P}^n$ be the morphism defined by the sections of $L^{\otimes 2}$. As in (2.20) let F_0 be the reduced connected fibre of f containing 0. Then $H(D)^0 = F_0 = K(L)^0_{\mathrm{red}}$, where the superscript "^{0}" denotes the connected component containing 0.*

Proof. Let $x \in F_0$. It follows from (2.20) that $f \circ t_x = f$. Hence if $s \in \Gamma(X, L^{\otimes 2})$ then s and $t_x^* s$ have the same zero divisor. We apply this to $s = t^2$, where t is a section of L with divisor D. This gives $t_x^* D = D$, i.e., $x \in H(D)$. This shows that $F_0 \subseteq H(D)$, and since F_0 is connected we find $F_0 \subseteq H(D)^0$. Next, it is obvious that $H(D)^0$ is contained in $K(L)^0_{\mathrm{red}}$. To prove that $K(L)^0_{\mathrm{red}} \subseteq F_0$, write L' for the restriction of L to $K(L)^0_{\mathrm{red}}$. By (2.13), $K(L)^0_{\mathrm{red}}$ is an abelian subvariety of X. Clearly it suffices to show that L' is trivial. Now L', hence also $(-1)^* L'$, has a non-trivial global section. On the other hand, $(-1)^* L' \cong (L')^{-1}$, as we have seen already in the proof of (2.19). Hence L' is trivial. □

As we shall see in the next chapters, there exists a quotient $X' := X/F_0$ which is again an abelian variety. The Stein factorisation of the morphism f is given by $X \twoheadrightarrow X' \to \mathbb{P}^n$, and L is the pull-back of a bundle on X'.

(2.22) Proposition. *Let L be a line bundle on an abelian variety X which has a non-zero global section. If $K(L)$ is a finite group scheme then L is ample.*

Proof. We may work over an algebraic closure of k. (Note that if a line bundle L becomes ample after extension of the ground field then it is already ample.) Let D be the divisor of the given section. By (2.21) the fibre F_0 is reduced to a point and by (2.20) it follows that f is quasi-finite. Since f is also proper, it is finite. By general theory (see HAG, Chap. III, Exercise 5.7), if the sections of $L^{\otimes 2}$ define a finite morphism $X \to \mathbb{P}^n$ then L is ample. □

(2.23) Corollary. *Let D be an effective divisor on an abelian variety X over an algebraically closed field. Set $L = O_X(D)$. Then the following are equivalent:*
(a) *$H(D)$ is finite,*
(b) *$K(L)$ is finite,*
(c) *L is ample.*

For later use we introduce some terminology.

(2.24) Definition. A line bundle L on an abelian variety is said to be *non-degenerate* if $K(L)$ is finite.

So, an *effective* line bundle is non-degenerate if and only if it is ample.

(2.25) Theorem. *An abelian variety is a projective variety.*

Proof. We first prove this for $k = \overline{k}$. Choose a quasi-affine open subset $U \subset X$ such that $X \setminus U = \cup_{i \in I} D_i$ for certain prime divisors D_i. Set $D = \sum_{i \in I} D_i$. By the preceding results it suffices to show that $H(D)$ is finite. If $x \in H(D)$ then t_x transforms U into itself. Assuming—as we may—that $0 \in U$, we find that $H(D)$ is contained in U. But $H(D)$ is proper, since $F_0 = H(D)^0$ (as in (2.21)). It follows that $H(D)$ is finite.

If k is arbitrary, we first choose an ample divisor $D \subset X_{\overline{k}}$. Then D is defined over a finite extension k' of k. If k' is Galois over k (which we may assume if k'/k is separable) then

$$\tilde{D} := \sum_{\sigma \in \mathrm{Gal}(k'/k)} {}^{\sigma}D$$

is an ample divisor on $X_{\overline{k}}$ which descends to X. If k'/k is purely inseparable such that $\alpha^{p^m} \in k$ for all $\alpha \in k'$ then $p^m \cdot D$ is an ample divisor which descends to X (clear from working at charts). Combination of these two cases gives the theorem. □

(2.26) We give another proof of the theorem. Choose a collection of prime divisors $D_1, \ldots, D_n$, all containing 0, such that the (scheme-theoretic) intersection $\cap_{i=1}^n D_i$ reduces to the single closed point 0. Set $D = \sum_{i=1}^n D_i$. We claim that $3D$ is a very ample divisor. To prove this we may pass to an algebraic closure of the ground field, so we will now assume that $k = \overline{k}$.

First let us show that the linear system $|3D|$ separates points. Thus, given points $P \neq Q$ of X we want to find a divisor Δ, linearly equivalent to $3D$, with $P \in \mathrm{Supp}(\Delta)$ but $Q \notin \mathrm{Supp}(\Delta)$. The divisor we take shall be of the form

$$\Delta = \sum_{i=1}^n t_{a_i}^* D_i + t_{b_i}^* D_i + t_{-a_i-b_i}^* D_i \tag{3}$$

for certain points a_i, $b_i \in X$. Note that by the Theorem of the Square, any divisor of this form is linearly equivalent to $3D$. As $P \neq Q$ and $\cap D_i = \{0\}$, one of the D_i does not contain $P - Q$. Say it is D_1. Take $a_1 = P$, and choose the points b_1, a_i and b_i (for $2 \leqslant i \leqslant n$) such that Q is not in the support of

$$t_{b_1}^* D_1 + t_{-P-b_1}^* D_1 + \sum_{i=2}^n t_{a_i}^* D_i + t_{b_i}^* D_i + t_{-a_i-b_i}^* D_i\,. \tag{4}$$

With these choices the divisor Δ given by (3) has the required properties.

Essentially the same argument shows that $|3D|$ also separates tangent vectors. Namely, suppose $P \in X$ and $0 \neq \tau \in T_{X,P}$. As the scheme-theoretic intersection $\cap_{i=1}^n D_i$ reduces to the single closed point 0, there is an index i such that $t_{-P}^* \tau \in T_{X,0}$ does not lie in the subspace $T_{D_i,0} \subset T_{X,0}$. Say this holds for $i = 1$. Take $a_1 = P$, and take the remaining points a_i and b_i such that P is not in the support of the divisor given by (4). This gives a divisor Δ with $P \in \mathrm{Supp}(\Delta)$ but τ not tangent to Δ. $\square$

Later we shall prove that if D is an ample divisor on an abelian variety, then $3D$ is very ample. In general $2D$ will not be very ample. For an example, take an elliptic curve E and let $D = P$, a point. Then $L(2P) = \Gamma\bigl(E, O(2P)\bigr)$ has dimension 2, and $|2P|$ defines a morphism $E \to \mathbb{P}^1$ of degree 2 with ramification divisor of degree 4. (In fact, if $\mathrm{char}(k) \neq 2$ this morphism is ramified in 4 points.)

§ 3. *Projective embeddings of abelian varieties.*

Any smooth projective variety of dimension g can be embedded into $\mathbb{P}^{2g+1}$, see [??]. We shall now show that an abelian variety of dimension g cannot be embedded into $\mathbb{P}^{2g-1}$ and that an embedding into $\mathbb{P}^{2g}$ exists only for elliptic curves and for certain abelian surfaces. So in some sense abelian varieties do not fit easily into projective space; this also helps to explain why it is so difficult to write down explicit examples of abelian varieties.

In the proof of the next result we shall use the Chow ring $\mathrm{CH}(X)$ of X; we could also work with a suitable cohomology theory (e.g., Betti cohomology or étale cohomology). In fact, all we need are a couple of basic formulas which can be found in Fulton's book [1]. The Chow ring of an abelian variety is further studied in Chap. 13.

(2.27) Theorem. *No abelian variety of dimension g can be embedded into $\mathbb{P}^{2g-1}$. No abelian variety of dimension $g \geqslant 3$ can be embedded into $\mathbb{P}^{2g}$.*

Proof. Let X be an abelian variety, $\dim(X) = g$, and suppose we have an embedding $i\colon X \hookrightarrow \mathbb{P} = \mathbb{P}^m$. Consider the exact sequence of sheaves ("adjunction sequence")

$$0 \to T_X \to i^*T_{\mathbb{P}} \to N \to 0\,, \tag{5}$$

where N is the normal bundle of X in $\mathbb{P}$ and T_X (resp. $T_{\mathbb{P}}$) is the tangent bundle of X (resp. $\mathbb{P}$). Write $h \in \mathrm{CH}(X)$ for the class of a hyperplane section and $c_i = c_i(N)$ (for $i = 1, \ldots, g-1$) for the ith Chern class of N. We know that the tangent bundle of X is trivial. Therefore, the equality of total Chern classes resulting from (5) reads:

$$(1+h)^{m+1} = 1 + \sum_{i=1}^{m-g} c_i\,.$$

(See Fulton [1], 3.2.12.) This implies immediately that $h^{m-g+1} = 0$ in $\mathrm{CH}^g(X)$. But $\deg(h^g)$ equals the degree, say d, of X in $\mathbb{P}^m$ which is non-zero. We thus find $m - g + 1 \geqslant g + 1$, i.e., $m \geqslant 2g$.

We now consider the case of an embedding into $\mathbb{P}^{2g}$. The previous argument gives

$$c_g = \binom{2g+1}{g} \cdot h^g\,.$$

Aplying the degree map we find

$$\deg(c_g) = \binom{2g+1}{g} \deg(h^g) = \binom{2g+1}{g} d\,. \tag{6}$$

But since $2\dim(X) = \dim(\mathbb{P}^g)$, the degree of the highest Chern class c_g of the normal bundle N on X is the self-intersection number of X in $\mathbb{P}^{2g}$, (see Fulton [1], §6.3), which is d^2. Together with (6) this gives

$$d = \binom{2g+1}{g}\,.$$

On the other hand, if we apply the Hirzebruch-Riemann-Roch theorem to the line bundle $L = O(1)$ and use that the Chern classes of X vanish we find that

$$\chi(L) = c_1(L)^g/g!\,,$$

where $\chi(L) = \sum_{i=0}^{g} (-1)^i \dim_k H^i(X, L)$ is the Euler-Poincaré characteristic of L. Since $\chi(L) \in \mathbb{Z}$ it follows that $g!$ divides $\deg(h^g) = d$. (For more details on Riemann-Roch see Chapter IX.) But one easily checks that

$$g! \text{ divides } \binom{2g+1}{g} \quad \Rightarrow \quad g < 3\,.$$

This finishes the proof. $\square$

The proof of the theorem shows that the possibilities for $g = 1$ and $g = 2$ are the cubic curves in $\mathbb{P}^2$ and abelian surfaces of degree 10 in $\mathbb{P}^4$. We have met the cubic curves in (1.7). That there exist abelian surfaces of degree 10 in $\mathbb{P}^4$ was shown first by Comessatti in 1909. He considered complex abelian surfaces $\mathbb{C}^2/\Lambda$, where $\Lambda \subset \mathbb{C}^2$ is the lattice obtained from a suitable embedding of $O_K \oplus O_K$, with O_K the ring of integers of $K = \mathbb{Q}(\sqrt{5})$. Horrocks and

Mumford found abelian surfaces in $\mathbb{P}^4$ as zero sets of sections of the Horrocks-Mumford bundle, an indecomposable rank two vector bundle on $\mathbb{P}^2$. For further discussion we refer to Chap. ??.

Exercises.

(2.1) Let $k \subset K$ be a field extension. Let X be a k-variety and F a sheaf of O_X-modules. Write X_K for the K-variety obtained from X by extension of scalars, and let $F_K := (X_K \to X)^*F$. Show that $\dim_k H^0(X,F) = \dim_K H^0(X_K, F_K)$. Also show that $F \cong O_X$ if and only if $F_K \cong O_{X_K}$.

(2.2) Show that the isomorphism in the Theorem of the Cube is canonical. By this we mean that to a given line bundle L on an abelian variety X we can associate an isomorphism $\tau_{X,L}\colon \Theta(L) \xrightarrow{\sim} O_{X\times X\times X}$ in a functorial way, i.e, such that for every homomorphism $f\colon Y \to X$ we have $f^*(\tau_{X,L}) = \tau_{Y,f^*L}$ (via the canonical isomorphisms $\Theta(f^*L) \cong (f\times f\times f)^*\Theta(L)$ and $O_{Y\times Y\times Y} \cong (f\times f\times f)^*O_{X\times X\times X}$).

(2.3) Let X be an abelian variety over an algebraically closed field. Show that every effective divisor on X is linearly equivalent to an effective divisor without multiple components.

(2.4) Prove that no abelian variety of dimension g can be embedded into $(\mathbb{P}^1)^{2g-1}$. Analyze when an abelian variety of dimension g can be embedded into $(\mathbb{P}^1)^{2g}$.

(2.5) Let A and B be two abelian groups, written additively, and let $n \geqslant 0$ be an integer. If $f\colon A \to B$ is a map (not necessarily a homomorphism), define a map $\theta_n(f)\colon A^n \to B$ by

$$\theta_n(f)\big(a_1,\ldots,a_n\big) = \sum_I (-1)^{n+\#I} f(a_I)\,,$$

where I runs over the non-empty subsets of $\{1,2,\ldots,n\}$ and $a_I := \sum_{i\in I} a_i$. For instance, $\theta_0(f)\colon \{0\} \to B$ is the map with value 0 (by convention), $\theta_1(f) = f$, and

$$\theta_2(f)\big(a,a'\big) = f(a+a') - f(a) - f(a')$$
$$\theta_3(f)\big(a,a',a''\big) = f(a+a'+a'') - f(a+a') - f(a+a'') - f(a'+a'') + f(a) + f(a') + f(a'')\,.$$

(i) Show that $\theta_n(f)\colon A^n \to B$ is symmetric, i.e., invariant under the action of the group S_n on A^n by permutation of the factors.

(ii) For $n \geqslant 1$, show that we have a relation

$$\theta_{n+1}(f)\big(a_1,\ldots,a_n,a_{n+1}\big) = \theta_n(f)\big(a_1,\ldots,a_n+a_{n+1}\big) - \theta_n(f)\big(a_1,\ldots,a_n\big) - \theta_n(f)\big(a_1,\ldots,a_{n+1}\big)\,.$$

(iii) Use (i) and (ii) to show that $\theta_{n+1}(f) = 0$ if and only if the map $\theta_n(f)\colon A^n \to B$ is n-linear.

(iv) Let L be a line bundle on an abelian variety X over a field k. If T is a k-scheme, show that the map $X(T)\times X(T) \to \mathrm{Pic}(T)$ given by $(x_1,x_2) \mapsto (x_1+x_2)^*L \otimes x_1^*L^{-1} \otimes x_2^*L^{-1}$ is bilinear.

Notes. The Theorem of the Square and of the Cube are the pivotal theorems for divisors or line bundles on abelian varieties. They are due to Weil [3]. Our discussion owes much to Mumford's book MAV. Solomon Lefschetz (1884–1972) gave a criterion for complex tori to be embeddable into projective space. This was remodelled by Weil to give the projectivity of abelian varieties; see Weil [5]. Our first proof of Theorem (2.25) follows MAV; the argument given in (2.26) is the one found in Lang [1]. The definition of $K(L)$ goes back to Weil. Proposition (2.20) is due to M.V. Nori. Theorem (2.27) is due to Barth [1] and Van de Ven [1].