Chapter VIII. The Theta group of a line bundle.

To a line bundle L on an abelian variety X we shall associate a group scheme, the *theta group* $\mathscr{G}(L)$ of L. If the class of L is in $\mathrm{Pic}^0_{X/k} = X^t$ then $\mathscr{G}(L)$ is an extension of X by the multiplicative group $\mathbb{G}_m$ and is commutative. If $[L] \notin \mathrm{Pic}^0_{X/k}$ then $\mathscr{G}(L)$ is much smaller and in general not commutative. The theta group is a convenient tool for studying when a line bundle descends over an isogeny. Further we study the structure of so-called non-degenerate theta groups, and their representations.

§ 1. *The theta group $\mathscr{G}(L)$.*

(8.1) Let X be an abelian variety over a field k. Let L be a line bundle on X. Write $\mathbb{L} = \mathbb{V}(L^\vee)$ for the corresponding geometric line bundle over X.

For a k-scheme T define $\mathscr{G}(L)(T)$ to be the set of pairs (x,φ) where $x \in X(T)$ and where $\varphi\colon L_T \to t_x^*L_T$ is an isomorphism. Geometrically this means that we have $\varphi_{\mathbb{L}}\colon \mathbb{L}_T \xrightarrow{\sim} \mathbb{L}_T$, fibrewise linear, fitting in a commutative diagram

$$\begin{array}{ccc} \mathbb{L}_T & \xrightarrow{\varphi_{\mathbb{L}}} & \mathbb{L}_T \\ \downarrow & & \downarrow \\ X_T & \xrightarrow[t_x]{} & X_T. \end{array}$$

Note that x is uniquely determined by φ, so that $\mathscr{G}(L)(T)$ is in natural bijection with the set of $\varphi_{\mathbb{L}}\colon \mathbb{L}_T \xrightarrow{\sim} \mathbb{L}_T$ lying over a translation on X_T.

The set $\mathscr{G}(L)(T)$ carries a natural group structure, with multiplication given by $(x_1,\varphi_1)\cdot(x_2,\varphi_2) = (x_1+x_2, t_{x_2}^*\varphi_1\circ\varphi_2)$. Since the association $T \mapsto \mathscr{G}(L)(T)$ is functorial in T we obtain a group functor $\mathscr{G}(L)\colon (\mathsf{Sch}_{/k})^0 \to \mathsf{Gr}$.

The (fibrewise linear) automorphisms of $\mathbb{L}_T$ lying over the identity on X_T are just the multiplications by elements of $\Gamma(X_T, O_{X_T})^* = \Gamma(T, O_T)^*$. This gives an identification of $\mathbb{G}_{m,k}$ with the kernel of the natural homomorphism (of group functors) $\mathscr{G}(L) \to K(L) \subset X$. Notice that $\mathbb{G}_{m,k}$ is central in $\mathscr{G}(L)$.

(8.2) Lemma. *The group functor $\mathscr{G}(L)$ is representable. There is an exact sequence of group schemes*

$$0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathscr{G}(L) \longrightarrow K(L) \longrightarrow 0\,, \qquad (1)$$

where the map $\mathscr{G}(L) \to K(L)$ is given on points by $(x,\varphi) \mapsto x$.

Proof. Since the functor $K(L)$ is representable, it suffices to show that the homomorphism $\pi\colon \mathscr{G}(L) \to K(L)$ is (relatively) representable by a $\mathbb{G}_m$-torsor. So, let T be a k-scheme and $x \in K(L)(T)$. Write

$$M := \mathrm{pr}_{T,*}(L_T^{-1} \otimes t_x^* L_T)\,,$$

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which is a line bundle on T. If T' is a T-scheme then the $\varphi\colon \mathbb{L}_{T'} \xrightarrow{\sim} \mathbb{L}_{T'}$ such that $(x,\varphi) \in \mathscr{G}(L)(T')$ are precisely the nowhere vanishing sections of $M_{T'}$. Thus, writing $\mathbb{M}^*$ for the $\mathbb{G}_{m,T}$-torsor corresponding to M (i.e., the T-scheme obtained from the geometric line bundle $\mathbb{M} := \mathbb{V}(M^\vee)$ by removing the zero section), we find that the fibre $\pi^{-1}(x)$ is representable by the T-scheme $\mathbb{M}^*$. That the sequence (1) is exact (even as a sequence of Zariski sheaves on $\mathsf{Sch}_{/k}$) is clear from the remarks preceding the lemma and the definition of $K(L)$. $\square$

(8.3) We indicate another proof of (8.2). For this, consider the $\mathbb{G}_m$-torsor $\mathbb{L}^*$ over X associated to L. Write $\xi\colon \mathbb{L}^* \to X$ for the structure morphism. Let $Y = \xi^{-1}(K(L)) = K(L) \times_X \mathbb{L}^*$, the scheme obtained by pulling back $\mathbb{L}^*$ via the inclusion map $K(L) \hookrightarrow X$. Choose a k-rational point $P \in \mathbb{L}^*(0)$. (This gives a trivialization $\mathbb{L}^*(0) \cong \mathbb{G}_{m,k}$.) We obtain a morphism $r_P\colon \mathscr{G}(L) \to Y$ by sending a point $(x,\varphi) \in \mathscr{G}(L)(T)$ to the image point $\varphi_{\mathbb{L}}(P) \in Y(T) \subset \mathbb{L}^*(T)$. It is not difficult to see that r_P gives an isomorphism of (set-valued) functors. So $\mathscr{G}(L)$ is represented by the scheme $Y = \xi^{-1}(K(L))$.

(8.4) Definition. Let L be a line bundle on an abelian variety. The group scheme $\mathscr{G}(L)$ is called the *theta group* of L.

(8.5) Consider the morphism $[\ ,\ ]\colon \mathscr{G}(L)\times\mathscr{G}(L) \to \mathscr{G}(L)$ given on points by $(g_1,g_2) \mapsto [g_1,g_2] = g_1g_2g_1^{-1}g_2^{-1}$. Since $K(L)$ is commutative this morphism factors through $\mathbb{G}_m \subset \mathscr{G}(L)$. The fact that $\mathbb{G}_m$ is central in $\mathscr{G}(L)$ then implies that $[\ ,\ ]$ factors modulo $\mathbb{G}_m \times \mathbb{G}_m$. We thus obtain a pairing

$$e^L\colon K(L) \times K(L) \to \mathbb{G}_m\,,$$

called the *commutator pairing* of the theta group. Note that e^L is alternating, meaning that $e^L(x,x) = 1$ for every $x \in K(L)$. For fixed $x \in K(L)(T)$ the morphisms $K(L)_T \to \mathbb{G}_{m,T}$ given by $y \mapsto e^L(x,y)$ resp. $y \mapsto e^L(y,x)$ are homomorphisms. In sum we find that the theta group $\mathscr{G}(L)$ gives rise to an alternating bilinear form e^L.

The alternating form e^L has the following properties.

(8.6) Proposition. (i) *If $f\colon X \to Y$ is a homomorphism of abelian varieties and L is a line bundle on Y then*

$$e^{f^*(L)} = e^L \circ (f,f) \qquad \text{on } f^{-1}(K(L)) \times f^{-1}(K(L))\,.$$

(ii) *If L and M are line bundles on X then $e^{L\otimes M} = e^L \cdot e^M$ on $K(L) \cap K(M)$.*
(iii) *If $[L] \in \mathrm{Pic}^0_{X/k}$ then $e^L = 1$.*
(iv) *For $x \in K(L)$ and $y \in n_X^{-1}(K(L)) = K(L^n)$ we have $e^{L^n}(x,y) = e^L(x,ny)$.*

Proof. (i) Note that $f^{-1}(K(L)) \subseteq K(f^*L)$, for if $x \in X$ then

$$t_x^* f^* L = f^*(t_{f(x)}^* L)\,. \tag{2}$$

Now suppose T is a k-scheme and $x_1, x_2 \in f^{-1}(K(L))(T)$. We can cover T by Zariski-open subsets U such that there exist automorphisms $\varphi_{1,\mathbb{L}}$ and $\varphi_{2,\mathbb{L}}$ of the geometric line bundle $\mathbb{L}_U$, lying over the translations $t_{f(x_1)}$ and $t_{f(x_2)}$, respectively. As it suffices to show that $e^{f^*(L)} = e^L \circ (f,f)$ on $[f^{-1}(K(L)) \times f^{-1}(K(L))](U)$ for every such U, we may replace T by U.

By construction, the automorphism $[\varphi_{1,\mathbb{L}}, \varphi_{2,\mathbb{L}}]$ of $\mathbb{L}$ is the (fibrewise) multiplication by $e^L(f(x_1), f(x_2))$. Then $f^*\varphi_{1,\mathbb{L}}$ and $f^*\varphi_{2,\mathbb{L}}$ are automorphisms of $f^*\mathbb{L}$ which, by formula (2), lie over the translations t_{x_1} resp. t_{x_2} on X. Since clearly $[f^*\varphi_{1,\mathbb{L}}, f^*\varphi_{2,\mathbb{L}}] = f^*[\varphi_{1,\mathbb{L}}, \varphi_{2,\mathbb{L}}]$, we find that $e^{f^*(L)}(x_1,x_2) = e^L(f(x_1), f(x_2))$.

(ii) If we have elements (φ_1, x), $(\varphi_2, y) \in \mathscr{G}(L)(T)$ and (ψ_1, x), $(\psi_2, y) \in \mathscr{G}(M)(T)$ then $(\varphi_{1,\mathbb{L}} \otimes \psi_{1,\mathbb{M}}) \circ (\varphi_{2,\mathbb{L}} \otimes \psi_{2,\mathbb{M}}) = (\varphi_{1,\mathbb{L}} \circ \varphi_{2,\mathbb{L}}) \otimes (\psi_{1,\mathbb{M}} \circ \psi_{2,\mathbb{M}})$ as (fibrewise linear) automorphisms of $\mathbb{L} \otimes \mathbb{M}$. The claim readily follows from this.

(iii) If the class of L is in Pic^0 then $K(L) = X$, and since X is complete the pairing $e^L \colon X \times X \to \mathbb{G}_m$ must be constant.

(iv) For $n \geqslant 0$ this follows by induction from (ii) and the bilinearity of the pairing e^L. The case $n \leqslant 0$ then follows from (ii) and (iii). □

Let k be a field. As we have seen in (4.41), the category $\boldsymbol{C}$ of commutative group schemes of finite type over k is abelian. In particular, given objects A and B of $\boldsymbol{C}$ we can form the groups $\mathrm{Ext}^n_{\boldsymbol{C}}(A,B)$ of n-extensions of A by B. If there is no risk of confusion we shall simply write $\mathrm{Ext}(A,B)$ for $\mathrm{Ext}^1_{\boldsymbol{C}}(A,B)$. Thus, the elements of $\mathrm{Ext}(A,B)$ are equivalence classes of exact sequences

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

where E is again an object of $\boldsymbol{C}$.

It can be shown (but this requires some work) that $\boldsymbol{C}$ does not contain any injective or projective objects. In particular, the computation of Ext-groups by homological methods cannot be done "directly" in $\boldsymbol{C}$. To repair this, one may work in an Ind- or Pro-category, cf. ?? and ??.

We shall further discuss extensions of group schemes in Chapter ??. In this chapter we only need the following two facts.

(8.7) Fact. *Let k be an algebraically closed field.*

(i) *Write $\boldsymbol{C}$ for the category of commutative group schemes of finite type over k. If G is a finite commutative k-group scheme then $\mathrm{Ext}^1_{\boldsymbol{C}}(G, \mathbb{G}_m) = 0$. In other words, for every extension $0 \to \mathbb{G}_{m,k} \to \mathscr{G} \to G \to 0$ with $\mathscr{G}$ commutative, there exists a section $s \colon G \to \mathscr{G}$ which is a homomorphism of group schemes.*

(ii) *Let G be a finite k-group scheme of prime order. If $0 \to \mathbb{G}_{m,k} \to \mathscr{G} \to G \to 0$ is an exact sequence of k-group schemes then $\mathscr{G}$ is commutative.*

(8.8) We shall use the notion of a theta group to obtain an interpretation of $X^t = \mathrm{Pic}^0_{X/k}$ as being $\mathrm{Ext}_{\boldsymbol{C}}(X, \mathbb{G}_m)$, where $\boldsymbol{C}$ is the category of commutative k-group schemes of finite type.

In one direction this is quite easy. Namely, suppose that L is a line bundle on X which gives a class in Pic^0. Then $K(L) = X$ and the pairing e^L is trivial. This means that $G = \mathscr{G}(L)$ is a commutative group scheme fitting in an exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow X \longrightarrow 0. \qquad (3)$$

Thus, if $[L] \in \mathrm{Pic}^0_{X/k}$ then $\mathscr{G}(L)$ gives an element of $\mathrm{Ext}(X, \mathbb{G}_m)$.

Conversely, suppose G is a commutative k-group scheme for which we have an exact sequence (3). Then G can be viewed as a $\mathbb{G}_m$-torsor over X. Write L_G for the corresponding line bundle on X. (See Appendix ??) We claim that L_G is a line bundle in $\mathrm{Pic}^0_{X/k}$ with theta group isomorphic to G. To see this, suppose that G_1 and G_2 are commutative k-group schemes and that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{j_1} & G_1 & \xrightarrow{\pi_1} & X & \longrightarrow & 0 \\ & & \gamma\big\downarrow & & \big\downarrow\varphi & & \big\downarrow f & & \\ 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{j_2} & G_2 & \xrightarrow{\pi_2} & X & \longrightarrow & 0 \end{array}$$

with exact rows. Here f is only required to be a morphism of schemes and φ is required to be "fibrewise linear" (meaning that $\varphi(j_1(c)\cdot g) = j_2\gamma(c)\cdot\varphi(g)$ for all $c \in \mathbb{G}_m$ and $g \in G_1$.) Then φ gives an isomorphism of $\mathbb{G}_m$-torsors $G_1 \xrightarrow{\sim} f^*G_2$, hence it induces a homomorphism $\varphi\colon L_{G_1} \xrightarrow{\sim} f^*L_{G_2}$. Now take $G_1 = G_2 = G$ and let $\varphi = t_g$ be the translation over an element $g \in G$. If $x \in X$ is the image of g then we obtain a pair $(x, \varphi) \in \mathscr{G}(L_G)$. Since this construction is obviously functorial, it gives a homomorphism $h\colon G \to \mathscr{G}(L_G)$, compatible with the projections to X. In particular this shows that $K(L_G) = X$, so that the class of L_G is in $\mathrm{Pic}^0_{X/k}$. Furthermore it is clear that h is injective, and it follows that h is an isomorphism.

In sum, we can pass from line bundles L on X with $[L] \in \mathrm{Pic}^0$ to commutative group schemes G as in (3) and vice versa.

(8.9) Theorem. *Let X be an abelian variety over a field k. Write $\boldsymbol{C}$ for the (abelian) category of commutative group schemes of finite type over k. Associating $\mathscr{G}(L)$ to a line bundle L with $[L] \in \mathrm{Pic}^0_{X/k}$ gives an isomorphism $X^t(k) \xrightarrow{\sim} \mathrm{Ext}^1_{\boldsymbol{C}}(X, \mathbb{G}_m)$.*

Proof. All that remains to be shown is that $L \cong L_{\mathscr{G}(L)}$ as line bundles on X. This follows from the construction in (8.3), as it shows that $\mathscr{G}(L)$ is (non-canonically) isomorphic to $\mathbb{L}^*$ as a $\mathbb{G}_m$-torsor. □

We shall later extend this result, obtaining an isomorphism of group schemes $X^t \xrightarrow{\sim} \mathscr{E}xt(X, \mathbb{G}_m)$. The main problem here is to set up a framework in which we can define $\mathscr{E}xt(X, \mathbb{G}_m)$ correctly.

§ 2. *Descent of line bundles over homomorphisms.*

Theta groups are a useful tool in studying when a line bundle on an abelian variety descends over an isogeny. The basic result is in fact just a reformulation of what we have seen in (7.2).

(8.10) Theorem. *Let $f\colon X \to Y$ be a surjective homomorphism of abelian varieties. Let L be a line bundle on X. Then there is a bijective correspondence between the $M \in \mathrm{Pic}(Y)$ with $f^*M \cong L$ and the homomorphisms $\mathrm{Ker}(f) \to \mathscr{G}(L)$ lying over the natural inclusion $\mathrm{Ker}(f) \hookrightarrow X$.*

Note that such homomorphisms $\mathrm{Ker}(f) \to \mathscr{G}(L)$ can only exist if $\mathrm{Ker}(f) \subseteq K(L)$ and $\mathrm{Ker}(f)$ is totally isotropic for the pairing e^L.

Proof. Write V_1 for the set of isomorphism classes of pairs (M, α) where M is a line bundle on Y and $\alpha\colon f^*M \xrightarrow{\sim} L$. Write V_2 for the set of isomorphism classes of line bundles M on Y such that $f^*M \cong L$. Using that $\mathrm{Aut}(M) = k^* = \mathrm{Aut}(L)$ we see that the natural map $V_1 \to V_2$ (forgetting α) is a bijection.

Write $H = \mathrm{Ker}(f)$. Then Y represents the fppf quotient of X by H. We have seen in (7.2) that the pairs $(M, \alpha) \in V_1$ correspond to the H-actions on L compatible with the natural action of H on X. It is an immediate translation of the definitions that such H-actions correspond to homomorphisms $H \to \mathscr{G}(L)$ lifting the inclusion $H \hookrightarrow X$. □

For isogenies over an algebraically closed field this leads to a handy criterion for when a line bundle descends. To prove it we shall make use of a result about extensions that we stated above.

(8.11) Corollary. *Let X and Y be abelian varieties over an algebraically closed field k. Let $f\colon X \to Y$ be an isogeny. Then a line bundle L on X is the pull-back of a line bundle on Y if and only if* $\mathrm{Ker}(f)$ *is a subgroup scheme of $K(L)$ which is totally isotropic with respect to the pairing e^L.*

Proof. According to the preceding theorem one must check whether $\mathrm{Ker}(f) \hookrightarrow X$ can be lifted to a homomorphism $\mathrm{Ker}(f) \to \mathscr{G}(L)$. If it can then $\mathrm{Ker}(f)$ is a subgroup scheme of $K(L)$ and e^L is trivial when restricted to $\mathrm{Ker}(f) \times \mathrm{Ker}(f)$.

Conversely, if $\mathrm{Ker}(f)$ is a totally isotropic subgroup scheme of $K(L)$ then we consider the extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \pi^{-1}(\mathrm{Ker}(f))) \longrightarrow \mathrm{Ker}(f) \longrightarrow 0\,,$$

where $\pi\colon \mathscr{G}(L) \to K(L)$ is the projection. Since we assume $\mathrm{Ker}(f)$ to be totally isotropic, the group scheme $G := \pi^{-1}(\mathrm{Ker}(f)))$ is commutative. By (8.7), the extension splits, i.e., there exists a (homomorphic) section $\mathrm{Ker}(f) \to G$. $\square$

(8.12) Remarks. (i) In the "if" statement of the theorem we really need the assumption that $k = \bar{k}$: if k is an arbitrary field and $\mathrm{Ker}(f)$ is a totally isotropic subgroup scheme of $K(L)$ then in general L descends to a line bundle on Y only after we pass to a finite extension of k.

(ii) The condition in (8.11) that the kernel of f is finite is necessary. If $K(L)$ is not finite (i.e., L is degenerate) then $Y := K(L)^0_{\mathrm{red}}$ is a nonzero abelian subvariety of X (assuming the ground field is perfect), and the quotient $Z = X/Y$ exists as an abelian variety; see Example (4.40). In this situation L is not, in general, the pullback of a line bundle on Z, even though $Y \subset K(L)$ is totally isotropic with respect to e^L. For example, if the class of L is in $\mathrm{Pic}^0_{X/k}$ then $Y = X$, so if L is non-trivial it is not a pullback from $Z = \{0\}$.

If $q\colon X \to Z$ is the quotient map then possibly after replacing the ground field by a finite separable extension it is still true that there exists a line bundle M on Z such that $L \otimes q^*M^{-1}$ is in $\mathrm{Pic}^0_{X/k}$; see Exercise (11.3) below.

(8.13) Definition. A *level subgroup* of the theta group $\mathscr{G}(L)$ is a subgroup scheme $\tilde{H} \subset \mathscr{G}(L)$ such that $\mathbb{G}_m \cap \tilde{H} = \{1\}$, i.e., $\tilde{H}$ maps isomorphically to its image $H \subset K(L)$ under π.

With this notion of a level subgroup we have the following corollary to the theorem.

(8.14) Corollary. *Let L be a line bundle on an abelian variety X over a field k. Then there is a bijective correspondence between the set of level subgroups $\tilde{H} \subset \mathscr{G}(L)$ and the set of isomorphism classes of pairs (f, M) where $f\colon X \to Y$ is a surjective homomorphisms and M is a line bundle on Y with $f^*M \cong L$. If $\tilde{H}$ corresponds to the pair (f, M) then* $\mathrm{Ker}(f) = \pi(\tilde{H})$.

Proof. Given a level subgroup $\tilde{H} \subset \mathscr{G}(L)$, set $H := \pi(\tilde{H}) \subset K(L)$ and write $\xi\colon H \xrightarrow{\sim} \tilde{H} \subset \mathscr{G}(L)$ for the inverse of $\pi_{|\tilde{H}}$. The projection $f\colon X \to X/H =: Y$ is a surjective homomorphism and Theorem (8.10) shows that ξ corresponds to a line bundle M on Y with $f^*M \cong L$.

Conversely, if $f\colon X \to Y$ is a surjective homomorphism and M is a line bundle on Y with $f^*M \cong L$ then the image of the corresponding homomorphism $\mathrm{Ker}(f) \to \mathscr{G}(L)$ is a level subgroup. One now easily verifies that these two constructions give the desired bijection. $\square$

Given a pair (f, M) as in the corollary, we can describe the theta group $\mathscr{G}(M)$ in terms of $\mathscr{G}(L)$ and the level subgroup $\tilde{H}$.

(8.15) Proposition. *Let $f\colon X \to Y$ be a surjective homomorphism of abelian varieties. Let L*

be a line bundle on X and let M be a line bundle on Y with $f^*M \cong L$. Write $\tilde{H} \subset \mathscr{G}(L)$ for the level subgroup corresponding to the pair (f, M). Then $f^{-1}\big(K(M)\big) \subseteq K(L)$, the centralizer $C_{\tilde{H}}$ of $\tilde{H}$ inside $\mathscr{G}(L)$ is given by

$$C_{\tilde{H}} = \{g \in \mathscr{G}(L) \mid \pi(g) \in f^{-1}(K(M))\}\,,$$

and $\mathscr{G}(M) \cong C_{\tilde{H}}/\tilde{H}$.

Proof. As already remarked in the proof of (8.6), we have $f^{-1}\big(K(M)\big) \subseteq K(L)$. Write $H = \mathrm{Ker}(f) = \pi(\tilde{H}) \subset K(L)$. Let $\xi\colon H \to \mathscr{G}(L)$ be the homomorphism giving the canonical H-action on L. By construction, $\tilde{H}$ is the image of ξ. As remarked after (7.2), such an H-action on L (compatible with the H-action on X by translations) is nothing but a descent datum on L with respect to the morphism f.

Let T be a k-scheme and $(x, \varphi) \in \mathscr{G}(L)(T)$. Write $y = f(x) \in Y(T)$. Then $t_x^*\xi\colon H \to \mathscr{G}(t_x^*L)$ gives a descent datum on t_x^*L. This descent datum corresponds to the line bundle t_y^*M on Y, and we have a natural identification $f^*(t_y^*M) = t_x^*L$. Now the isomorphism $\varphi\colon L \xrightarrow{\sim} t_x^*L$ descends to an isomorphism $\psi\colon M \xrightarrow{\sim} t_y^*M$ if and only if φ is equivariant with respect to the descent data ξ and $t_x^*\xi$. This last condition precisely means that $(x, \varphi) \cdot (h, \xi(h)) = (h, \xi(h)) \cdot (x, \varphi)$ for all $(h, \xi(h)) \in \tilde{H}$, i.e., $(x, \varphi) \in C_{\tilde{H}}$. Thus we obtain a homomorphism $\gamma\colon C_{\tilde{H}} \to \mathscr{G}(M)$.

By construction, if (x, φ) maps to (y, ψ) then $f^*\psi = \varphi$ as homomorphisms from L to t_x^*L. Thus, if $(x, \varphi) \in \mathrm{Ker}(\gamma)$ then $x \in H = \mathrm{Ker}(f)$ and $\varphi = \xi(x)\colon L \xrightarrow{\sim} t_x^*L$. This means precisely that $(x, \varphi) \in \tilde{H} \subset C_{\tilde{H}}$. (Note that $\tilde{H}$ is commutative, being isomorphic to H, so that $\tilde{H}$ is indeed contained in $C_{\tilde{H}}$.)

Conversely, if $(\psi, y) \in \mathscr{G}(M)(T)$, then there is an fppf cover $T' \to T$ and an $x \in X(T')$ with $f(x) = y$. Then $(f^*\psi, x)$ is an element of $C_{\tilde{H}}(T')$ with $\gamma(f^*\psi, x) = (\psi, y)$. Thus γ is surjective and $\mathscr{G}(M) \cong C_{\tilde{H}}/\tilde{H}$.

Finally, it is clear from the above that $C_{\tilde{H}} \subseteq \{g \in \mathscr{G}(L) \mid \pi(g) \in f^{-1}(K(M))\}$. Conversely, if $g = (\varphi, x) \in \mathscr{G}(L)$ with $f(x) \in K(M)$ then we have shown that there exists an element of the form (φ', x) in $C_{\tilde{H}}$. As $C_{\tilde{H}}$ clearly contains the central subgroup $\mathbb{G}_{m,k} \subset \mathscr{G}(L)$, it follows that also $g \in C_{\tilde{H}}$. □

If x is a T-valued point of $K(L)$ for some k-scheme T then $y \mapsto e^L(x, y)$ defines a homomorphism $K(L)_T \to \mathbb{G}_{m,T}$. The bilinearity of the pairing e^L implies that the map $K(L) \to \mathit{Hom}\big(K(L), \mathbb{G}_m\big)$ given on points by $x \mapsto e^L(x, -)$ is a homomorphism of group schemes.

(8.16) Corollary. *In the situation of the proposition we have $f^{-1}(K(M)) = H^\perp := \{k \in K(L) \mid e^L(k, h) = 1$ for every $h \in H\}$. We have $K(M) \cong H^\perp/H$.*

Proof. It easily follows from the definition of e^L that $C_{\tilde{H}} = \{(x, \varphi) \in \mathscr{G}(L) \mid x \in H^\perp\}$. With this remark the corollary directly follows from the proposition. □

§ 3. *Theta groups of non-degenerate line bundles.*

It will be helpful to reformulate some of the notions we have encountered without reference to line bundles.

(8.17) Definition. Let k be a field. A *theta-group over k* is an exact sequence of k-group schemes

$$0 \longrightarrow \mathbb{G}_{m,k} \xrightarrow{i} \mathscr{G} \xrightarrow{\pi} K \longrightarrow 0$$

such that $i(\mathbb{G}_{m,k})$ is contained in the center of $\mathscr{G}$ and K is commutative. The commutator pairing e: $K \times K \to \mathbb{G}_{m,k}$ of the theta group is the alternating bilinear pairing induced by the commutator $[\ ,\ ]$: $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$. We say that two theta groups are isomorphic if they are isomorphic as extensions of a group scheme K by $\mathbb{G}_{m,k}$.

Suppose that we have a theta group as above such that K is finite. If T is a k-scheme and $x \in K(T)$ then $y \mapsto e(x,y)$ defines a homomorphism $K_T \to \mathbb{G}_{m,T}$, i.e., an element of $K^D(T)$. In this way the pairing e gives a homomorphism ν: $K \to K^D$. The relation $e(x,y) = e(y,x)^{-1}$ gives that $\nu^D = \nu^{-1}$.

(8.18) Definition. A theta group $\mathscr{G}$ as above is said to be *non-degenerate* if K is finite and if ν: $K \to K^D$ is an isomorphism.

Notice that the non-degeneracy condition can also be expressed by saying that $i(\mathbb{G}_{m,k})$ is the center of $\mathscr{G}$.

As the terminology suggests, the theta group of a non-degenerate line bundle is non-degenerate. This is a consequence of the following result.

(8.19) Proposition. *Let L be a non-degenerate line bundle on an abelian variety X. If $H \subset K(L)$ is a subgroup scheme which is maximal totally isotropic with respect to the pairing e^L then $H = H^\perp$ and* $\mathrm{rank}(H)^2 = \mathrm{rank}(K(L))$.

Proof. It suffices to prove this over an algebraically closed field. Write f: $X \to X/H =: Y$ for the projection. By (8.11) there is a line bundle M on Y with $f^*M \cong L$.

We claim that $K(M) = \{1\}$. Suppose not. Then there is a subgroup scheme $K' \subset K(M)$ of prime order. By (8.7) this K' is totally isotropic for e^M. Then (i) of (8.6) shows that $f^{-1}(K')$ is totally isotropic for e^L. As $H \subsetneq f^{-1}(K')$ this contradicts our choice of H. So indeed $K(M) = \{1\}$. It then follows from (8.16) that $H^\perp = H$ and by (7.6) we have $\mathrm{rank}(K(L)) = \mathrm{rank}(H)^2$. ☐

(8.20) Corollary. *If L is a non-degenerate line bundle on an abelian variety then the theta group $\mathscr{G}(L)$ is non-degenerate.*

Proof. Choose H as above. Remark that ν: $K(L) \to K(L)^D$ fits in a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^\perp & \longrightarrow & K(L) & \longrightarrow & K(L)/H^\perp & \longrightarrow & 0 \\
 & & \big\downarrow \nu' & & \big\downarrow \nu & & \big\downarrow \bar{\nu} & & \\
0 & \longrightarrow & [K(L)/H]^D & \longrightarrow & K(L)^D & \longrightarrow & H^D & \longrightarrow & 0\,.
\end{array}$$

By definition of $H^\perp$ the homomorphism $\bar{\nu}$ is injective. Now $\nu^D = \nu^{-1}$ so that $\bar{\nu}^D$: $H \to [K(L)/H^\perp]^D$ is the map obtained by restricting ν^{-1} to H. But $H = H^\perp$, so we find that $\nu' = (\bar{\nu}^D)^{-1}$ is surjective. By rank considerations it follows that ν' and $\bar{\nu}$ are isomorphisms. Hence ν is an isomorphism. ☐

(8.21) *Heisenberg groups.* We now discuss an important example of non-degenerate theta-groups, the so-called Heisenberg groups.

We work over a field k. Let H be a finite abelian group; we shall view it as a (constant) k-group scheme. Write $H^D := \mathrm{Hom}(H, \mathbb{G}_{m,k})$ for its Cartier dual. (If $k = \overline{k}$ one would also refer to H^D as the character group of H.) So

$$H \cong (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_n\mathbb{Z})\,, \qquad H^D \cong \mu_{d_1,k} \times \cdots \times \mu_{d_n,k}\,,$$

with $d_1|d_2|\cdots|d_n$. To the pair (H, H^D) we associate a Heisenberg group $\mathscr{H}$; it is defined by $\mathscr{H} = \mathbb{G}_{m,k} \times H \times H^D$ as a k-scheme, with multiplication given by

$$(\lambda, x, \chi) \cdot (\lambda', x', \chi') = (\lambda\lambda'\chi'(x), x + x', \chi\chi')\,. \tag{4}$$

Then $\mathscr{H}$ is a theta-group: we have an exact sequence

$$0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathscr{H} \longrightarrow H \times H^D \longrightarrow 0\,.$$

The commutator pairing $e\colon (H \times H^D) \times (H \times H^D) \to \mathbb{G}_{m,k}$ is given by $e((x,\chi),(x',\chi')) = \chi'(x)\chi(x')^{-1}$. As this is clearly a perfect pairing, $\mathscr{H}$ is non-degenerate.

The construction clearly generalizes to the case where we start with an arbitrary finite commutative k-group scheme H. For $\mathscr{H}$ we now take $\mathbb{G}_{m,k} \times H \times H^D$, and the group structure is again given on points by (4). We refer to the resulting theta group as the Heisenberg group associated to the group scheme H (or to the pair (H, H^D)).

Our next goal is to show that under suitable assumptions the theta group of a line bundle can be described as a Heisenberg group.

(8.22) Lemma. *Let* $0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathscr{G} \xrightarrow{\pi} K \longrightarrow 0$ *be a non-degenerate theta group over a field* k. *Let* $H \subset K$ *be a subgroup scheme. Consider the following conditions.*

(i) H *is maximal totally isotropic w.r.t. the commutator pairing* $e\colon K \times K \to \mathbb{G}_{m,k}$,

(ii) H *is totally isotropic and* $\mathrm{rank}(H)^2 = \mathrm{rank}(K)$,

(iii) $H = H^\perp$.

Then (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i). *If* k *is algebraically closed the three conditions are equivalent.*

Proof. The isomorphism $\nu\colon K \xrightarrow{\sim} K^D$ induces an isomorphism $K/H^\perp \xrightarrow{\sim} H^D$. In particular, $\mathrm{rank}(K) = \mathrm{rank}(H) \cdot \mathrm{rank}(H^\perp)$. Now H is totally isotropic precisely if $H \subseteq H^\perp$. This readily gives (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i).

To see that (i) $\Rightarrow$ (iii) if $k = \overline{k}$, let H be maximal totally isotropic and assume that $H \subsetneq H^\perp$. By (8.7) the extension $0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \pi^{-1}(H) \xrightarrow{\pi} H \longrightarrow 0$ splits, so there exists a level subgroup $\tilde{H} \subset \mathscr{G}$ with $\pi(\tilde{H}) = H$. Writing $\mathscr{G}' := \pi^{-1}(H^\perp)/\tilde{H}$ we obtain a new theta group $0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathscr{G}' \xrightarrow{\pi'} H^\perp/H \longrightarrow 0$. As $H \neq H'$ and $k = \overline{k}$ we can choose a subgroup scheme $\Gamma \subset H^\perp/H$ of prime order. By (8.7) ${\pi'}^{-1}(\Gamma)$ is commutative. It follows that the inverse image of Γ under $H^\perp \to H^\perp/H$ is totally isotropic. This contradicts the assumption that H is maximal totally isotropic. $\square$

(8.23) Definition. Let $\mathscr{G}$ be a non-degenerate theta group over a field k.

(i) A k-subgroup scheme $H \subset K$ satisfying (ii) and (iii) in (8.22) is called a *Lagrangian subgroup*. If $\tilde{H} \subset \mathscr{G}$ is a level subgroup then we say that $\tilde{H}$ is a *Lagrangian level subgroup* if $\pi(\tilde{H}) \subset K$ is Lagrangian.

(ii) A *Lagrangian decomposition* of K is an isomorphism $K \xrightarrow{\sim} H_1 \times H_2$ such that $\nu\colon K \xrightarrow{\sim} K^D$ induces an isomorphism $\bar{\nu}\colon H_1 \xrightarrow{\sim} H_2^D$.

Condition (i) in (8.22) shows that for every non-degenerate theta group over $k = \overline{k}$ there exist Lagrangian subgroups $H \subset K$. By (8.7) every such H can be lifted (still with $k = \overline{k}$) to a Lagrangian level subgroup of $\mathscr{G}$.

If $\mathscr{H}$ is a Heisenberg group then, with the notations of (8.21), H and H^D are Lagrangian subgroups. So, a necessary condition for a theta group $\mathscr{G}$ to be a Heisenberg group is that there exists a Lagrangian decomposition. This is not always the case. For instance, suppose that E is a supersingular elliptic curve over a field k of characteristic $p > 0$ and that $\mathscr{G}$ is a theta group with finite quotient equal to $E[p]$. One can show that $E[p]$ has a *unique* non-trivial subgroup scheme, isomorphic to α_p. It follows that $\mathscr{G}$ is not a Heisenberg group $\mathscr{H}$ as in (8.21).

If the ground field is algebraically closed and K admits a Lagrangian decomposition then we can describe $\mathscr{G}$ as a Heisenberg group.

(8.24) Lemma. *Suppose $\mathscr{G}$ is a non-degenerate theta group over an algebraically closed field k.*

(i) *Assume that $K = \mathscr{G}/\mathbb{G}_{m,k}$ admits a Lagrangian decomposition, say $K \cong H_1 \times H_2$. Then $\mathscr{G}$ is isomorphic, as a theta group, to the Heisenberg group associated to the pair (H_1, H_1^D).*

(ii) *If* $\mathrm{rank}(K)$ *is prime to* $\mathrm{char}(k)$ *then K admits a Lagrangian decomposition.*

Proof. (i) Lift H_i $(i = 1, 2)$ to a Lagrangian level subgroup $\tilde{H}_i \subset \mathscr{G}$ and write $\xi_i\colon H_i \xrightarrow{\sim} \tilde{H}_i$ for the inverse of the projection. Write $\mathscr{H} = \mathbb{G}_{m,k} \times H_1 \times H_1^D$ for the Heisenberg group associated to the pair (H_1, H_1^D). If $\alpha\colon H_1^D \xrightarrow{\sim} H_2$ is the inverse of $\bar{\nu}^D\colon H_2 \xrightarrow{\sim} H_1^D$ then the map $\mathscr{H} \to \mathscr{G}$ given by

$$(\lambda, x, \chi) \mapsto i(\lambda) \cdot \xi_2(\alpha(\chi)) \cdot \xi_1(x)$$

gives the desired isomorphism of theta groups.

The proof of (ii) is done by the usual procedure of putting a symplectic pairing in canonical form. For details we refer to Exercise ??. □

We apply this to non-degenerate line bundles L such that $K(L)$ is finite and prime to $\mathrm{char}(k)$. This last condition is equivalent to saying that the isogeny $\varphi_L\colon X \to X^t$ is separable (see Exercise ??), hence we say that L is a non-degenerate line bundle *of separable type*.

(8.25) Corollary. *Let k be algebraically closed field. Let X be an abelian variety over k and let L be a non-degenerate line bundle on X of separable type. Then there is a sequence of integers $d_1|d_2|\cdots|d_n$, called the type of L, such that $\mathscr{G}(L)$ is isomorphic to the Heisenberg group associated to the group $H = (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_n\mathbb{Z})$.*

Here is another case where a theta group can be descibed as a Heisenberg group.

(8.26) Example. Let X be an abelian variety. If $\mathscr{P} = \mathscr{P}_X$ is the Poincaré bundle on $X \times X^t$ then we know from Exercise (7.5) that $\varphi_{\mathscr{P}}\colon X \times X^t \to X^t \times X$ is given by $(x, y) \mapsto (y, x)$. Hence $K(\mathscr{P}) = \{0\}$ and $\mathscr{G}(\mathscr{P}) = \mathbb{G}_m$ is the trivial theta group.

Next consider an isogeny $h\colon X \to X^t$, and let $M := (1 \times h)^*\mathscr{P}_X$ on $X \times X$. (If $h = \varphi_L$ for some non-degenerate line bundle L then $M = \Lambda(L)$.) Note that also $M = (h^t \times 1)^* s^* \mathscr{P}$, where $s\colon X \times X^t \xrightarrow{\sim} X^t \times X$ is the isomorphism switching the two factors. Identifying $\mathrm{Ker}(h^t) = \mathrm{Ker}(h)^D$ as in Thm. (7.5) we find that $\{0\} \times \mathrm{Ker}(h) \subset K(M)$ and $\mathrm{Ker}(h)^D \times \{0\} \subset K(M)$, and by comparing ranks it follows that in fact $K(M) = \mathrm{Ker}(h)^D \times \mathrm{Ker}(h)$.

We claim that the theta group $\mathscr{G}(M)$ is naturally isomorphic to the Heisenberg group $\mathscr{H}$ associated to the pair $\big(\mathrm{Ker}(h)^D, \mathrm{Ker}(h)\big)$. We already have natural actions of $\mathrm{Ker}(h)^D \times \{0\}$ and $\{0\} \times \mathrm{Ker}(h)$ on M, compatible with the actions on the basis by translations; this realizes

$\mathrm{Ker}(h)^D$ and $\mathrm{Ker}(h)$ as Lagrangian level subgroups of $\mathscr{G}(M)$. An isomorphism $\mathscr{H} \xrightarrow{\sim} \mathscr{G}(M)$ is then obtained in the same manner as in the proof of (8.24).

§ 4. *Representation theory of non-degenerate theta groups.*

As we have seen in (2.27), an abelian variety X of dimension g can only be embedded in projective spaces of dimension at least $2g$. Hence we will need, at least for large g, a rather large number of equations to describe X. In a beautiful series of papers, Mumford [1] showed how one can nevertheless set up a systematic study of the equations defining an abelian variety. Theta groups play a crucial role in this. To explain why, suppose we choose an ample line bundle L on X. To find the equations for X in the projective embedding defined by (some power of) L, we must try to describe the kernel of the map

$$\mathrm{Sym}^\bullet H^0(X,L) \longrightarrow \bigoplus_{n\geqslant 0} H^0(X,L^n)$$

given by cup-product. The key observation is that $H^0(X,L)$ has a natural action of $\mathscr{G}(L)$. Under suitable assumptions we can identify $\mathscr{G}(L)$ with a Heisenberg group, in which case the representation $H^0(X,L)$ can be described very precisely. This then allows to choose a basis for $H^0(X,L)$ (the elements of which are referred to as theta functions) that has particular properties.

What is sketched here is discussed in much greater detail in Chapter ?? below. First, however, we shall study representations of non-degenerate theta groups.

(8.27) Definition. Let $\mathscr{G}$ be a theta group over a field k. If $\rho\colon \mathscr{G} \to \mathrm{GL}(V)$ is a representation of $\mathscr{G}$ then we say that ρ is a representation *of weight* n ($n \in \mathbb{Z}$) if $\rho\circ i\colon \mathbb{G}_{m,k} \to \mathrm{GL}(V)$ is given by $z \mapsto z^n \cdot \mathrm{id}_V$.

We shall mainly be interested in representations of weight 1.

(8.28) Theorem. *Let k be an algebraically closed field. Let $0 \to \mathbb{G}_{m,k} \to \mathscr{G} \to K \to 0$ be a non-degenerate theta group over k such that* $\mathrm{rank}(K)$ *is prime to* $\mathrm{char}(k)$. *Then $\mathscr{G}$ has a unique irreducible representation $\rho = \rho_{\mathscr{G}} : \mathscr{G} \to \mathrm{GL}(V)$ of weight 1 (up to isomorphism). We have* $\dim(V)^2 = \mathrm{rank}(K)$.

If W is any representation of $\mathscr{G}$ of weight 1 *then W is isomorphic to a direct sum of copies of $\rho_{\mathscr{G}}$. More precisely, if $\tilde{H} \subset \mathscr{G}$ is a maximal level subgroup then $W \cong V^{\oplus a}$ with $a = \dim_k(W^{\tilde{H}})$ equal to the dimension of the subspace of $\tilde{H}$-invariants in W.*

Proof. Choose a maximal level subgroup $\tilde{H} \subset \mathscr{G}$. (By (8.22) it is Lagrangian.) As $\mathrm{rank}(K)$ is prime to $\mathrm{char}(k)$ and $k = \overline{k}$ we can view K and $\tilde{H}$ as constant groups.

Let $\tau\colon \mathscr{G} \to \mathrm{GL}(W)$ be a representation of weight 1. Viewing W as a module under $\tilde{H}$ (which is abelian) it decomposes as a direct sum of character spaces:

$$W = \bigoplus_{\chi\in\mathrm{Hom}(\tilde{H},k^*)} W_\chi\,, \qquad \text{with} \quad W_\chi := \{w \in W \mid \tau(h)(w) = \chi(h)\cdot w \text{ for all } h \in \tilde{H}\}\,.$$

An element $g \in \mathscr{G}$ defines a character $\chi_g \in \mathrm{Hom}(\tilde{H},k^*)$ by $g^{-1}\cdot h\cdot g = \chi_g(h)\cdot h$. (That is, $\chi_g(h) = [g^{-1},h] \in \mathbb{G}_m$.) Then $\tau(g)W_\chi = W_{\chi\cdot\chi_g}$.

As $\tilde{H}$ is Lagrangian, its centralizer $C_{\tilde{H}} \subset \mathscr{G}$ equals $\mathbb{G}_m \cdot \tilde{H}$ (cf. the proof of (8.16)) and $g \mapsto \chi_g$ gives an isomorphism

$$\gamma\colon C_{\tilde{H}}\backslash\mathscr{G} \xrightarrow{\sim} \mathrm{Hom}(\tilde{H}, k^*)\,.$$

As furthermore the elements of $C_{\tilde{H}} = \mathbb{G}_m \cdot \tilde{H}$ act on each W_χ through scalar multiplications, it follows that: (a) all W_χ have the same dimension, and (b) if $0 \neq w \in W_\chi$ then the elements $\tau(g)(w)$ span a $\mathscr{G}$-submodule $V \subset W$ with $\dim(V \cap W_\chi) = 1$ for all χ.

Choose a section $\sigma\colon \mathrm{Hom}(\tilde{H}, k^*) \cong C_{\tilde{H}}\backslash\mathscr{G} \to \mathscr{G}$ of the projection $\mathscr{G} \twoheadrightarrow C_{\tilde{H}}\backslash\mathscr{G}$. Suppose that W is irreducible. Choose $0 \neq w_1 \in W_1$, where $1 \in \mathrm{Hom}(\tilde{H}, k^*)$ is the trivial character. For $\chi \in \mathrm{Hom}(\tilde{H}, k^*)$ set $w_\chi := \tau(\sigma(\chi))(w_1) \in W_\chi$. Then $\{w_\chi\}$ is a k-basis of W. If $g \in \mathscr{G}$ has image η in $\mathrm{Hom}(\tilde{H}, k^*)$ then there is a unique $c = c(g,\chi) \in C_{\tilde{H}}$ such that $g \cdot \sigma(\chi) = c \cdot \sigma(\eta \cdot \chi)$. Then the representation τ is completely described by $\tau(g)(w_\chi) = \tau(c(g,\chi))(w_{\eta\chi})$. (Note that $c(g,\chi) \in \mathbb{G}_m \cdot \tilde{H}$, so we know how it acts on the spaces W_ψ.) As the elements $c(g,\chi)$ only depend on the structure of $\mathscr{G}$ and the chosen section σ, it follows that there is at most one irreducible representation of weight 1, up to isomorphism. Conversely, our description gives a simple recipe of how to construct one. (See also (8.29) below.) This shows that there is a unique irreducible representation $\rho\colon \mathscr{G} \to \mathrm{GL}(V)$ of weight 1.

To prove the last assertions, write $r = \mathrm{rank}(K) = \mathrm{rank}(\tilde{H})^2$ and consider the subgroup $\mathscr{G}[r] \subset \mathscr{G}$ of elements g with $g^r = 1$. As $\mathscr{G}$ is generated by $\mathbb{G}_m$ and $\mathscr{G}[r]$, a weight 1 representation W of $\mathscr{G}$ is completely reducible (i.e., a direct sum of irreducible representations) if and only if it is completely reducible as a representation of $\mathscr{G}[r]$. But $\mathscr{G}[r]$ is a finite group of order not divisible by $\mathrm{char}(k)$. Therefore all k-representations of $\mathscr{G}[r]$ are completely reducible. If $W \cong V^{\oplus a}$ then $a = \dim(W_1) = \dim(W^{\tilde{H}})$. $\square$

(8.29) *The standard representation of a Heisenberg group.* By (8.24), the theta group $\mathscr{G}$ in the theorem is isomorphic to a Heisenberg group $\mathscr{H} = \mathbb{G}_{m,k} \times H \times H^D$ with

$$H \cong (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_n\mathbb{Z})\,, \qquad H^D = \mathrm{Hom}(H, k^*) \cong \mu_{d_1}(k) \times \cdots \times \mu_{d_n}(k)\,.$$

We can take $\tilde{H} := \{(1,h,1) \mid h \in H\}$ as a Lagrangian level subgroup. In the proof of the theorem we have seen how to construct an irreducible weight 1 representation. The result can be described as follows.

Let V be the space of functions on H with values in k. Then we have a representation $\rho\colon \mathscr{H} \to \mathrm{GL}(V)$ given by $[\rho(\lambda,x,\chi)(f)](h) = \lambda \cdot \chi(h) \cdot f(x+h)$ for $f \in V$ and $h \in H$. One easily checks that this indeed gives an irreducible representation of weight 1.

More generally, let H be an arbitrary finite commutative group scheme over a field k. Write $A_H := \Gamma(H, O_H)$ for its affine algebra and let $\mathscr{H} = \mathbb{G}_{m,k} \times H \times H^D$ be the associated Heisenberg group, as defined in (8.21). Then we have a representation

$$\rho\colon \mathscr{H} \longrightarrow \mathrm{GL}(A_H)$$

by letting $(\lambda, x, \chi) \in \mathscr{H}$ act on A_H by

$$f \mapsto \lambda \cdot \chi \cdot t_x^*(f)\,,$$

where we view χ as an invertible element of A_H. More precisely, we should write $(\lambda,x,\chi) \in \mathscr{H}(T)$, where T is a k-scheme. For simplicity, assume that $T = \mathrm{Spec}(R)$ is affine. Then $\rho(\lambda,x,\chi)$ is an R-linear automorphism of $A_H \otimes_k R$. Now notice that χ is given by an invertible element of $A_H \otimes_k R$.

Again this resentation ρ is irreducible of weight 1. We shall refer to it as ??.

(8.30) We now wish to lift the restrictions on the characteristic of k. In the case considered above the desired representation was realized as a representation on the space of functions on a Lagrangian level subgroup. Therefore is natural to consider representations of $\mathscr{G}$ on spaces of functions on $\mathscr{G}$.

We work over an algebraically closed field k. As K is a semi-local scheme, we can trivialize $\mathscr{G}$ as a $\mathbb{G}_m$-torsor over K. So, we can choose an isomorphism $\mathscr{G} \xrightarrow{\sim} \mathbb{G}_m \times K$ of K-schemes via which the $\mathbb{G}_m$-action on $\mathscr{G}$ corresponds to multiplication in $\mathbb{G}_m$ on the right hand term. Writing $K = \mathrm{Spec}(A_0)$ this gives $\mathscr{G} = \mathrm{Spec}(B)$, with

$$B = A_0[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} A_i\,, \qquad \text{where we set} \quad A_i := A_0 \cdot t^i\,.$$

We can view A_i as the vector space of those functions $f\colon \mathscr{G} \to \mathbb{A}^1$ such that $f(\lambda x) = \lambda^i \cdot f(x)$ for all $\lambda \in \mathbb{G}_m$. (This has to be read functorially: if R is a k-algebra then $A_i \otimes_k R = \{f \in \mathrm{Hom}_R(\mathscr{G}_R, \mathbb{A}^1_R) \mid f(\lambda x) = \lambda^i \cdot f(x) \text{ for all } \lambda \in \mathbb{G}_m\}$.)

As will become clear in the proof of (8.32), the space A_1 is the most interesting for us. (That is, if we want to study representations of weight 1.) Note that $\dim(A_1)$ is the square of the dimension of the irreducible $\mathscr{G}$-representation that we are looking for. We consider the action of $\mathscr{G} \times \mathscr{G}$ on A_1 given by

$$[(g,h) \cdot f](x) = f(h^{-1}xg) \qquad \text{for} \quad f \in A_1\,, (g,h) \in \mathscr{G} \times \mathscr{G} \quad \text{and} \quad x \in \mathscr{G}\,.$$

(Again this has to be read functorially.)

(8.31) Lemma. *With this action A_1 is an irreducible $\mathscr{G} \times \mathscr{G}$-module.*

Proof. First we look at the diagonal $\mathscr{G}$-action. If $g \in \mathscr{G}$ and $f \in A_1$ then $(g,g) \cdot f$ is the function given by

$$\begin{aligned} x \mapsto f(g^{-1}xg) &= f(g^{-1}xgx^{-1} \cdot x) \\ &= f([g^{-1}, x] \cdot x) \\ &= [g^{-1}, x] \cdot f(x) \qquad\qquad \text{(because } f \in A_1 \text{ and } [g^{-1},x] \in \mathbb{G}_m) \\ &= e(\pi(g)^{-1}, \pi(x)) \cdot f(x)\,. \end{aligned}$$

Each $\gamma \in K$ defines a character $e(\gamma, -)\colon K \to \mathbb{G}_m$. If χ is any such character then $\chi \circ \pi\colon \mathscr{G} \to \mathbb{G}_m \subset \mathbb{A}^1$ can be viewed as an element of A_0. The previous calculations show that $\mathscr{G} \xrightarrow{\Delta} \mathscr{G} \times \mathscr{G} \to \mathrm{GL}(A_1)$ factors through $\mathscr{G} \twoheadrightarrow K$ and that the resulting action of K on A_1 is given by

$$\gamma \cdot f = [\pi \circ e(\gamma^{-1}, -)] \cdot f \qquad \text{for} \quad \gamma \in K\,, f \in A_1\,.$$

Suppose that $E \subset A_1$ is a $\mathscr{G} \times \mathscr{G}$-submodule. The non-degeneracy of our theta group means that every character $\chi\colon K \to \mathbb{G}_m$ is of the form $e(\gamma^{-1}, -)$ for some $\gamma \in K$. Furthermore, the k-characters χ form a k-basis of the k-algebra A_0; see Exercise ??. It thus follows from the above that E is an A_0-submodule of A_1. But $A_1 = A_0 \cdot t$, so E is of the form IA_1 with I an ideal of A_0.

So far we have only used the diagonal action of $\mathscr{G}$. Using the full action of $\mathscr{G} \times \mathscr{G}$ again we see that E is stable under all translations by elements of $\mathscr{G}$. Combined with the previous it now readily follows that $E = (0)$ or $E = A_1$ $\square$

We can now generalize Theorem (8.28).

(8.32) Theorem. *Let $\mathscr{G}$ be a non-degenerate theta group over an algebraically closed field k. Then $\mathscr{G}$ has a unique irreducible representation $\rho = \rho_{\mathscr{G}}\colon \mathscr{G} \to \mathrm{GL}(V)$ of weight 1 (up to isomorphism). We have $\dim(V)^2 = \mathrm{rank}(K)$. If W is any representation of $\mathscr{G}$ of weight 1 then W is isomorphic to a direct sum of copies of $\rho_{\mathscr{G}}$.*

Proof. Let $\tau\colon \mathscr{G} \to \mathrm{GL}(W)$ be a representation of $\mathscr{G}$ of weight 1. Then τ gives rise to a homomorphism of $\mathscr{G} \times \mathscr{G}$-modules $r\colon W^* \otimes W \to A_1$, by

$$r(\varphi \otimes w)(g) = \varphi(\tau(g)(w)) \qquad \text{for} \quad \varphi \in W^*\,, w \in W \text{ and } \quad g \in \mathscr{G}\,.$$

Suppose that W is irreducible. Then $W^* \otimes_k W$ is an irreducible $\mathscr{G} \times \mathscr{G}$-module. (Here we need that $k = \overline{k}$! The point is that $\mathrm{End}_{\mathscr{G}\times\mathscr{G}}(W^* \otimes W) = \mathrm{End}_{\mathscr{G}}(W^*) \otimes_k \mathrm{End}_{\mathscr{G}}(W)$. As $k = \overline{k}$ the irreducibility of W implies that $\mathrm{End}_{\mathscr{G}}(W) = k = \mathrm{End}_{\mathscr{G}}(W^*)$.) As r is obviously not the zero map it follows from the lemma that r is an isomorphism. We conclude that $A_1 \cong W \oplus \cdots \oplus W$ ($\dim(W)$ factors) as a $\mathscr{G}$-module, that there is a unique irreducible $\mathscr{G}$-module of weight 1, and that $\mathrm{rank}(K) = \dim(A_1)$ is the square of its dimension. We also see that A_1 is completely reducible as a $\mathscr{G}$-module.

Now let W be an arbitrary $\mathscr{G}$-module of weight 1 again. Then r gives a k-linear map $r'\colon W \to \mathrm{Hom}(W^*, A_1) = W \otimes A_1$, sending $w \in W$ to $\varphi \mapsto r(\varphi \otimes w)$. From $r(\varphi \otimes w)(1) = \varphi(w)$ we see that r' is injective. Moreover, if we let $\mathscr{G}$ act on $W \otimes A_1$ through its action on A_1 then r' is $\mathscr{G}$-equivariant. We conclude that W is isomorphic to a $\mathscr{G}$-submodule of $A_1^{\dim(W)}$. As A_1 is a completely reducible $\mathscr{G}$-module, W is also completely reducible. $\square$

Exercises.

(8.1) Let k be an algebraically closed field. Let K be a finite commutative group scheme of order prime to $\mathrm{char}(k)$. Let $e\colon K \times K \to \mathbb{G}_m$ be a non-degenerate alternating bilinear pairing, i.e., e is a morphism of k-schemes such that (a) $e(x,y) = e(y,x)^{-1}$, (b) for fixed $x \in K$ the maps $y \mapsto e(x,y)$ and $y \mapsto e(y,x)$ are homomorphisms, and (c) the homomorphism $K \to K^D$ given by $x \mapsto e(x,-)$ is an isomorphism.

(i) Show that K is isomorphic to a constant group scheme of the form

$$K \cong (\mathbb{Z}/d_1\mathbb{Z}) \times (\mathbb{Z}/d_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_n\mathbb{Z})\,,$$

where we may require that $d_1|d_2|\cdots|d_n$. (And if $\mathrm{char}(k) = p > 0$ then $p \nmid d_n$.)

(ii) Choose an element $a \in K$ such that K is a product $K = \langle a\rangle \times K'$. Let d be the order of a and let $\zeta_d \in k$ be a primitive dth root of unity. Show that there is a unique $b \in K$ with $e(a,b) = \zeta_d$ and $e(k,b) = 1$ for all $k \in K'$.

(iii) Let $K'' := \{k \in K' \mid e(a,k) = 1\}$. Show that K decomposes as a product of groups $K = \langle a\rangle \times \langle b\rangle \times K''$. Also show that the restriction of e to $K'' \times K''$ is again non-degenerate.

(iv) Prove that there exists a finite commutative k-group scheme H and an isomorphism $K \xrightarrow{\sim} H \times H^D$ via which the pairing e corresponds to the pairing on $H \times H^D$ given by

$$\big((x,\chi),(x',\chi')\big) \mapsto \chi'(x)\cdot\chi(x')^{-1}\,.$$