CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for April 26

Exercise 1. Let R be a principal ideal domain. If M is a finitely generated R-module, show that

$$M \text{ is a projective } R\text{-module} \iff M \text{ is a free } R\text{-module} \iff M \text{ is a flat } R\text{-module.}$$

Exercise 2. Let R be a commutative ring. Let I and J be ideals of R that are coprime, which means that $I + J = R$.

(a) Show that we have an exact sequence of R-modules

$$0 \longrightarrow I \cdot J \xrightarrow{f} I \oplus J \xrightarrow{g} R \longrightarrow 0\,,$$

where $f(x) = (x, -x)$ and $g(x, y) = x + y$.

(b) Prove that $I \oplus J \cong R \oplus (I \cdot J)$ as R-modules. [*Hint:* Use Exercise 6 from last week.]

(c) Suppose R is a domain and $I \cdot J$ is a principal ideal. Prove that I and J are projective R-modules. [*Hint:* Again use Exercise 6 from last week.]

Exercise 3. Let $R = \mathbb{Z}\big[\sqrt{-5}\,\big]$. Let $N\colon R \to \mathbb{Z}$ be the map given by $N\big(a+b\sqrt{-5}\big) = a^2+5b^2$. It has the property that $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$ for all α, $\beta \in R$. Further, $N(\alpha) = 1$ if and only if $\alpha \in R^*$. Consider the ideals $I = \big(3, 1 + \sqrt{-5}\big)$ and $J = \big(3, 1 - \sqrt{-5}\big)$.

(a) Prove that I and J are coprime and that $I \cdot J = (3)$.

(b) Prove that I and J are not principal ideals. [*Hint:* Suppose $I = (\alpha)$. Show that we must have $N(\alpha) = 3$, and remark that this is impossible.]

(c) Prove that I and J are projective R-modules. [*Hint:* Use the previous exercise.]

(d) Prove that I and J are not free R-modules. [*Hint:* First show that if one of the two is free, so is the other. Then show that if I and J are free, they must have rank 1, contradicting (b).]

Exercise 4. Let R be a ring, and let

$$(*) \qquad M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a sequence of R-modules. Prove that $(*)$ is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M', N)$$

is exact.

Exercise 5. Let R be a ring, and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a *split exact* sequence of R-modules.

(a) Prove that for every R-module N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M', N) \longrightarrow 0$$

is exact.

(b) Prove that for every R-module N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, M') \longrightarrow \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_R(N, M'') \longrightarrow 0$$

is exact.

(c) Assume R is commutative. Prove that for every R-module N the induced sequence

$$0 \longrightarrow M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0$$

is exact.