ERRATUM

In October 2005, Christophe Cornut pointed out to me that the argument given on page 12 of my PhD thesis is not correct. The problem comes from the centre of the group G. (If $G = G^{\mathrm{ad}}$ then what I wrote is correct.) I believe there are several possible arguments to prove the asserted finiteness. Here is an argument that repairs the one given in my thesis.

(1) We start with a Shimura datum (G, X) and a closed subgroup $H \subset G$ over $\mathbb{Q}$. Notation:

$$Y_H := \{x \in X \mid h_x \text{ factors through } H_{\mathbb{R}}\}.$$

If $Y_H = \emptyset$ there is nothing to prove, so we assume $Y_H \neq \emptyset$. Claim: this implies that H is reductive. To prove this claim, it suffices to show that the image of H in G^{ad} is reductive. Let us write H' for this image. Choose any $x \in Y_H$, and write C for the image of $h_x(i)$ in $H'(\mathbb{R}) \subset G^{\mathrm{ad}}(\mathbb{R})$. Then $\theta := \mathrm{Inn}(C)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$. By definition this means that the inner form $G^{\mathrm{ad},\theta}_{\mathbb{R}}$ of $G^{\mathrm{ad}}_{\mathbb{R}}$ that is determined by θ has the property that $G^{\mathrm{ad},\theta}(\mathbb{R})$ is compact as a real Lie group. But clearly θ also gives an involution of $H'_{\mathbb{R}}$, and we get an inner form $H'^{,\theta}_{\mathbb{R}} \subset G^{\mathrm{ad},\theta}_{\mathbb{R}}$. Hence $H'_{\mathbb{R}}$ has a compact inner form, and this implies that H' is reductive.

(2) *Notation:* Given an algebraic group Q over $\mathbb{R}$ and a maximal torus $T \subset Q$, let us write $W_{\mathbb{R}}(T) := N_Q(T)(\mathbb{R})/Z_Q(T)(\mathbb{R})$ for the associated "real Weyl group".

As on page 12 of my thesis, let $S_1, \ldots, S_k$ be a set of representatives for the conjugacy classes of maximal tori in $H_{\mathbb{R}}$.

Given the $H(\mathbb{R})$-conjugacy class $Y_\alpha \subset Y_H$, we can choose a maximal torus S_i in our list, such that for any element $x \in Y_\alpha$ some $H(\mathbb{R})$-conjugate of $h_x \colon \mathbb{S} \to H_{\mathbb{R}}$ factors through S_i. For each $H(\mathbb{R})$-conjugacy class $Y_\alpha \subset Y_H$ we choose an index $i = i(\alpha)$ for which this holds. Next we can choose an element $x \in Y_\alpha$ such that h_x itself (rather than some conjugate) factors through S_i.

Let $S'_i \cong S_i/(S_i \cap Z(G)_{\mathbb{R}})$ be the image of S_i under $\mathrm{ad}_G \colon G \to G^{\mathrm{ad}}$, and write $h'_x := \mathrm{ad}_G \circ h_x$. Claim: the class of h'_x in

$$\mathrm{Hom}(\mathbb{S}, S'_i)/W_{\mathbb{R}}(S'_i) \qquad (*)$$

is independent of the choice of x. To see this, suppose we have another element $y \in Y_\alpha$ such that h_y factors through S_i too. Choose $h \in H(\mathbb{R})$ such that $y = hx$. Let $M \subset H_{\mathbb{R}}$ be the centralizer of h_x, which is a connected reductive subgroup of $H_{\mathbb{R}}$. Let $M' \cong M/(M \cap Z(G)_{\mathbb{R}})$ be the image of M in $H' \subset G^{\mathrm{ad}}_{\mathbb{R}}$. Then S'_i is a maximal torus of M'. Also $T := h^{-1}S'_i h$ is a maximal torus of M'. But $M'(\mathbb{R})$ is compact, because the Cartan involution θ of $G^{\mathrm{ad}}_{\mathbb{R}}$ that we considered in point (1) restricts to the identity on M'. So S'_i and T are conjugate: there exists an element $m \in M'(\mathbb{R})$ such that $h^{-1}S'_i h = m^{-1}S'_i m$. Then $z := hm^{-1}$ is an element of $N_{H'_i}(S'_i)(\mathbb{R})$, and because $y = hx = zx$ we see that h'_y gives the same class in $(*)$ as h'_x. This proves our claim. Let us denote that class in $(*)$ associated to Y_α by $\mathrm{cl}(Y_\alpha)$.

(3) Choose a maximal torus $T_i \subset G_{\mathbb{R}}$ containing S_i, and write $T^{\mathrm{ad}}_i \subset G^{\mathrm{ad}}_{\mathbb{R}}$ for its image under ad_G. Applying the arguments in (2) with $H = G$ (in which case $Y_H = X$) we find that the conjugacy class of homomorphisms $\mathrm{ad}_g \circ h_x$ for $x \in X$ gives a well-determined element $\mathrm{cl}(X) \in \mathrm{Hom}(\mathbb{S}, T^{\mathrm{ad}}_i)/W_{\mathbb{R}}(T^{\mathrm{ad}}_i)$.

Given an index $i \in \{1, \ldots, k\}$ there are finitely many elements in $(*)$ that can occur as an element of the form $\mathrm{cl}(Y_\alpha)$. To see this, suppose γ is such an element. Choose any element $\tilde{\gamma} \in \mathrm{Hom}(\mathbb{S}, S'_i)$ that map to γ. We have a natural map $\mathrm{Hom}(\mathbb{S}, S'_i) \to \mathrm{Hom}(\mathbb{S}, T^{\mathrm{ad}}_i)/W_{\mathbb{R}}(T^{\mathrm{ad}}_i)$ with finite fibres, and $\tilde{\gamma}_j$ maps to $\mathrm{cl}(X)$. Hence indeed there are only finitely many possibilities for $\mathrm{cl}(Y_\alpha)$.

(4) We are going to prove that the $H'(\mathbb{R})$-orbit of Y_α is determined by the element $\mathrm{cl}(Y_\alpha)$. Note first that H' acts on Y_H. The natural map $H(\mathbb{R}) \to H'(\mathbb{R})$ need not be surjective, but in any case the image of $H(\mathbb{R})$ in $H'(\mathbb{R})$ has finite index. (In fact, $H'(\mathbb{R})$ has finitely many components for the analytic topology, and on the (analytic) identity components the quotient map is surjective.) So it suffices if we can show that there are only finitely many $H'(\mathbb{R})$-orbits in Y_H.

Now suppose we have $H(\mathbb{R})$-conjugacy classes Y_α and Y_β in Y_H such that $i(\alpha) = i(\beta)$ and such that $\mathrm{cl}(Y_\alpha) = \mathrm{cl}(Y_\beta)$. Choose $x \in Y_\alpha$ and $y \in Y_\beta$ such that h_x and h_y both factor through S_i. Possibly after changing y by an element of $N_{H'}(S'_i)(\mathbb{R})$, the assumption that $\mathrm{cl}(Y_\alpha) = \mathrm{cl}(Y_\beta)$ gives us that $h'_x = h'_y$. This means that h_x and h_y differ by a homomorphism $\chi\colon \mathbb{S} \to Z(G) \cap S_i$, i.e., $h_y(s) = h_x(s) \cdot \chi(s)$ for all $s \in \mathbb{S}$. Now we look at the abelianization map $\mathrm{ab}_G\colon G \to G^{\mathrm{ab}} := G/G^{\mathrm{der}}$. Then $\mathrm{ab}_G \circ h_\xi$ is independent of the choice of $\xi \in X$. Hence we find that χ takes values in $Z(G) \cap G^{\mathrm{der}}$. But this is a finite group and $\mathbb{S}$ is connected, so it follows that χ is trivial and therefore $Y_\alpha = Y_\beta$. Conclusion: $H'(\mathbb{R}) \cdot Y_\alpha$ is determined by the class $\mathrm{cl}(Y_\alpha)$, and as explained this implies that there are finitely many $H(\mathbb{R})$-conjugacy classes in Y_H.