ERRATUM

In October 2005, Christophe Cornut pointed out to me that the argument given on page 12 of my PhD thesis is not correct. The problem comes from the centre of the group $G$. (If $G = G^{\text{ad}}$ then what I wrote is correct.) I believe there are several possible arguments to prove the asserted finiteness. Here is an argument that repairs the one given in my thesis.

(1) We start with a Shimura datum $(G, X)$ and a closed subgroup $H \subset G$ over $\mathbb{Q}$. Notation:

\[ Y_H := \{ x \in X \mid h_x \text{ factors through } H_{\mathbb{R}} \} . \]

If $Y_H = \emptyset$ there is nothing to prove, so we assume $Y_H \neq \emptyset$. Claim: this implies that $H$ is reductive. To prove this claim, it suffices to show that the image of $H$ in $G^{\text{ad}}$ is reductive. Let us write $H'$ for this image. Choose any $x \in Y_H$, and write $C$ for the image of $h_x(i)$ in $H'(\mathbb{R}) \subset G^{\text{ad}}(\mathbb{R})$. Then $\theta := \text{Im}(C)$ is a Cartan involution of $G_{\mathbb{R}}^{\text{ad}}$. By definition this means that the inner form $G_{\mathbb{R}}^{\text{ad}, \theta}$ of $G_{\mathbb{R}}^{\text{ad}}$ that is determined by $\theta$ has the property that $G_{\mathbb{R}}^{\text{ad}, \theta}$ is compact as a real Lie group. But clearly $\theta$ also gives an involution of $H'_{\mathbb{R}}$, and we get an inner form $H'_{\mathbb{R}} \subset G_{\mathbb{R}}^{\text{ad}}$. Hence $H'_{\mathbb{R}}$ has a compact inner form, and this implies that $H'$ is reductive.

(2) Notation: Given an algebraic group $Q$ over $\mathbb{R}$ and a maximal torus $T \subset Q$, let us write $W_Q(T) := N_Q(T)(\mathbb{R})/Z_Q(T)(\mathbb{R})$ for the associated “real Weyl group”.

As on page 12 of my thesis, let $S_1, \ldots, S_k$ be a set of representatives for the conjugacy classes of maximal tori in $H_{\mathbb{R}}$.

Given the $H(\mathbb{R})$- conjugacy class $Y_{\alpha} \subset Y_H$, we can choose a maximal torus $S_i$ in our list, such that for any element $x \in Y_{\alpha}$ some $H(\mathbb{R})$- conjugate of $h_x : S \to H_{\mathbb{R}}$ factors through $S_i$. For each $H(\mathbb{R})$- conjugacy class $Y_{\alpha} \subset Y_H$ we choose an index $i = i(\alpha)$ for which this holds. Next we can choose an element $x \in Y_{\alpha}$ such that $h_x$ itself (rather than some conjugate) factors through $S_i$.

Let $S_i' \cong S_i/(S_i \cap Z(G)_{\mathbb{R}})$ be the image of $S_i$ under $\text{ad}_G : G \to G^{\text{ad}}$, and write $h_x' := \text{ad}_{G^{\text{ad}}} h_x$. Claim: the class of $h_x'$ in

\[ \text{Hom}(S, S_i')/W_{\mathbb{R}}(S_i') \quad (*) \]

is independent of the choice of $x$. To see this, suppose we have another element $y \in Y_{\alpha}$ such that $h_y$ factors through $S_i$ too. Choose $h \in H(\mathbb{R})$ such that $y = h x$. Let $M \subset H_{\mathbb{R}}$ be the centralizer of $h_x$, which is a connected reductive subgroup of $H_{\mathbb{R}}$. Let $M' \cong M/(M \cap Z(G))_{\mathbb{R}}$ be the image of $M$ in $H'(\mathbb{R}) \subset G^{\text{ad}}$. Then $S_i'$ is a maximal torus of $M'$. Also $T := h^{-1} S_i' h$ is a maximal torus of $M'$. But $M'(\mathbb{R})$ is compact, because the Cartan involution $\theta$ of $G_{\mathbb{R}}^{\text{ad}}$ that we considered in point (1) restricts to the identity on $M'$. So $S_i'$ and $T$ are conjugate: there exists an element $m \in M'(\mathbb{R})$ such that $h^{-1} S_i' h = m^{-1} S_i' m$. Then $z := \text{Ad}_m^{-1}$ is an element of $N_{H'_{\mathbb{R}}}(S_i')_{\mathbb{R}}$, and because $y = h x = z x$ we see that $h_y'$ gives the same class in $(*)$ as $h_x'$. This proves our claim. Let us denote that class in $(*)$ associated to $Y_{\alpha}$ by $\text{cl}(Y_{\alpha})$.

(3) Choose a maximal torus $T_i \subset G_{\mathbb{R}}$ containing $S_i$, and write $T_i^{\text{ad}} \subset G_{\mathbb{R}}^{\text{ad}}$ for its image under $\text{ad}_G$. Applying the arguments in (2) with $H = G$ (in which case $Y_H = X$) we find that the conjugacy class of homomorphisms $\text{ad}_g \cdot h_x$ for $x \in X$ gives a well-determined element $\text{cl}(X) \in \text{Hom}(S, T_i^{\text{ad}})/W_{\mathbb{R}}(T_i^{\text{ad}})$.

Given an index $i \in \{1, \ldots, k\}$ there are finitely many elements in $(*)$ that can occur as an element of the form $\text{cl}(Y_{\alpha})$. To see this, suppose $\gamma$ is such an element. Choose any element $\tilde{\gamma} \in \text{Hom}(S, S_i')$ that maps to $\gamma$. We have a natural map $\text{Hom}(S, S_i') \to \text{Hom}(S, T_i^{\text{ad}})/W_{\mathbb{R}}(T_i^{\text{ad}})$ with finite fibres, and $\tilde{\gamma}_j$ maps to $\text{cl}(X)$. Hence indeed there are only finitely many possibilities for $\text{cl}(Y_{\alpha})$.

(4) We are going to prove that the $H'(\mathbb{R})$-orbit of $Y_{\alpha}$ is determined by the element $\text{cl}(Y_{\alpha})$. Note first that $H'$ acts on $Y_H$. The natural map $H(\mathbb{R}) \to H'(\mathbb{R})$ need not be surjective, but in any case the image of $H(\mathbb{R})$ in $H'(\mathbb{R})$ has finite index. (In fact, $H'(\mathbb{R})$ has finitely many components for the analytic topology, and on the (analytic) identity components the quotient map is surjective.) So it suffices if we can show that there are only finitely many $H'(\mathbb{R})$-orbits in $Y_H$. 

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Now suppose we have $H(\mathbb{R})$-conjugacy classes $Y_\alpha$ and $Y_\beta$ in $Y_H$ such that $i(\alpha) = i(\beta)$ and such that $\text{cl}(Y_\alpha) = \text{cl}(Y_\beta)$. Choose $x \in Y_\alpha$ and $y \in Y_\beta$ such that $h_x$ and $h_y$ both factor through $S_i$. Possibly after changing $y$ by an element of $N_{H'}(S'_i)(\mathbb{R})$, the assumption that $\text{cl}(Y_\alpha) = \text{cl}(Y_\beta)$ gives us that $h'_x = h'_y$. This means that $h_x$ and $h_y$ differ by a homomorphism $\chi: S \to Z(G) \cap S_i$, i.e., $h_y(s) = h_x(s) \cdot \chi(s)$ for all $s \in S$. Now we look at the abelianization map $ab_G: G \to G^{ab} := G/G^{\text{der}}$. Then $ab_G \circ h_\xi$ is independent of the choice of $\xi \in X$. Hence we find that $\chi$ takes values in $Z(G) \cap G^{\text{der}}$. But this is a finite group and $S$ is connected, so it follows that $\chi$ is trivial and therefore $Y_\alpha = Y_\beta$. Conclusion: $H'(\mathbb{R}) \cdot Y_\alpha$ is determined by the class $\text{cl}(Y_\alpha)$, and as explained this implies that there are finitely many $H(\mathbb{R})$-conjugacy classes in $Y_H$. 
