DISCRETE INVARIANTS OF VARIETIES IN POSITIVE CHARACTERISTIC

ERRATA, AND AN EXAMPLE

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In Section 2.6 of the published paper (B. Moonen and T. Wedhorn, Discrete invariants of varieties in positive characteristic, IMRN 2004, no. 72, 3855–3901) there is a discussion of standard F-zips. Not only this section is very difficult to read, it also seems to contain mistakes. Probably the only way to get all details correct is to check them in an explicit example, which in the paper we did not do. The following comments are intended to make up for this.

1. The relative position of two parabolics (or two flags). Let G be a connected reductive group over an algebraically closed field. Fix a borus $T_0 \subset B_0 \subset G$. The Weyl group $W = W_G$ can be defined in many ways but our choice of a borus simply gives $W = N_G(T_0)$. Writing $\mathrm{Par}_\emptyset$ for the variety of Borel subgroups we have $W \xrightarrow{\sim} G\backslash(\mathrm{Par}_\emptyset \times \mathrm{Par}_\emptyset)$ by sending $w \in N_G(T_0)$ to the G-orbit of $(B_0, {}^wB_0)$.

Let $I \subset W$ be the set of simple reflections (w.r.t. the chosen borus), and for $J \subset I$ let P_J be the standard parabolic of type J. Also let $W_J \subset W$ be the subgroup generated by the elements in J. We have $W_J = \bigl\{w \in N_G(T) \bigm| {}^wP_J = P_J\bigr\}$. Let Par_J be the variety of parabolics of type J.

Perhaps the simplest way to define the relative position of two parabolics is to use, for J, $K \subset I$, the bijection

$$W_J\backslash W/W_K \xrightarrow{\sim} G\backslash\bigl(\mathrm{Par}_J \times \mathrm{Par}_K\bigr)$$

that sends the double coset of an element $w \in N_G(T_0)$ to the G-orbit of the pair $(P_J, {}^wP_K)$. Let us do some sanity checks: The map is well-defined, for if $x \in W_J$ and $y \in W_K$ then

$$(P_J, {}^{xwy}P_K) = (P_J, {}^{xw}P_K) = ({}^xP_J, {}^{xw}P_K) = x \cdot (P_J, {}^wP_K)\,.$$

The inverse map is the relative position; so if P and Q are parabolics P and Q of types J and K, respectively, then we define $\mathrm{relpos}(P,Q) \in W_J\backslash W/W_K$ as the class that maps to the G-orbit of (P,Q). In practice it is often convenient to work with the minimal representative of this double coset, which lives in ${}^JW^K$. To describe this a bit more concretely: Choose $g \in G$ such that ${}^gP = P_J$ and $T_0 \subset {}^gQ$. Then ${}^gQ = {}^wP_K$ for some uniquely determined $w \in W/W_K$ and $\mathrm{relpos}(P,Q)$ is the class of w in $W_J\backslash W/W_K$. Again it is not so hard to see that this is independent of choices, for if $h \in G$ is another element with ${}^hP = P_J$ and $T_0 \subset {}^hQ$ then $h = xg$ for some $x \in W_J$ and ${}^hQ = {}^{xw}P_K$, so we find the same class in $W_J\backslash W/W_K$.

This leads to a first correction to our paper:

Correction to Section 3.6 of the paper: Given two parabolics P and Q and a maximal torus T contained in $P \cap Q$, there exists an element $w \in N_G(T)$ such that $w(P) = {}^wP$ and Q have a Borel in common, and then $\mathrm{relpos}(P,Q)$ is the class of w in $W_J\backslash W/W_K$. [In the paper we took an $n \in N_G(T)$ such that P and $n(Q)$ have a common Borel and we said that this n represents the relative position. Correct is: the *inverse* of n represents the relative position.]

In our discussion about standard F-zips, we will have $G = \mathrm{GL}_n$ for some n and we identify $W = S_n$ in the usual way. We will want to calculate the relative position of two flags $C^\bullet$ and $D(\infty)_\bullet$ (see below). With respect to a basis $e_1, \ldots, e_n$ of the underlying vector space, the situation will be such that the ascending flag $(0) \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \cdots$ is a refinement of $D(\infty)_\bullet$ and that there is a permutation $u \in S_n$ such that the descending flag $M \supset \langle e_{u(1)}, \ldots, e_{u(n-1)} \rangle \supset \cdots \supset \langle e_{u(1)}, e_{u(2)} \rangle \supset \langle e_{u(1)} \rangle \supset 0$ is a refinement of $C^\bullet$. In this case, we find that $\mathrm{relpos}(C^\bullet, D(\infty)_\bullet)$ is represented by u^{-1}.

2. How the classification of F-zips works. In concrete terms, the classifying element of an F-zip $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ is obtained as follows. Start with the flags $C(0)^\bullet = C^\bullet$ and $D(0)_\bullet = D_\bullet$. One iterates a procedure that replaces the given flags $C(n)^\bullet$ and $D(n)_\bullet$ by refinements $C(n+1)^\bullet$ and $D(n+1)_\bullet$. In each step we first refine the $C(n)^\bullet$-flag using $D(n)_\bullet$, and then transfer this to the D-side using the φ_i. Note that in this process we only care about the flags, not about their numbering. Refinement of $C^\bullet$ using $D_\bullet$ means that we replace $C^\bullet$ by the flag that consists of all terms $(C^i \cap D_j) + C^{i+1}$. Transfer to the D-side means that if $C^i \supset V \supset C^{i+1}$ is one of the terms that has been added then in the $D_\bullet$-flag we add the subspace $D_{i-1} \subset V' \subset D_i$ such that V'/D_{i-1} is the image of V/C^{i+1} under $\varphi_i \colon \mathrm{gr}^i_C \xrightarrow{\sim} \mathrm{gr}^D_i$.

Iterate this until you get flags $C(\infty)^\bullet$ and $D(\infty)_\bullet$. Let $\tilde{D}(\infty)_\bullet$ be any refinement of $D(\infty)_\bullet$ to a full flag. Then take $\mathrm{relpos}(C^\bullet, \tilde{D}(\infty)_\bullet) \in W_J\backslash W$, which does not depend on how the refinement $\tilde{D}(\infty)_\bullet$ is chosen. (You can also just take the minimal representative of $\mathrm{relpos}(C^\bullet, D(\infty)_\bullet) \in W_J\backslash W/W_{K(\infty)}$, where $K(\infty)$ is the type of $D(\infty)_\bullet$.)

3. Standard F-zips—corrections to our paper. As in Section 2.6 of the paper, fix $n \geqslant 1$ and a type τ. Let $i_1 > i_2 > \cdots > i_r$ be the support of τ. (See Remark 4 for more on the numbering scheme we use.) Let $n_j = \tau(i_j)$, so that $J = (n_1, n_2, \ldots, n_r)$ is an ordered partition of n. Write $m_j = n_1 + \cdots + n_j$, with the convention that $m_0 = 0$. Let $W = S_n \supset W_J = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$. To an element $u \in {}^JW$ we want to associate a standard F-zip of type u.

Let $x \in {}^JW$ be the inverse of the minimal representative of the class $[w_0] \in W_J\backslash W$, where $w_0 \in W$ is the longest element. Explicitly (see (2.4) of the paper), $x(i) = i + n - m_j - m_{j-1}$ if $m_{j-1} < i \leqslant m_j$.

Given $u \in {}^JW$ the associated standard F-zip M^u is the following:

— The underlying vector space is $M^u = \mathbb{F}_p^n$ with basis $e_1, \ldots, e_n$.

— The filtration $D_\bullet$ is the unique ascending filtration of type τ such that the standard flag $0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots$ is a refinement of the associated flag.

— The filtration $C^\bullet$ is the unique descending filtration of type τ such that the standard flag $0 \subset \langle e_{u^{-1}(1)} \rangle \subset \langle e_{u^{-1}(1)}, e_{u^{-1}(2)} \rangle \subset \cdots$ is a refinement of the associated flag.

— Finally, if $j = i_s$ then

$$\varphi_j \colon \mathrm{gr}_C^{j,(p)} = \langle e_{u^{-1}(m_{s-1}+1)}, \ldots, e_{u^{-1}(m_s)} \rangle \xrightarrow{\sim} \mathrm{gr}_j^D = \langle e_{n-m_s+1}, \ldots, e_{n-m_{s-1}} \rangle$$

is given by the permutation matrix associated with $x \cdot u$.

For concreteness: if $\mathrm{Supp}(\tau) = \{0, 1, \ldots, r-1\}$ then we have

$$\begin{aligned} D_{-1} = 0 \quad \subset \quad D_0 = \langle e_1, \ldots, e_{n-m_{r-1}} \rangle \quad \subset \quad D_1 = \langle e_1, \ldots, e_{n-m_{r-2}} \rangle \quad \subset \quad \cdots \\ \cdots \quad \subset \quad D_{r-2} = \langle e_1, \ldots, e_{n-m_1} \rangle \quad \subset \quad D_{r-1} = M \end{aligned}$$

and

$$C^0 = M \quad \supset \quad C^1 = \big\langle e_{u^{-1}(1)}, \ldots, e_{u^{-1}(m_{r-1})} \big\rangle \quad \supset \quad C^2 = \big\langle e_{u^{-1}(1)}, \ldots, e_{u^{-1}(m_{r-2})} \big\rangle \quad \supset \\ \cdots \quad \supset \quad C^{r-1} = \big\langle e_{u^{-1}(1)}, \ldots, e_{u^{-1}(m_1)} \big\rangle \quad \supset \quad C^r = 0\,.$$

To summarize: the changes with respect to Section 2.6 of our paper is that we use the base vectors $e_{u^{-1}(i)}$ in the description of the C-filtration, and that the Frobenii φ_j are described by $x \cdot u$, instead of the $x^{-1}u^{-1}$ that we had in the paper.

4. Remark. We have chosen to number the support of the function τ as $i_1 > i_2 > \cdots > i_r$, so if we think of the numbers $n_j = \tau(i_j)$ as the "Hodge numbers", it means we are reading the Hodge numbers from right to left. At first glance this may not seem very natural. However, what really matters is the stabilizer of a flag, and this does not see the difference between a descending and an ascending flag. Moreover, with our numbering the subgroup $W_J \subset W = S_n$ becomes $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$, which is convenient.

5. Example. Take $n = 8$. We are going to consider 8-dimensional F-zips with type τ given by

$$\tau(0) = 1\,, \quad \tau(1) = 2\,, \quad \tau(2) = 5\,, \quad \tau(i) = 0 \quad \text{if } i \notin \{0, 1, 2\}.$$

So

$$n_1 = 5\,, \quad n_2 = 2\,, \quad n_3 = 1 \qquad \text{and} \qquad m_0 = 0\,, \quad m_1 = 5\,, \quad m_2 = 7\,, \quad m_2 = 8\,.$$

We have $J = (5, 2, 1)$,

$$x = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 2 & 3 & 1 \end{bmatrix}\,.$$

and $W_J = S_5 \times S_2 \times S_1 \subset W = S_8$.

We take

$$u = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 8 & 6 & 3 & 4 & 5 & 7 \end{bmatrix}\,,$$

which gives

$$u^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 7 & 4 & 8 & 3 \end{bmatrix} \qquad \text{and} \qquad x \cdot u = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 2 & 6 & 7 & 8 & 3 \end{bmatrix}\,.$$

Let $e_1, \ldots, e_8$ be the standard basis of $M^u = \mathbb{F}_p^8$. To simplify notation, if a, b, $c, \ldots$ are indices then we write $\langle a, b, c, \ldots \rangle$ instead of $\langle e_a, e_b, e_c, \ldots \rangle$, and we use a–b to indicate a range $a, a+1, \ldots, b$. The filtrations $C(0)^\bullet = C^\bullet$ and $D(0)_\bullet = D_\bullet$ on M^u are given by

$$C^0 = M \supset C^1 = \big\langle 1, 2, 4\text{–}8 \big\rangle \supset C^2 = \big\langle 1, 2, 5, 6, 7 \big\rangle \supset C^3 = 0$$

and

$$D_{-1} = 0 \subset D_0 = \big\langle 1 \big\rangle \subset D_1 = \big\langle 1, 2, 3 \big\rangle \subset D_2 = M\,.$$

The maps φ_i are the Frobenius-linear maps given by

$$\varphi_0 \colon e_3 \mapsto e_1\,, \qquad \varphi_1 \colon \begin{cases} e_4 \mapsto e_2 \\ e_8 \mapsto e_3 \end{cases}, \qquad \varphi_2 \colon \begin{cases} e_1 \mapsto e_4 \\ e_2 \mapsto e_5 \\ e_5 \mapsto e_6 \\ e_6 \mapsto e_7 \\ e_7 \mapsto e_8 \end{cases}.$$

To check that this is correct, we calculate by hand what happens in the refinement-transfer procedure that was described in point 2 above. After the first iteration, we obtain the flags

$$
\begin{aligned}
C(1): &\qquad M \supset \langle 1,2,4\text{–}8\rangle \supset \langle 1,2,5,6,7\rangle \supset \langle 1,2\rangle \supset \langle 1\rangle \supset 0\\
D(1): &\qquad 0 \subset \langle 1\rangle \subset \langle 1,2,3\rangle \subset \langle 1\text{–}4\rangle \subset \langle 1\text{–}5\rangle \subset M\,.
\end{aligned}
$$

At the next stage we get

$$
\begin{aligned}
C(2): &\quad M \supset \langle 1,2,4\text{–}8\rangle \supset \langle 1,2,4\text{–}7\rangle \supset \langle 1,2,5,6,7\rangle \supset \langle 1,2,5\rangle \supset \langle 1,2\rangle \supset \langle 1\rangle \supset 0\\
D(2): &\quad 0 \subset \langle 1\rangle \subset \langle 1,2\rangle \subset \langle 1\text{–}3\rangle \subset \langle 1\text{–}4\rangle \subset \langle 1\text{–}5\rangle \subset \langle 1\text{–}6\rangle \subset M\,.
\end{aligned}
$$

After one more iteration :

$$
\begin{aligned}
&C(3): M \supset \langle 1,2,4\text{–}8\rangle \supset \langle 1,2,4\text{–}7\rangle \supset \langle 1,2,5,6,7\rangle \supset \langle 1,2,5,6\rangle \supset \langle 1,2,5\rangle \supset \langle 1,2\rangle \supset \langle 1\rangle \supset 0\\
&D(3): 0 \subset \langle 1\rangle \subset \langle 1,2\rangle \subset \langle 1\text{–}3\rangle \subset \langle 1\text{–}4\rangle \subset \langle 1\text{–}5\rangle \subset \langle 1\text{–}6\rangle \subset \langle 1\text{–}7\rangle \subset M\,.
\end{aligned}
$$

As these are complete flags, the procedure stops here. As explained above we find that the relative position of $C(0)$ and $D(3) = D(\infty)$ is represented by the permutation u that we started with, because u applied to $C(0)$ gives a flag of which $D(\infty)$ is a refinement.