SEMINAR ON PERVERSE SHEAVES AND SOME APPLICATIONS

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Website and mailing list. I maintain a website about the seminar at

http://www.math.ru.nl/personal/bmoonen/Seminars/PerverseSheaves.html

If you wish to receive announcements, let me know by email (bmoonen@science.ru.nl), and I'll put you on the mailing list.

References. There is a lot of interesting literature, and what you want/need to read may depend on your background. In any case we will use the following texts:

BBD A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque 100.
Stacks The Stacks project. `http://stacks.math.columbia.edu`

On the seminar website, I'll put links to some other relevant papers.

Tentative planning for the first lectures.

Note: The following only attempts to give a rough indication of what could/should be covered. The lecturers are free to diverge from this.

Lecture 1. (Johan Commelin) *The étale site of a scheme.*

— Recap: étale morphisms, basic properties. Point out that étale morphisms need not be separated and need not be of finite type. On affines, it may be instructive to recall [Stacks], Lemma 10.141.2 (tag 00U9).
— The small étale site $X_{\text{ét}}$.
— Sheaves on $X_{\text{ét}}$. Functors induced by a morphism $f\colon X \to Y$. A useful remark to make is that if $f\colon X \to Y$ is a universal homeomorphism of schemes then f_* gives an equivalence on the categories of étale sheaves; cf. [Stacks], Thm. 50.46.2 (tag 04DZ). For instance, this applies to the canonical morphism $X_{\text{red}} \hookrightarrow X$, or we can pass from schemes over k^{sep} to schemes over $\bar{k}$.
— Example: for $X = \text{Spec}(k)$ with k a field, describe $X_{\text{ét}}$ and sheaves on in terms of Galois theory.
— Discuss constant sheaves, locally constant (= "lisse") sheaves, and constructible sheaves. Throughout, assume X is qcqs (quasi-compact + quasi-separated) and noetherian; this simplifies things and is more than enough for our purposes. Recommended sources: [Stacks], Sections 50.68 (tag 05BE) and 50.71 (tag 03RY); or SGA4, Exposé IX, Section 2. Note that we only require the basic notions, so don't spend much time on details.
— If time permits, and to put things in perspective, mention [Stacks], Lemma 50.67.4 (tag 03RV). Explain that, if X is a connected scheme and Λ is a finite ring, finite locally constant

sheaves of Λ-modules can be described by giving a finite Λ-module with continuous action of $\pi_1(X, \bar{x})$.

Lecture 2. (Steffen Sagave) *Triangulated categories and t-structures.*

— Recall the definition of a triangulated category. Key examples: the homotopy category of complexes $\mathsf{K}(\mathsf{A})$ and the derived category $\mathsf{D}(\mathsf{A})$, for A an abelian category.
— The truncation functors $\tau_{\geqslant n}$ and $\tau_{\leqslant n}$ on $\mathsf{D}(\mathsf{A})$.
— Definition of a t-structure and its heart.
— The truncation functors associated with a t-structure; BBD, Prop. 1.3.3.
— BBD, Thm. 1.3.6.
— Gluing of t-structures. The main result is BBD, Thm. 1.4.10. Try to outline the proof, starting from the axiomatic situation in BBD 1.4.3. (The full details may be too cumbersome.) Caution: there are some typos and inaccuracies in BBD. In 1.4.3, when they say that functors (such as j_* and $i^!$) are exact, this means these are triangulated functors; this may create some confusion. An alternative reference is Section 2 of the paper *Perverse t-structure on Artin stacks* by Laszlo and Olsson.

Lectures 3. (Ben Moonen) *The category $D^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$ and the six functors.*

The goal of this lecture is to explain how to obtain a good category with $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$-coefficients. For this we will use the pro-étale topology introduced by Bhatt and Scholze, thereby simplifying the approach taken in the earlier literature.

Lecture 4. (Milan Lopuhaä) *The perverse t-structure.*

— The main goal of the lecture is to define the perverse t-structure and to show (using the last result of Lecture 2) that it indeed is a t-structure. There are two main settings that we want to consider: (1) working with complex algebraic varieties and sheaves of $\mathbb{Q}$-vector spaces; (2) working with separated schemes of finite type over a field k that is either finite or separably closed, taking $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$ (with ℓ invertible in k) as coefficient field. It can be mentioned that one can do this with an arbitrary perversity, but we will be using only the middle perversity. Because of the work that was done in lectures 2 and 3, most of the subtleties from BBD, Section 2.2 can now be skipped. Define the perverse t-structure directly, i.e., by taking the result of BBD, Prop. 2.2.2 as a definition (cf. BBD, Section 4.0). See also Laszlo-Olsson, Section 3.
— Write $\mathsf{Perv}(X)$ (or if the context requires it, $\mathsf{Perv}(X, \overline{\mathbb{Q}}_\ell)$, etc.) for the abelian category of perverse sheaves. Explain that we have functors like ${}^{\mathrm{p}}f_*$, ${}^{\mathrm{p}}f^*$, etc. Define (left/right) exactness with respect to a t-structure, and give BBD, Propositions 1.4.16 and 1.4.17.
— Give BBD, Thm. 4.1.1 (without complete details), and its Corollaries 4.1.2 and 4.1.3.

Lecture 5. (Salvatore Floccari) *The middle extension functor.*

Throughout, work with separated schemes of finite type over a field k. Fix a prime number ℓ that is invertible in this field and take $\overline{\mathbb{Q}}_\ell$ as coefficient field. (We may also take $\mathbb{Q}_\ell$, and for complex algebraic varieties we can use $\mathbb{Q}$-coeffcients, etc. But we cannot do all variants.) Throughout, we work with the middle perversity, indicated by the letter p.

Introduce (or recall?) the notation $\mathsf{D}^{\leqslant \mathrm{p}+n} = {}^{\mathrm{p}}\mathsf{D}^{\leqslant n}$ and $\mathsf{D}^{\geqslant \mathrm{p}+n} = {}^{\mathrm{p}}\mathsf{D}^{\geqslant n}$, for n an integer. A standard situation that we will have to consider is that $j\colon U \hookrightarrow X$ is the inclusion of a (non-empty) Zarisk-open subset and $i\colon Z \hookrightarrow X$ is the closed complement. (Do not follow BBD in using F for the closed subscheme.) If the context allows it, we may simply write D_X for $\mathsf{D}(X, \overline{\mathbb{Q}}_\ell)$, etc.

— Explain that for $F \in \mathsf{Perv}(U)$ the morphism $j_!F \to j_*F$ factors through a morphism ${}^{\mathrm{p}}j_!F \to {}^{\mathrm{p}}j_*F$. As the latter is a morphism in the abelian category $\mathsf{Perv}(X)$, we can define $j_{!*}F = \mathrm{Im}({}^{\mathrm{p}}j_!F \to {}^{\mathrm{p}}j_*F)$. This gives a functor $j_{!*}\colon \mathsf{Perv}(U) \to \mathsf{Perv}(X)$ that commutes with Verdier duality.
— Explain that we have an isomorphism of functors $i^*j_* \cong i^!j_![1]$; see BBD, Section 1.4.6. If we temporarily abbreviate $i^*j_* = S$ then for F in D_U then we have a distinguished triangle $j_!F \to j_*F \to i_*S(F) \to$.
— Prove BBD, Corollaries 1.4.24 and 1.4.25, adapted to our setting. (It seems best to avoid introducing the truncation functors τ^F used in BBD.) The relevant parts of BBD are a little hard to read. That $j_{!*}B$ has the desired property is not so hard. To show the uniqueness, which boils down to BBD, Prop. 1.4.14, the strategy is to use that for $G \in \mathsf{D}_X$ with $F = j^*G$, we have a distinguished triangle $j_!F \to G \to i_*i^*G \to$ and to use the uniqueness of cones. So one needs to understand the homomorphism $i_*i^*G \to j_!F[1]$. The point is then that if the extension G has the desired cohomological properties, i^*G is a truncation of $S(F)$, where S is as in the previous point. (In BBD, S is called $(j_*/j_!)$.)
— Discuss some examples over the punctured disc, and if possible explain that $j_{!*}$ preserves injections and surjections but is not exact.
— Deduce BBD, Lemmas 4.3.2 and 4.3.3.
— Now prove BBD, Thm. 4.3.1.

Lecture 6. (Pol van Hoften) *Mixed complexes.*

— Give BBD, Prop. 5.1.2. It may be instructive to give at least a sketch of the proof.
— Notation as in BBD, Section 5.1. Define when a sheaf F_0 on X_0 is pointwise pure. Next: F_0 is mixed if it admits a finite filtration by pure sheaves.
— Introduce the subcategories $\mathsf{D}_{\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell) \subset \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$ of mixed complexes. Summarize the basic properties: these are triangulated subcategories that are stable under the six functors (we don't need vanishing cycles at this stage) and under the (standard) truncation functors $\tau_{\leqslant i}$ and $\tau_{\geqslant i}$. In BBD, 5.1.6 it is stated that it follows from this that the $\mathsf{D}_{\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ are also stable under the perverse truncation functors ${}^{\mathrm{p}}\tau_{\leqslant i}$ and ${}^{\mathrm{p}}\tau_{\geqslant i}$. State this as a proposition, and explain the proof. See for instance Section III.3 in the Kiehl-Weissauer book. Perhaps make this explicit if $\dim(X) = 1$.
— Define what it means for an object K in $\mathsf{D}_{\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ to be of weight $\leqslant w$ or $\geqslant w$.
— State the Main Theorem of Deligne's "Weil II", and give the list of properties in BBD, 5.1.14.
— If times permits, give Prop. 5.1.15, in particular parts (ii) and (iii).

Lecture 7. (Arne Smeets) *The weight filtration.*

Lecture 8. (Wessel Bindt) *The decomposition theorem.*

Lecture 9. *Passing to* $\mathbf{C}$*, and first applications.*