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Sums of Nearest Integer and Complex Continued Fractions

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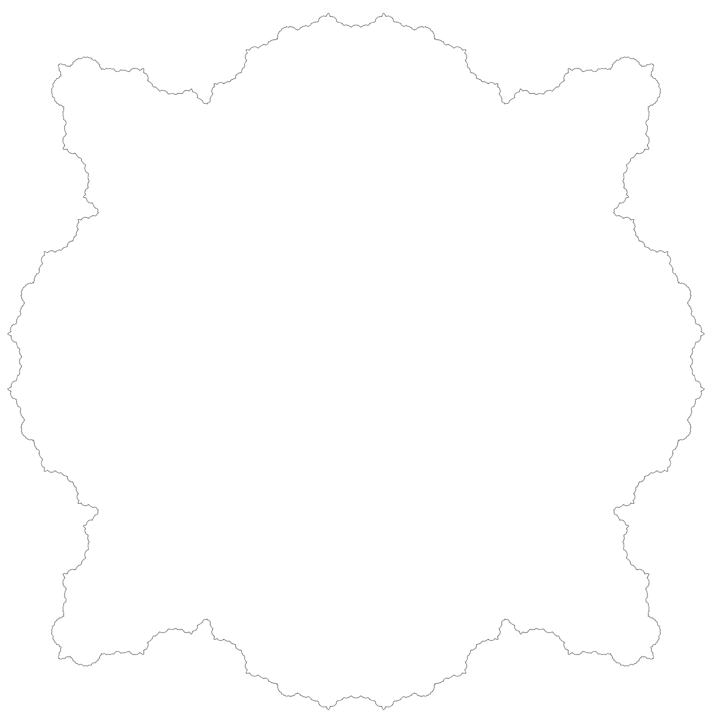

Mathematical Foundations of Computer Science
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Abstract

Let NICF_n be the set of real numbers with nearest integer continued fractions with absolute value of the partial coefficients no greater than n and let

$$\mathrm{NICF}_m + \mathrm{NICF}_n = \{a + b \mid a \in \mathrm{NICF}_m, b \in \mathrm{NICF}_n\}.$$

We show that

$$\mathrm{NICF}_5 + \mathrm{NICF}_5 = \mathbb{R}$$

and

$$\mathrm{NICF}_4 + \mathrm{NICF}_4 \neq \mathbb{R}.$$

Let HCF_n be the set of complex numbers with Hurwitz complex continued fractions with absolute value of the partial coefficients no greater than n and let

$$\mathrm{HCF}_m + \mathrm{HCF}_n = \{a + b \mid a \in \mathrm{HCF}_m, b \in \mathrm{HCF}_n\}.$$

We show that

$$\mathrm{HCF}_{\sqrt{5}} + \mathrm{HCF}_{\sqrt{5}} = \mathbb{C}.$$

Preface

This thesis started as a subject for a small Research Seminar course based on the master thesis of Noud Aldenhoven [1]:

> "Using two nearest integer continued fractions per axis, we know how we can write every complex number as a sum of four bounded complex continued fractions. Is it possible with three?"

It took a long time to find out I never saw a correct depiction of sets of bounded complex continued fractions, but after that, I did two things:

1. I improved the bounds needed on nearest integer continued fractions.
2. I solved my main question without referring to non-complex continued fractions.

I achieved both of this in a matter of weeks, and I remember sending an email to Wieb saying: "Ok, I'm done!"... except, I was not. This is when the work started, as it took me over a year to write it down...
Maybe that does seem long, but I really liked the subject. It has captured my attention and focus from the start and I am sure my friends and family have been at the receiving end of my enthusiasm more than once.

But this long year was quite the roller-coaster ride, as some might know. And this is where I share the wisdom I gained and thank the people who helped me. So here we go:

- "Always check your sources!"
- "Don't just work towards a positive result."
- "Don't underestimate how long the writing will take." (But did anyone expect it to take a year! Got you! You underestimated it too!)

But also:

- "It's OK that you came up with the result in three weeks."
- "Maak van je scriptie hooguit een meesterwerk, niet een levenswerk." ("This should be your master piece, not your life's work")
- "Don't be too critical, talk to people, also about your feelings."

And I followed this advice. So, I would like to thank everyone who let me tell, scream or cry out about my experiences writing this thesis.

Especially Wieb, for being my advisor (you must have recognised a lot of the advice I mentioned above), someone to talk with, to brainstorm with, and my mentor. You encouraged me, and I was so proud when you chose to present my work at a conference. I could not have asked for anyone better. Thank you, Wieb.

Yet I did not only have one great help and supervisor, I had another: Jan. You were always there to help me puzzle, plan and to make me take care of myself. I am sure that you recognise some of the sage advice mentioned above too. You are an amazing friend. Thank you for everything.

Bastiaan, for letting me use his tikz templates [2] for creating the shapes involved in complex continued fractions.

Kim, not only for the mind clearing dance sessions, but also for the times we talked about my life and problems instead of training for a competition. Thank you for always being there to listen and to offer your caring advice.

Suzan, for my daily dose of hugs. Thank you for your protective warmth.

Vincent, for a (promise of) pie, you firmly but kindly made me make small steps towards my goal, without letting me worry about what I had and had not achieved. Thank you for motivating me, and being my friend.

Nienke, for believing in me, even when I didn't. For listening to, and understanding, the vague outlines of the proofs, even though you had little prior knowledge. Thank you for joining the fight against my negative inner voices, and keeping me on my feet.

My parents, for never really giving up on me, no matter how long I took. I love you, thank you for all you have done for me.

And last, but not least, my boss, Paddy, for pretty much the same reason as my parents. You were patient and supporting no matter how long I took, and were happy to hear of intermediary progress. I am glad and proud to work for you.

Contents

Chapter 1

Continued fractions

A *continued fraction* is an expression of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

Continued fractions can be used as representations of numbers. But they can also be seen as representations of converging sequences of good rational approximations of a number.

Definition 1. *A* good approximation *of a real number x is a rational number $\frac{p}{q}$ such that for every rational number $\frac{p'}{q'}$ if $q' \leq q$, then $|x - \frac{p}{q}| \leq |x - \frac{p'}{q'}|$.*

Every real number x_0 can be represented by a continued fraction as follows, recursively, starting with $k = 0$:

- find an integer close to x_k, call it a_k;
- define x_{k+1} as $\frac{1}{x_k - a_k}$;
- continue when $x_{k+1} \neq 0$.

Note that, in almost all cases, x_k will never be 0, in which case the continued fraction representation will be infinite.
To better define what we mean by close to, we are going to describe this more thoroughly:

Definition 2. *We call a function $f \colon \mathbb{R} \to \mathbb{Z}$ an* approximation function *if for all $x \in \mathbb{R}$:*

$$|f(x) - x| < 1.$$

Given an approximation function apx, we describe an algorithm $\mathrm{GCF}(x)$ which returns a sequence of integers:

```
GCF(x) :=
  if x equals 0
    return []; \\empty list
  else
    \\ list form: head = apx(x), tail = GCF(1/(x - apx(x)))
    return [apx(x) :: GCF(1/(x - apx(x)))];
```

or:

```
GCF(x) :=
  while not(x equals 0) {
    yield return apx(x);
    x := 1/(x - apx(x));
  }
  yield return break;
```

The result will be a sequence of integers $a_0, a_1, a_2, \ldots$, which we will call the coefficients of x. When $x \in \mathbb{Q}$, the sequence will be finite. GCF(x) represents x when written as the *continued fraction*:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

Because continued fractions take a lot of space to write in this shape, we will use the abbreviated notation $[a_0; a_1, a_2, a_3, a_4, \ldots]$ in this thesis.
For construction purposes, it is useful to define a shorthand for continued fractions. For every n, we can write $x = [a_0; a_1, \ldots, a_n : \mu]$ when $x = [a_0; a_1, \ldots]$ and $\mu = [a_{n+1}; a_{n+2}, \ldots]$. So, when for all i, $x_i = [a_i; a_{i+1}, \ldots]$, then $x = [a_0 : x_1] = [a_0; a_1 : x_2] = \ldots = [a_0; a_1, a_2, \ldots, a_n : x_{n+1}]$. Note that this is equal to $x = [a_0; a_1, a_2, \ldots, a_{i-1} : a_i + \frac{1}{x_{i+1}}]$.

1.1 Complex continued fractions

We can extend our definition of an approximation function to a *complex approximation function*:

Definition 3. *We call a function $f\colon \mathbb{C} \to \mathbb{Z}[i]$ a* complex approximation function *if for all $x \in \mathbb{C}$: $|f(x) - x| < 1$.*

In combination with the GCF algorithm, we can create complex continued fractions, in which all the $a_j \in \mathbb{Z}[i]$.
For every $x \in \mathbb{R}$ we know that $f(x) \in \mathbb{Z}$. So every complex approximation function canonically induces a regular approximation function.

1.2 Rational approximations

A continued fraction of a real number x can be truncated to find a rational approximation of x. We will now take a closer look at how to calculate such an approximation.
Given $x = [a_0; a_1, a_2, \ldots] \in \mathbb{R}$, we want to find $p_n, q_n \in \mathbb{Z}$ with

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n].$$

For this we have ($i \geq 0$):

$$p_i = a_i p_{i-1} + p_{i-2} \qquad\qquad q_i = a_i q_{i-1} + q_{i-2},$$

with initial values:

$$p_{-2} = 0, \qquad p_{-1} = 1; \qquad q_{-2} = 1, \qquad q_{-1} = 0.$$

We can extend our definitions to find Gaussian rational approximations of complex numbers.
Given $x = [a_0; a_1, a_2, \ldots] \in \mathbb{C}$, we find $p_n, q_n \in \mathbb{Z}[i]$ with

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n].$$

The following theorems have been known for quite a time for real continued fractions, we will extend them to complex continued fractions:
Let f be a complex approximation function, and $x = [a_0; a_1, \ldots] \in \mathbb{C}$. Let $p_j, q_j \in \mathbb{Z}[i]$ be as described, then:

Theorem 4. *For all $j \geq -2 \in \mathbb{Z}$, we have that:*

$$p_j q_{j+1} - p_{j+1} q_j = (-1)^{j+1}.$$

Proof. This proof is similar to [7, §1.1 eq 3].
We will prove this by induction over j:

- When $j = -2$, we have that $p_{-2}q_{-1} - p_{-1}q_{-2} = -1 = (-1)^{-1}$;
- Assume $p_j q_{j+1} - p_{j+1} q_j = (-1)^{j+1}$, then we have:

$$\begin{aligned} p_{j+1}q_{j+2} - p_{j+2}q_{j+1} &= p_{j+1}(a_{j+2}q_{j+1} + q_j) - (a_{j+2}p_{j+1} + p_j)q_{j+1} \\ &= p_{j+1}q_j - p_j q_{j+1} = -1 \cdot (-1)^{j+1} = (-1)^{j+2}. \end{aligned}$$ □

Theorem 5. *For every $j \geq -1 \in \mathbb{Z}$, when $x = [a_0; a_1, a_2, \ldots, a_j : y]$, then $x = \frac{yp_j + p_{j-1}}{yq_j + q_{j-1}}$.*

Proof. This proof is similar to [7, §1.2 eq 5].
We will prove this by induction over j:

- When $j = -1$, we have that

$$\frac{yp_{-1} + p_{-2}}{yq_{-1} + q_{-2}} = y.$$

- Let $x = [a_0; a_1, a_2, \ldots, a_j : y]$, and assume that

$$x = \frac{(a_j + \frac{1}{y})p_{j-1} + p_{j-2}}{(a_j + \frac{1}{y})q_{j-1} + q_{j-2}}.$$

Then

$$x = \frac{(a_j + \frac{1}{y})p_{j-1} + p_{j-2}}{(a_j + \frac{1}{y})q_{j-1} + q_{j-2}} = \frac{a_j p_{j-1} + p_{j-2} + \frac{p_{j-1}}{y}}{a_j q_{j-1} + q_{j-2} + \frac{q_{j-1}}{y}} = \frac{p_j + \frac{p_{j-1}}{y}}{q_j + \frac{q_{j-1}}{y}} = \frac{yp_j + p_{j-1}}{yq_j + q_{j-1}}.$$ □

Theorem 6.

$$\lim_{j\to\infty} \frac{p_j}{q_j} = x.$$

Proof. This proof is based on [2, Lemma 3.38].
For every j, let $x_j = [a_j; a_{j+1}, a_{j+2}, \ldots]$, and notice that

$$\begin{aligned} x_{j+1} &= \frac{x_{j+1}(p_j q_{j-1} - p_{j-1} q_j)}{p_j q_{j-1} - p_{j-1} q_j} \\ &= -\frac{x_{j+1} p_j q_{j-1} + p_{j-1} q_{j-1} - x_{j+1} p_{j-1} q_j - p_{j-1} q_{j-1}}{x_{j+1} p_j q_j + p_{j-1} q_j - x_{j+1} p_j q_j - p_j q_{j-1}} \\ &= -\frac{\frac{x_{j+1} p_j + p_{j-1}}{x_{j+1} q_j + q_{j-1}} q_{j-1} - p_{j-1}}{\frac{x_{j+1} p_j + p_{j-1}}{x_{j+1} q_j + q_{j-1}} q_j - p_j} \\ &\overset{(*)}{=} -\frac{x q_{j-1} - p_{j-1}}{x q_j - p_j}. \end{aligned}$$

In (*) we used Theorem 5.
Now

$$\prod_{j=0}^{n} x_{j+1} = (-1)^{n+1} \prod_{j=0}^{n} \frac{x q_{j-1} - p_{j-1}}{x q_j - p_j} = (-1)^{n+1} \frac{x q_{-1} - p_{-1}}{x q_n - p_n} = \frac{(-1)^n}{x q_n - p_n},$$

and therefore we have

$$\left| x - \frac{p_n}{q_n} \right| = \left| \frac{x q_n - p_n}{q_n} \right| = \left| \frac{1}{q_n} \frac{1}{\prod_{j=0}^{n} x_{j+1}} \right|.$$

As $|x_j| \geq 1$ for $j \geq 1$, and $\lim_{j\to\infty} |q_j| = \infty$ by [2, Lemma 3.36][1], we have that:

$$\lim_{j\to\infty} \left| x - \frac{p_j}{q_j} \right| = \lim_{j\to\infty} \left| \frac{1}{q_n} \frac{1}{\prod_{j=0}^{n} x_{j+1}} \right| \leq \left| \frac{1}{q_n} \right| = 0.$$ □

Theorem 7. *If $x = [a_0; a_1, a_2, \ldots, a_n : \mu]$ and $y = [a_0; a_1, a_2, \ldots, a_n : \nu]$, then*

$$|x - y| = \frac{|\mu - \nu|}{q_n^2 \left(\mu + \frac{q_{n-1}}{q_n}\right)\left(\nu + \frac{q_{n-1}}{q_n}\right)}.$$

[1] Different definitions of q_n are used, but they only differ by a factor $\prod_{j=1}^{n} e_j$, with for every $j \geq 1$, $|e_j| = 1$.

Proof. This proof is similar to [4, Lemma 4].

$$\begin{aligned} x - y &\overset{(*)}{=} \frac{p_n\mu + p_{n-1}}{q_n\mu + q_{n-1}} - \frac{p_n\nu + p_{n-1}}{q_n\nu + q_{n-1}} \\ &= \frac{p_nq_{n-1}\mu - p_nq_{n-1}\nu + p_{n-1}q_n\nu - p_{n-1}q_n\mu}{(q_n\mu + q_{n-1})(q_n\nu + q_{n-1})} \\ &= \frac{1}{q_n^2}\frac{(p_nq_{n-1} - p_{n-1}q_n)(\mu - \nu)}{(\mu + \frac{q_{n-1}}{q_n})(\nu + \frac{q_{n-1}}{q_n})} \\ &\overset{(**)}{=} \frac{(-1)^{n+1}(\mu - \nu)}{q_n^2(\mu + \frac{q_{n-1}}{q_n})(\nu + \frac{q_{n-1}}{q_n})}. \end{aligned}$$

In (*) we used Theorem 5 and in (**) we used Theorem 4. □

1.3 Nearest integer continued fractions

There are many types of real continued fractions, we will focus on one in particular.

Definition 8. *The* nearest integer continued fraction (NICF) *of* $x \in \mathbb{R}$ *is the result of* GCF(x) *with* $f(y) = \lfloor y \rceil = \lfloor y + \frac{1}{2} \rfloor$.

There are a few things to point out involving the range of NICF.
For all $i \geq 1$:

- $a_i \in \mathbb{Z} \setminus \{-1, 0, 1\}$;
- if $a_i = 2$, then $a_{i+1} \geq 2$;
- if $a_i = -2$, then $a_{i+1} \leq -2$.

Usually, finite nearest integer continued fractions will not end with a 2. In this thesis, we choose to ignore this. This way every truncated infinite nearest integer continued fraction is accepted as a finite nearest integer continued fraction. It will not lead to problems, as our proofs only use finite continued fractions to work towards infinite continued fractions.
Note that for every $x = [a_0; a_1, a_2, \ldots] \in$ NICF, we have that $-x = [-a_0; -a_1, -a_2, \ldots] \in$ NICF.
Furthermore, there exists a bound on $\frac{q_{n-1}}{q_n}$ [6, p. 378]:

$$\left|\frac{q_{n-1}}{q_n}\right| \leq \frac{1}{\varphi} = \frac{\sqrt{5} - 1}{2}. \tag{1.1}$$

Definition 9. NICF$_r$ *is a subset of* $\mathbb{R}$*, containing only the numbers representable by a nearest integer continued fraction where every coefficient except possibly the first has absolute value less than or equal to* r*:*

$$\text{NICF}_r = \{x : x \in \mathbb{R} \mid x = [a_0; a_1, a_2, \ldots] \in \text{NICF} \ \ \textit{and} \ \forall_{j \geq 1} |a_j| \leq r\}$$

1.4 Hurwitz complex continued fractions

There are also many types of complex continued fractions, we will focus on one in particular which was first defined in [5]:

Definition 10. *The* Hurwitz complex continued fraction (HCF) *of x is the result of* GCF(x) *with* $f(a+bi) = \lfloor a \rceil + \lfloor b \rceil i = \lfloor a+\frac{1}{2} \rfloor + \lfloor b+\frac{1}{2} \rfloor i$.

Remark 11. *For $x \in \mathbb{R}$, the Hurwitz complex continued fraction of x is equal to the nearest integer continued fraction of x.*

It is difficult to describe the range of HCF, especially when, for any j the real or imaginary part of x_j is equivalent to $\frac{1}{2}$ modulo 1. These points are tie-breaking points of the approximation, therefore we have difficulties determining if these points are part of the range. Therefore, for the moment, we will only consider the range of elements where the real and imaginary part of x_j is not equivalent to $\frac{1}{2}$ modulo 1 for all j.
Let ρ be the rotation function that rotates a complex number 90 degrees counterclockwise around the origin:

$$\rho(a+bi) = (a+bi) \cdot i = -b + ai.$$

We see that for our specific representation of elements of $\mathbb{C}$:

$$\rho\left(x + \frac{1}{y}\right) = \rho(x) + \rho\left(\frac{1}{y}\right) = \rho(x) + \frac{1}{\rho^3(y)},$$

which leads to:

$$\rho([a+bi : \mu]) = [\rho(a+bi) : \rho^{-1}(\mu)] = [-b+ai : \rho^3(\mu)].$$

Definition 12. HCF$_r$ *is a subset of $\mathbb{C}$, containing only the numbers representable by a Hurwitz complex continued fraction where every coefficient except possibly the first has absolute value less than or equal to r:*

$$\text{NICF}_r = \{x : x \in \mathbb{C} \mid x = [a_0; a_1, a_2, \ldots] \in \text{NICF} \;\; \textit{and} \; \forall_{j \geq 1} |a_j| \leq r\}$$

1.5 Shapes

We will use *shapes* to describe the ways you can continue sequences of HCF. Each shape corresponds with a set of possible continuations.

Definition 13. *A* shape *is a connected open subset of* $\{a+bi \in \mathbb{C} \mid |a|, |b| < \frac{1}{2}\}$.

We will show that there are five *shapes* that occur as possible continuations in the construction of HCF. Every shape, except the first, has four orientations, which we will describe using rotations. Note that when we rotate the shape clockwise, the reciprocal shape rotates counterclockwise. By reciprocal shape we mean the set containing the reciprocal of all elements in the corresponding shape.

The shapes are called:

- S for Square;
- M for Moon;
- W for Without-a-corner;
- J for Just-a-corner;
- E for Empty.

The boundaries of the shapes are defined by the lines

$$\mathcal{Z}_k = \left\{ \rho^k(\frac{1}{2} + xi) \middle| x \in [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R} \right\},$$

and the circles

$$\mathcal{C}_k = \left\{ x \middle| |x - i^k| = 1 \right\}$$

and

$$\mathcal{C}'_k = \left\{ x \middle| |x - (1+i)^k| = 1 \right\}.$$

The rotation function ρ can be extended to shapes.
Let X be a shape, then:

$$\rho(X) = \{x \cdot i : x \in X\}.$$

In particular we have:

- $\rho(S) = S$;
- $\rho^4(M) = M$;
- $\rho^4(W) = W$;
- $\rho^4(J) = J$;
- $\rho(E) = E$.

Definition 14. *$\mathbb{S}$ is the set of the shapes S, M, W, J and E with their rotations.*

With this, we can build an automaton.

Definition 15. *A deterministic automaton is a 5-tuple, consisting of:*

- *a finite set of states Q;*
- *a set of input symbols Σ, the alphabet of the automaton;*
- *a transition function δ $(Q, \Sigma \to Q)$;*
- *an initial state $q_0 (\in Q)$;*
- *a set of accepting states $F (\subseteq Q)$.*

Given a string $w = a_1a_2a_3 \ldots a_n$ with $a_i \in \Sigma$, we say an automaton accepts *w if there exists a finite sequence of states r_i such that:*

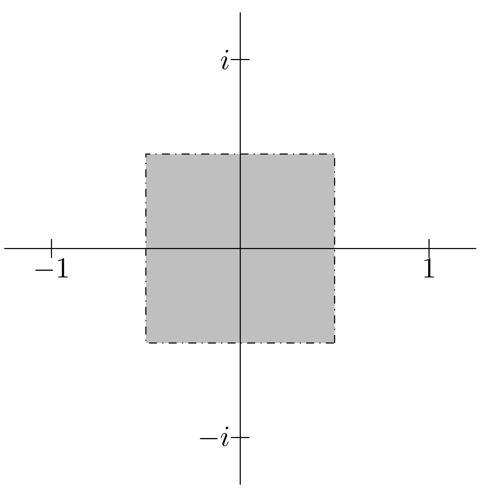
- $r_1 = q_0$.
- $\delta(r_i, a_i) = r_{i+1}$ *for* $0 < i < n$.
- $r_n \in F$.

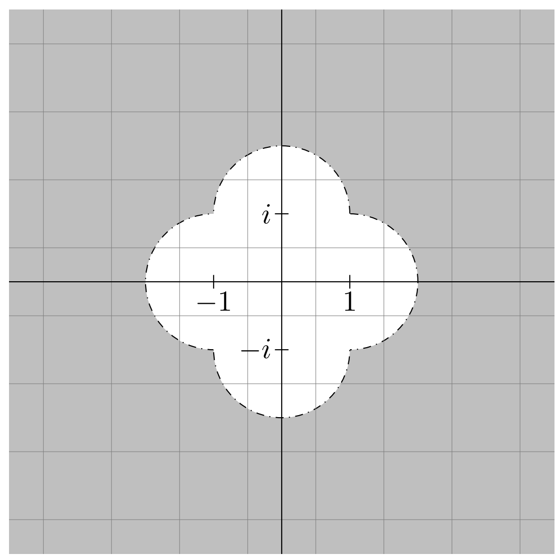
In our automaton, these will be described as follows:

- $\mathbb{S}$ is the finite set of states;
- the set of Gaussian integers is the alphabet;
- S (Square) is the initial state;
- the accepting states are the non-empty shapes, $\mathbb{S} \setminus E$.

We still have to construct our transition function $\delta\colon (\mathbb{S}, \mathbb{Z}[i]) \to \mathbb{S}$. We will do this per shape. For every shape, the left picture depicts the shape, while the right picture depicts the (pointwise) reciprocal of the shape.

1.5.1 Square



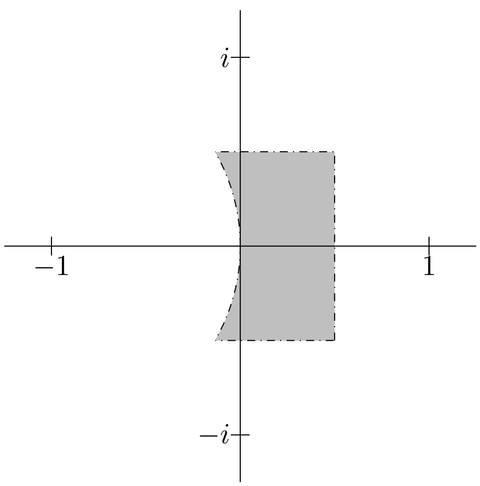


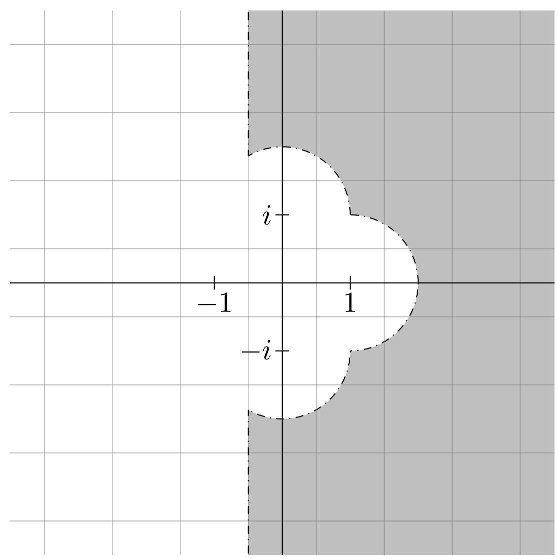
Given the coefficient $a_j + b_j \cdot i$, the transition function on S is defined by:
$\delta(S, a_j + b_j \cdot i) =$

$$\begin{aligned} S \text{ if } & |a_j| > 2, & \\ \text{or } & |b_j| > 2, & \\ \text{or } & |a_j| = 2 & \text{and } |b_j| = 2; \\ M \text{ if } & a_j = 2 & \text{and } b_j = 0; \end{aligned}$$

$$
\begin{array}{lll}
\rho(M) & \text{if } a_j = 0 & \text{and } b_j = 2; \\
\rho^2(M) & \text{if } a_j = -2 & \text{and } b_j = 0; \\
\rho^3(M) & \text{if } a_j = 0 & \text{and } b_j = -2; \\
W & \text{if } a_j = 1 & \text{and } b_j = 2, \\
 & \text{or } a_j = 2 & \text{and } b_j = 1; \\
\rho(W) & \text{if } a_j = -1 & \text{and } b_j = 2, \\
 & \text{or } a_j = -2 & \text{and } b_j = 1; \\
\rho^2(W) & \text{if } a_j = -1 & \text{and } b_j = -2, \\
 & \text{or } a_j = -2 & \text{and } b_j = -1; \\
\rho^3(W) & \text{if } a_j = 1 & \text{and } b_j = -2, \\
 & \text{or } a_j = 2 & \text{and } b_j = -1; \\
J & \text{if } a_j = 1 & \text{and } b_j = 1; \\
\rho(J) & \text{if } a_j = -1 & \text{and } b_j = 1; \\
\rho^2(J) & \text{if } a_j = -1 & \text{and } b_j = -1; \\
\rho^3(J) & \text{if } a_j = 1 & \text{and } b_j = -1.
\end{array}
$$

1.5.2 Moonshaped



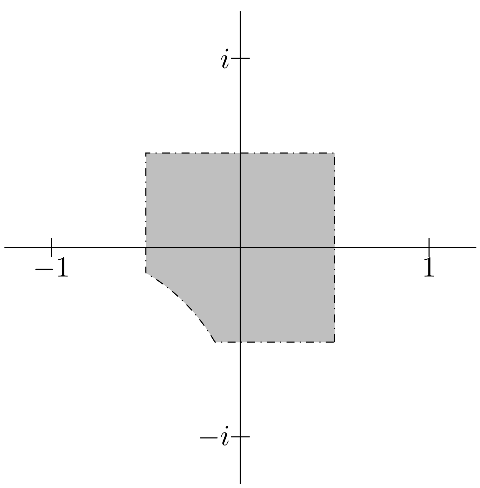


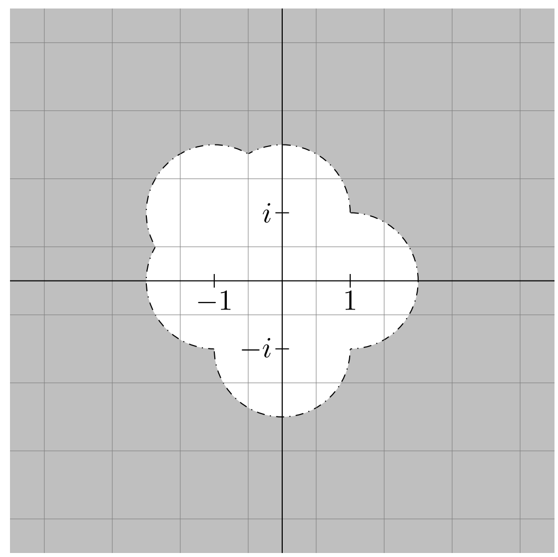
Given the coefficient $a_j + b_j \cdot i$, the transition function on M is defined by:
$\delta(M, a_j + b_j \cdot i) =$

$$
\begin{array}{lll}
S & \text{if } a_j > 2, & \\
 & \text{or } a_j \geq 0 & \text{and } |b_j| > 2, \\
 & \text{or } a_j = 2 & \text{and } |b_j| = 2;
\end{array}
$$

$$
\begin{array}{rr}
M \text{ if } a_j = 2 & \text{and } b_j = 0; \\
\rho(M) \text{ if } a_j = 0 & \text{and } b_j = 2; \\
\rho^3(M) \text{ if } a_j = 0 & \text{and } b_j = -2; \\
W \text{ if } a_j = 1 & \text{and } b_j = 2, \\
\text{or } a_j = 2 & \text{and } b_j = 1; \\
\rho^3(W) \text{ if } a_j = 1 & \text{and } b_j = -2, \\
\text{or } a_j = 2 & \text{and } b_j = -1; \\
J \text{ if } a_j = 1 & \text{and } b_j = 1; \\
\rho^3(J) \text{ if } a_j = 1 & \text{and } b_j = -1.
\end{array}
$$

1.5.3 Without-a-corner



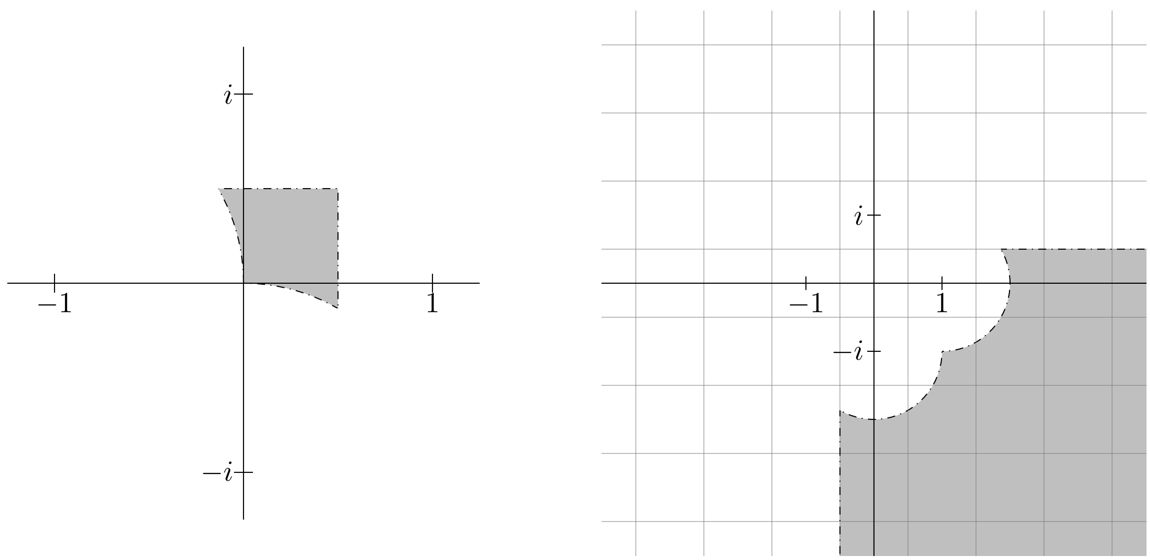


Given the coefficient $a_j + b_j \cdot i$, the transition function on W is defined by:
$\delta(W, a_j + b_j \cdot i) =$

$$
\begin{array}{rr}
S \text{ if } |a_j| > 2 & \text{or } |b_j| > 2, \\
\text{or } a_j = 2 & \text{and } |b_j| = 2; \\
\text{or } a_j = -2 & \text{and } b_j = -2; \\
M \text{ if } a_j = 2 & \text{and } b_j = 0; \\
\rho(M) \text{ if } a_j = 0 & \text{and } b_j = 2; \\
\text{or } a_j = -1 & \text{and } b_j = 2; \\
\rho^2(M) \text{ if } a_j = -2 & \text{and } b_j = 0; \\
\text{or } a_j = -2 & \text{and } b_j = 1; \\
\rho^3(M) \text{ if } a_j = 0 & \text{and } b_j = -2; \\
W \text{ if } a_j = 1 & \text{and } b_j = 2,
\end{array}
$$

$$\begin{array}{rl}
\text{or } a_j = 2 & \text{and } b_j = 1; \\
\rho(W) \text{ if } a_j = -2 & \text{and } b_j = 2, \\
\rho^2(W) \text{ if } a_j = -1 & \text{and } b_j = -2, \\
\text{or } a_j = -2 & \text{and } b_j = -1; \\
\rho^3(W) \text{ if } a_j = 1 & \text{and } b_j = -2, \\
\text{or } a_j = 2 & \text{and } b_j = -1; \\
J \text{ if } a_j = 1 & \text{and } b_j = 1; \\
\rho^2(J) \text{ if } a_j = -1 & \text{and } b_j = -1; \\
\rho^3(J) \text{ if } a_j = 1 & \text{and } b_j = -1.
\end{array}$$

1.5.4 Just-a-corner



Given the coefficient $a_j + b_j \cdot i$, the transition function on J is given by:
$\delta(J, a_j + b_j \cdot i) =$

$$\begin{array}{rl}
S \text{ if } a_j > 2 & \text{and } b_j \leq 0, \\
\text{or } a_j \geq 0 & \text{and } b_j < -2; \\
\text{or } a_j = 2 & \text{and } b_j = -2; \\
M \text{ if } a_j = 2 & \text{and } b_j = 0; \\
\rho^3(M) \text{ if } a_j = 0 & \text{and } b_j = -2; \\
\rho^3(W) \text{ if } a_j = 1 & \text{and } b_j = -2, \\
\text{or } a_j = 2 & \text{and } b_j = -1; \\
\rho^3(J) \text{ if } a_j = 1 & \text{and } b_j = -1.
\end{array}$$

1.5.5 Shape function

For every shape $X \in \mathbb{S}$, the transition function of $\rho(X)$ is defined by

$$\delta(\rho(X), a+bi) := (\rho^3 \circ \delta)(X, \rho(a+bi)) = (\rho^3 \circ \delta)(X, -b+ai).$$

For all other values, let $\delta(X, a+bi)$ be the empty shape E; in particular for all $a+bi \in \mathbb{Z}[i]$:

$$\delta(E, a+bi) = E.$$

With this, we can create a function Shape which gives the shape corresponding to a sequence of Gaussian integers. We use λ to denote the empty sequence.

Definition 16. *Let* $\text{Shape}\colon \mathbb{Z}[i]^* \to \mathbb{S}$ *be inductively defined by:*

$$\text{Shape}(\lambda) := S$$

and

$$\text{Shape}(a_1 \dots a_n) := \delta(\text{Shape}(a_1 \dots a_{n-1}), a_n).$$

The accepting function of the built automaton can now be used to determine which sequences occur in HCF. For every sequence $a_0, a_1, \dots, a_n \in \mathbb{Z}[i]$:

$$\text{Shape}(a_1 \dots a_n) \neq E \iff [a_0; a_1, \dots, a_n] \in \text{HCF}\,.$$

And for all infinite sequences $a_0, a_1, \dots \in \mathbb{Z}[i]$:

$$[a_0; a_1, \dots] \in \text{HCF} \iff \forall_n [a_0; a_1, \dots, a_n] \in \text{HCF}\,.$$

Lemma 17. *For* $a = [a_0; a_1, \dots, a_n] \in \text{HCF}$ *with* $\text{Shape}(a_1 \dots a_n) = X$ *we have,*

- *if n is even, then* $\rho(a) = [\rho(a_0); \rho^3(a_1), \rho(a_2), \dots, \rho^3(a_{n-1}), \rho(a_n)]$ *and*

$$\text{Shape}(\rho^3(a_1)\rho(a_2)\dots\rho^3(a_{n-1})\rho(a_n)) = \rho(X);$$

- *if n is odd, then* $\rho(a) = [\rho(a_0); \rho^3(a_1), \rho(a_2), \dots, \rho(a_{n-1}), \rho^3(a_n)]$ *and*

$$\text{Shape}(\rho^3(a_1)\rho(a_2)\dots\rho(a_{n-1})\rho^3(a_n)) = \rho^3(X).$$

Proof. Induction over n:

- When $n = 1$, we have $\rho(a) = [\rho(a_0); \rho^3(a_1)]$:

$$\text{Shape}(\rho^3(a_1)) = \delta(\text{Shape}(\lambda), \rho^3(a_1)) = \delta(S, \rho^3(a_1))$$

$$= \delta(\rho(S), \rho^3(a_1)) = \rho^3(\delta(S, a_1)).$$

- Induction step, n is even: $a = [a_0; a_1, \ldots, a_n]$, let $b = [a_0; a_1, \ldots, a_{n-1}]$, and let

$$\text{Shape}(a_0 a_1 \ldots a_{n-1}) = X_b,$$

so let $X = \text{Shape}(a_1 \ldots a_n) = \delta(X_b, a_n)$. Our Induction Hypothesis tells us that

$$\text{Shape}(\rho^3(a_1)\rho(a_2) \ldots \rho^3(a_{n-1})) = \rho^3(X_b).$$

This gives us:

$$\text{Shape}(\rho^3(a_1)\rho(a_2) \ldots \rho^3(a_{n-1})\rho(a_n)) = \delta(\rho^3(X_b), \rho(a_n)) = \rho(\delta(X_b, a_n)) = \rho(X).$$

- Induction step, n is odd: $a = [a_0; a_1, \ldots, a_n]$, let $b = [a_0; a_1, \ldots, a_{n-1}]$, and let

$$\text{Shape}(a_0 a_1 \ldots a_{n-1}) = X_b,$$

so let $X = \text{Shape}(a_1 \ldots a_n) = \delta(X_b, a_n)$. Our Induction Hypothesis tells us that

$$\text{Shape}(\rho^3(a_1)\rho(a_2) \ldots \rho(a_{n-1})) = \rho(X_b).$$

This gives us:

$$\text{Shape}(\rho^3(a_1)\rho(a_2) \ldots \rho(a_{n-1})\rho^3(a_n)) = \delta(\rho(X_b), \rho^3(a_n)) = \rho^3(\delta(X_b, a_n)) = \rho^3(X). \qquad \square$$

Chapter 2

Cantor sets

A Cantor set C of real numbers is defined by an initial closed interval I_1 and a gap function g.

Definition 18. *A* gap function *on a closed interval I_1 is a function from closed subintervals of I_1 to open subintervals of I_1 such that for every closed subinterval I of I_1, the open subinterval $g(I)$ is contained in I. The closed subintervals I_L and I_R of I_1 for which $I \setminus g(I) = I_L \cup I_R$ are called the* left *and* right remainder*.*

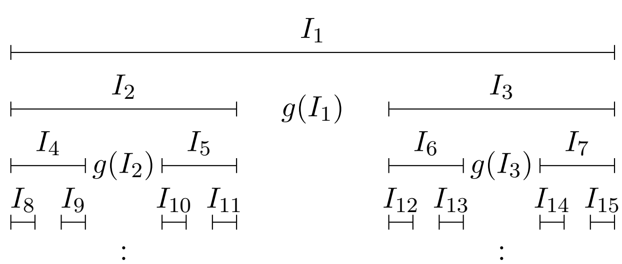
Constructing the Cantor set is done iteratively by the following protocol:

1. Create a set S with only the initial interval I_1.
2. For every interval I_i in S, remove the open subinterval (gap) $g(I_i)$.
3. Replace I_i in S by I_{2i} and I_{2i+1} for respectively the left and right remainder of I_i.
4. Repeat from 2.

The Cantor set (I_1, g) consists of all the points not contained in a gap:

$$(I_1, g) = I_1 \setminus \bigcup_{i=1}^{\infty} g(I_i).$$

The construction of a Cantor set can be drawn in a treelike picture:



Example 19. For $x, y \in [0, 1]$ with $x < y$ define

$$h([x, y]) = \left(\sqrt{\frac{(2x+y)(x+y)}{6}}, \sqrt{\frac{(x+2y)(x+y)}{6}} \right).$$

Because

$$x = \sqrt{\frac{(2x+x)(x+x)}{6}} < \sqrt{\frac{(2x+y)(x+y)}{6}} < \sqrt{\frac{(x+2y)(x+y)}{6}} < \sqrt{\frac{(y+2y)(y+y)}{6}} = y$$

we have that h is a gap function on $[0,1]$, and $H = ([0,1], h)$ is a Cantor set.

Given $a_l, a_r \in \mathbb{R}$ with $a_l \leq a_r$, an interval $A = [a_l, a_r] \subset \mathbb{R}$ is defined by

$$A = \{x : a_l \leq x \leq a_r\}.$$

The length of an interval $[a_l a_r]$ is defined by

$$\overline{[a_l, a_r]} = a_r - a_l.$$

The length of a gap (g_l, g_r) is defined by

$$\overline{(g_l, g_r)} = g_r - g_l.$$

Note that all the $g(I_i)$ are disjoint, so $\sum_{i=1}^{\infty} \overline{g(I_i)} \leq \overline{I_1}$. Because $\overline{I_1}$ is finite, we have

$$\lim_{i \to \infty} \overline{g(I_i)} = 0.$$

We are going to look when the sums of multiple Cantor sets are equal to the sum of their initial intervals (Theorem 28). This largely depends on the size of the gaps compared to the corresponding remainders. We will therefore define a *density ratio* of a Cantor set as follows:

Definition 20. *The* density ratio, $\mathrm{dr}((I_1, g))$ *is defined as:*

$$\mathrm{dr}((I_1, g)) = \inf_{i=1}^{\infty} \frac{\min(\overline{I_{2i}}, \overline{I_{2i+1}})}{\overline{g(I_i)}}.$$

It follows that $\overline{I_{2i}} \geq \mathrm{dr}((I_1, g)) \cdot \overline{g(I_i)}$ and $\overline{I_{2i+1}} \geq \mathrm{dr}((I_1, g)) \cdot \overline{g(I_i)}$, for all i.

2.1 Hole-decreasing Cantor sets

Definition 21. *We call a Cantor set* hole-decreasing *if for all i we have $\overline{g(I_i)} \geq \overline{g(I_{2i})}$ and $\overline{g(I_i)} \geq \overline{g(I_{2i+1})}$.*

Theorem 22. *For every Cantor set C, there exists a hole-decreasing Cantor set D such that $D = C$ and $\mathrm{dr}(D) \geq \mathrm{dr}(C)$.*

First, note that for all a, b: $I_b \subseteq I_a$, if and only if there exist $k_1, k_2, \ldots, k_n$ such that $I_a = I_{k_1}$, $I_b = I_{k_n}$, and for all $i < n : k_{i+1} = 2 \cdot k_i$ or $k_{i+1} = 2 \cdot k_i + 1$.
With this, we can rewrite the definition of a hole-decreasing Cantor Set:

Property 23. A Cantor set is *hole-decreasing* if and only if for every i, for every k such that $I_k \subseteq I_i$ we have $\overline{g(I_i)} \geq \overline{g(I_k)}$.

With this, we can define a n-hole-decreasing Cantor set:

Definition 24. *We call a Cantor set* (I_1, g) n-hole-decreasing *if for every* $i \leq n$, *for every* k *such that* $I_k \subseteq I_i$ *we have* $\overline{g(I_i)} \geq \overline{g(I_k)}$.

The Cantor set C is hole-decreasing if and only if C is n-hole-decreasing for every n.
We will first prove that we can interchange (by tree-rotation) an interval with one of its remainders, if that remainder has a larger gap. We will only prove this for the left remainder, for the proof of the right remainder is similar.

Lemma 25. *Given a Cantor set* (I_1, g), *let* i *be such that* $\overline{g(I_{2i})} > \overline{g(I_i)}$. *We can create* J_1 *and* g' *such that* $(J_1, g') = (I_1, g)$, *and for each* $j < i$ *we have* $g'(J_j) = g(I_j)$, $g'(J_i) = g(I_{2i})$, *and* $\mathrm{dr}((J_1, g')) \geq \mathrm{dr}((I_1, g))$.

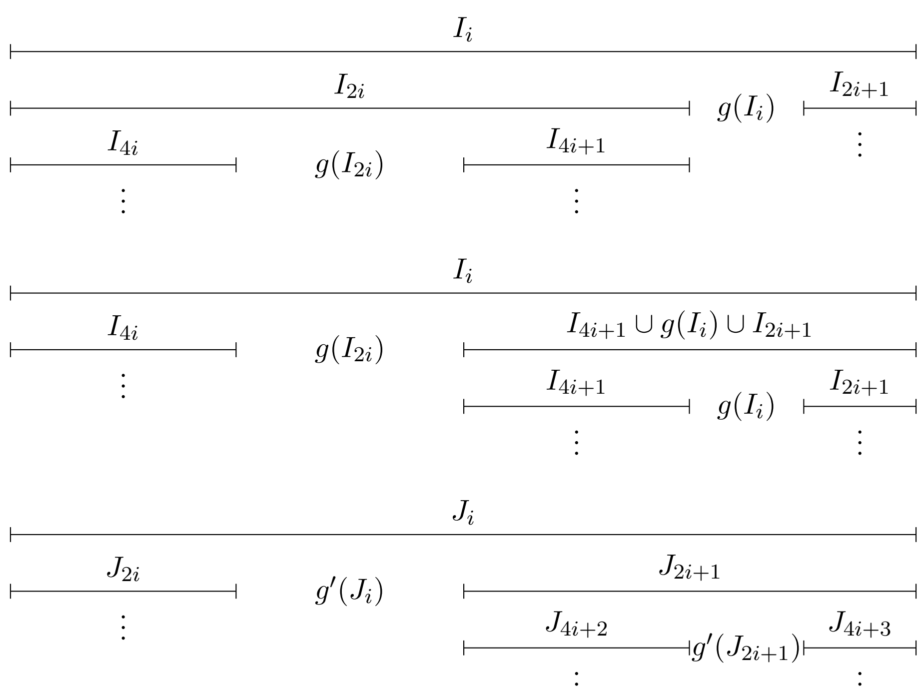
Proof. We define J_1 and g' by:

$$\begin{aligned} J_1 &= I_1 \\ g'(J_i) &= g(I_{2i}) \\ g'(J_{2i+1}) &= g'(I_{4i+1} \cup g(I_i) \cup I_{2i+1}) = g(I_i) \end{aligned}$$

For all other intervals, we let $g'(I) = g(I)$, which corresponds to:

$$\begin{aligned} g'(J_k) &= g(I_k) && \text{if } J_k \not\subseteq J_i; \\ g'(J_{ak+b}) &= g(I_{2ak+b}) && \text{if } J_{ak+b} \subseteq J_{2i}; \\ g'(J_{ak+b}) &= g(I_{ak+b-1}) && \text{if } J_{ak+b} \subseteq J_{4i+2}; \\ g'(J_{ak+b}) &= g(I_{\frac{a}{2}k+b-2}) && \text{if } J_{ak+b} \subseteq J_{4i+3}. \end{aligned}$$

for all $a, b, k \in \mathbb{Z}^+$.
Pictorially:



For the density ratio we note that the following (in-)equalities hold:

$$\begin{aligned}
\frac{\overline{J_{2i}}}{\overline{g'(J_i)}} &= \frac{\overline{I_{4i}}}{\overline{g'(J_i)}} = \frac{\overline{I_{4i}}}{\overline{g(I_{2i})}} \\
\frac{\overline{J_{2i+1}}}{\overline{g'(J_i)}} &= \frac{\overline{I_{4i+1} \cup g(I_i) \cup I_{2i+1}}}{\overline{g'(J_i)}} > \frac{\overline{I_{4i+1}}}{\overline{g'(J_i)}} = \frac{\overline{I_{4i+1}}}{\overline{g(I_{2i})}} \\
\frac{\overline{J_{4i+2}}}{\overline{g'(J_{2i+1})}} &= \frac{\overline{I_{4i+1}}}{\overline{g'(J_{2i+1})}} = \frac{\overline{I_{4i+1}}}{\overline{g(I_i)}} > \frac{\overline{I_{4i+1}}}{\overline{g(I_{2i})}} \\
\frac{\overline{J_{4i+3}}}{\overline{g'(J_{2i+1})}} &= \frac{\overline{I_{2i+1}}}{\overline{g'(J_{2i+1})}} = \frac{\overline{I_{2i+1}}}{\overline{g(I_i)}}
\end{aligned}$$

Now we have shown that for each interval for which g' is defined differently from g, the ratio with g' is greater than or equal to the ratio of an interval with g. Therefore the density ratio of (J_1, g') is greater than or equal to the density ratio of (I_1, g). □

Proof of Theorem 22. Since $\lim_{n\to\infty} \overline{g(I_n)} = 0$, we have that for every i, there exists a j with $I_j \subseteq I_i$ such that for every m with $I_m \subseteq I_i$ we have $\overline{g(I_j)} \geq \overline{g(I_m)}$. So, there exist $k_0, k_1, \ldots, k_n$ such that $I_i = I_{k_0}$, $I_j = I_{k_n}$, and for all $i < n$ we have either $k_{i+1} = 2 \cdot k_i$ or $k_{i+1} = 2 \cdot k_i + 1$.
Because $\overline{g(I_j)}$ is bigger than every $\overline{g(I_{k_i})}$, rotations can be used to first interchange I_{k_n} with $I_{k_{n-1}}$, without interfering with any of the other I_{k_i}. Then we can interchange $I_{k_{n-1}}$ with $I_{k_{n-2}}$, and so on until we interchange I_{k_1} with I_{k_0}. This means, we can use n rotations to interchange I_i with I_j.
We can therefore create a function that performs n rotations: $R((I, g), i) = (J, g')$ such that $\overline{(I, g)} = \overline{(J, g')}$, for every $j < i$: $g'(J_j) = g(I_j)$, and for every k with $J_k \subseteq J_i$: $\overline{g'(J_i)} \geq \overline{g'(J_k)}$.
We will use this to create a sequence $(I^0, g_0), (I^1, g_1), (I^2, g_2), \ldots$ such that for every $i, \overline{(I_1, g)} = \overline{(I^i, g_i)}$, and (I^n, g_n) is n-hole-decreasing.

$$\begin{aligned}
(I^0, g_0) &= (I, g) \\
(I^1, g_1) &= R((I^0, g_0), 1) \\
(I^2, g_2) &= R((I^1, g_1), 2) \\
&\vdots \\
(I^n, g_n) &= R((I^{n-1}, g_{n-1}), n)
\end{aligned}$$

We know that for all i, j with $0 < i < j$ we have $I^i_i = I^j_i$, and $g_i(I^i_i) = g_j(I^j_i)$. With this we can create (J, g'), where $J = I$, and for $n \geq 1$:

$$g'(J_n) = g_n(I^n_n)$$

This is a Cantor set, as for all i, j with $i \leq j$ we have $J_i = I^j_i$ and $g'(J_i) = g_j(I^j_i)$. Because (I^n, g_n) is n-hole-decreasing for every n, (J, g') is hole-decreasing. □

2.2 Intervals

We define the pointwise addition of two intervals as:

$$A + B = \{x + y : x \in A, y \in B\}.$$

Hence

$$[a_l, a_r] + [b_l, b_r] = [a_l + b_l, a_r + b_r].$$

Trivial results:

- $+$ is commutative: $A + B = B + A$;
- $+$ is associative: $(A + B) + C = A + (B + C)$;
- $\overline{A + B} = \overline{A} + \overline{B}$.

Lemma 26. *Let A and B be intervals; if $\overline{A} + \overline{B} \geq \overline{[\min(a_l, b_l), \max(a_r, b_r)]}$, then $A \cup B$ is an interval, namely $[\min(a_l, b_l), \max(a_r, b_r)]$.*

Proof. By contraposition.
Suppose that $A \cup B$ is not an interval. Then since $A \cup B \subseteq [\min(a_l, b_l), \max(a_r, b_r)]$ there exists an $x \in [\min(a_l, b_l), \max(a_r, b_r)]$ such that $x \notin A \cup B$. Then A and B are disjoint and

$$\overline{A} + \overline{B} < \overline{[\min(a_l, b_l), x]} + \overline{[x, \max(a_r, b_r)]} = \overline{[\min(a_l, b_l), \max(a_r, b_r)]}. \qquad \square$$

From this follows:

Corollary 27. *For I_i an interval of a Cantor set with gap function g, and J an interval, if*

$$\overline{J} \geq \overline{g(I_i)}$$

then

$$(I_{2i} + J) \cup (I_{2i+1} + J) = I_i + J. \tag{2.1}$$

Proof. Apply Lemma 26 with $A = I_{2i} + J$ and $B = I_{2i+1} + J$ together with:

$$\overline{I_i + J} = \overline{I_{2i}} + \overline{g(I_i)} + \overline{I_{2i+1}} + \overline{J} \leq \overline{I_{2i} + J} + \overline{I_{2i+1} + J}. \qquad \square$$

2.3 Comparable and Dividable

In this section, we will show the density requirements needed for sums of Cantor sets to ignore gaps. This is done in a similar way as done by Hlavka in [4], with *comparable* and *dividable* intervals. We aim to prove the following theorem:

Theorem 28. *Let $C^1 = (I^1, g^1), \ldots, C^n = (I^n, g^n)$ be hole-decreasing Cantor sets. If the density ratios satisfy* $\mathrm{dr}(C^j) \geq x_j$ *with*

$$\sum_{i=1}^{n} \frac{x_i}{x_i + 1} \geq 1; \tag{2.2}$$

and

$$\forall_{j,k}\ \overline{I_1^j} \geq \frac{x_j}{x_j + 1}(x_k + 1)\overline{I_1^k}, \tag{2.3}$$

then

$$\sum_{i=1}^{n} I_1^i = \{\sum_{i=1}^{n} x_i : x_i \in C^i\}.$$

We will use some definitions and lemmas to prove this theorem. Up to the proof, assume we work with hole-decreasing Cantor sets. Let $I_{i_j}^j$ be in the construction of C^j, with $x_1, \ldots, x_n$ such that $\mathrm{dr}(C^j) \geq x_j$, and (2.2) and (2.3) hold:

Definition 29. *We call intervals $I_{i_1}^1, I_{i_2}^2, \ldots, I_{i_n}^n$* comparable *if*

$$\forall_{j,k}\ \overline{I_{i_j}^j} \geq \frac{x_j}{x_j + 1}(x_k + 1)\overline{g^k(I_{i_k}^k)}.$$

Trivially, we see that $I_1^1, I_1^2, \ldots, I_1^n$ are comparable.

Lemma 30. *If $I_{i_1}^1, I_{i_2}^2, \ldots, I_{i_n}^n$ are comparable intervals, then for all k:*

$$\sum_{j=1}^{n} I_{i_j}^j = \left(\sum_{j=1; j \neq k}^{n} I_{i_j}^j + I_{2i_k}^k \right) \cup \left(\sum_{j=1; j \neq k}^{n} I_{i_j}^j + I_{2i_k+1}^k \right).$$

Proof. We will use (2.1) with $\sum_{j=1;j\neq k}^{n} I_{i_j}^j$ and $I_{i_k}^k$ for respectively J and I_i. Hence, what remains to be shown is that $\sum_{j=1;j\neq k}^{n} \overline{I_{i_j}^j} \geq \overline{g^k(I_{i_k}^k)}$. Indeed,

$$\begin{aligned}
\sum_{j=1; j \neq k}^{n} \overline{I_{i_j}^j} &\geq \left(\sum_{j=1; j \neq k}^{n} \frac{x_j}{x_j + 1} \right) (x_k + 1)\overline{g^k(I_{i_k}^k)} \\
&= \left(\sum_{j=1}^{n} \frac{x_j}{x_j + 1} - \frac{x_k}{x_k + 1} \right) (x_k + 1)\overline{g^k(I_{i_k}^k)} \\
&\overset{(2.2)}{\geq} \left(1 - \frac{x_k}{x_k + 1} \right) (x_k + 1)\overline{g^k(I_{i_k}^k)} \\
&= \overline{g^k(I_{i_k}^k)}.
\end{aligned}$$

□

Definition 31. *We call comparable intervals $I_{i_1}^1, I_{i_2}^2, \ldots, I_{i_n}^n$* j-dividable *if both $I_{i_1}^1, I_{i_2}^2, \ldots, I_{2i_j}^j, \ldots, I_{i_n}^n$ and $I_{i_1}^1, I_{i_2}^2, \ldots, I_{2i_j+1}^j, \ldots, I_{i_n}^n$ are comparable intervals.*

Lemma 32. *Given comparable intervals $I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}$. If j is such that $(x_j+1)\overline{g^j(I^j_{i_j})}$ is maximal, i.e.,*

$$\forall_k : (x_j+1)\overline{g^j(I^j_{i_j})} \geq (x_k+1)\overline{g^k(I^k_{i_k})},$$

then $I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}$ are j-dividable.

Proof. Since the Cantor sets are hole-decreasing, we have that $\overline{g^j(I^j_{2i_j})} \leq \overline{g^j(I^j_{i_j})}$ and $\overline{g^j(I^j_{2i_j+1})} \leq \overline{g^j(I^j_{i_j})}$. We still have to prove

$$\forall_k : \overline{I^j_{2j}} \geq \frac{x_j}{x_j+1}(x_k+1)\overline{g^k(I^k_{i_k})}$$

and

$$\forall_k : \overline{I^j_{2j+1}} \geq \frac{x_j}{x_j+1}(x_k+1)\overline{g^k(I^k_{i_k})}.$$

As both $\overline{I^j_{2i_j}} \geq x_j\overline{g^j(I^j_{i_j})}$ and $\overline{I^j_{2i_j+1}} \geq x_j\overline{g^j(I^j_{i_j})}$, it is sufficient to remark that:

$$x_j\overline{g^j(I^j_{i_j})} \geq x_j\overline{g^j(I^j_{i_j})}\frac{(x_k+1)\overline{g^k(I^k_{i_k})}}{(x_j+1)\overline{g^j(I^j_{i_j})}} = \frac{x_j}{x_j+1}(x_k+1)\overline{g^k(I^k_{i_k})}.$$

□

Proof of Theorem 28. We will recursively create sets G_m on n-tuples of intervals.

$$G_0 = \{\langle I^1_1, I^2_1, \ldots, I^n_1\rangle\};$$

$$G_{m+1} = \Big\{\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^k_{2i_k}, \ldots, I^n_{i_n}\rangle, \langle I^1_{i_1}, I^2_{i_2}, \ldots, I^k_{2i_k+1}, \ldots, I^n_{i_n}\rangle$$
$$\Big| \langle I^1_{i_1}, I^2_{i_2}, \ldots, I^k_{i_k}, \ldots, I^n_{i_n}\rangle \in G_m \wedge \forall l : x_k \cdot \overline{g^k(I^k_{i_k})} \geq x_l \cdot \overline{g^l(I^l_{i_l})}\Big\}.$$

Trivially, every $\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}\rangle \in G_0$ is comparable. By induction and Lemma 32, we have that for every m $\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}\rangle \in G_m$ is comparable. And therefore, for every m:

$$\bigcup\Big\{\sum I^j_{i_j}\Big|\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}\rangle \in G_m\Big\} = \sum I^j_1.$$

Let $g^k(I^k_{i_k})$ be a gap of C^k, then for every j there are only a finite number a_j of gaps $g^j(I^j_l)$ such that $\overline{g^j(I^j_{l_j})} \geq \frac{x_k}{x_j}\overline{g^k(I^k_{i_k})}$. For every $a \geq \sum a_j$, there will be no $\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}\rangle \in G_a$ such that $g^k(I^k_{i_k}) \subseteq I^k_{i_k}$.
Let us define G as:

$$G = \lim_{i\to\infty} G_i.$$

Then for each $\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}\rangle \in G$ we have $I^j_{i_j} \subset C^j$. Therefore

$$\left\{\sum_{i=1}^n x_i\middle| x_i \in C^i\right\} \subseteq \bigcup\Big\{\sum c_j\Big|\langle I^1_{i_1}, I^2_{i_2}, \ldots, I^n_{i_n}\rangle \in G \wedge c_j \in I^j_{i_j}\Big\} = \sum I^j_1.$$

□

If we drop the condition that the Cantor sets are hole-decreasing, we have the following result:

Theorem 33. *Given Cantor sets* $C^1, C^2, \ldots, C^n$ *with density ratios greater than or equal to* $x_1, x_2, \ldots, x_n$ *with*

$$\sum_{i=1}^{n} \frac{x_i}{x_i+1} \geq 1; \tag{2.4}$$

$$\forall_{j,k} \overline{I_1^j} \geq \frac{x_j}{x_j+1}(x_k+1) \max_i \overline{g(I_i^k)}. \tag{2.5}$$

Then

$$\sum_{i=1}^{n} I_1^i = \left\{ \sum_{i=1}^{n} x_i \;\middle|\; x_i \in C^i \right\}.$$

Proof. For every $C^j = (I^j, g^j)$, we can create a hole-decreasing Cantor set $D^j = (J^j, g'^j)$, such that $D^j = C^j$. We know the density ratio of D^j is greater than or equal to the density ratio of C^j, so x_j is smaller than or equal to the density ratio of D^j.
We also know that $\overline{g'(J_1^j)} = \max_i \overline{g(I_i^j)}$.
This gives all the premises needed in this chapter:

$$\left\{ \sum_{i=1}^{n} x_i \;\middle|\; x_i \in C^i \right\} = \left\{ \sum_{i=1}^{n} x_i \;\middle|\; x_i \in D^i \right\} \subseteq \sum J_1^j = \sum I_1^j. \qquad \square$$

We will add one extra theorem, which is easier to use than Theorem 33, but less general.

Theorem 34. *Given Cantor sets* $C^1, C^2, \ldots C^n$ *with density ratios equal to* $x_1, x_2, \ldots, x_n$ *with*

$$\sum_{i=1}^{n} \frac{x_i}{x_i+1} \geq 1; \tag{2.6}$$

$$\forall_{j,k} \overline{I_1^j} \geq \frac{x_j}{x_j+1} \frac{x_k+1}{2x_k+1} \overline{I_1^k}. \tag{2.7}$$

Then

$$\sum_{i=1}^{n} I_1^i = \{\sum_{i=1}^{n} x_i : x_i \in C^i\}.$$

Proof. First, notice that:

$$\begin{aligned}
\overline{I_{i_k}^k} &= \overline{I_{2i_k}^k} + \overline{g(I_{i_k}^k)} + \overline{I_{2i_k+1}^k} \\
&= \frac{\overline{I_{2i_k}^k} + \overline{g(I_{i_k}^k)} + \overline{I_{2i_k+1}^k}}{\overline{g(I_{i_k}^k)}} \cdot \overline{g(I_{i_k}^k)} \\
&= \left(\frac{\overline{I_{2i_k}^k}}{\overline{g(I_{i_k}^k)}} + \frac{\overline{I_{2i_k+1}^k}}{\overline{g(I_{i_k}^k)}} + 1 \right) \cdot \overline{g(I_{i_k}^k)} \\
&\geq (2x_k+1)\overline{g(I_i^k)}.
\end{aligned}$$

Since $\overline{I_1^k} \geq \overline{I_j^k}$ for all j, $\max_i \overline{g(I_i^k)} \leq \frac{1}{2x_k+1}\overline{I_1^k}$.
So, for all j, k:

$$\overline{I_1^j} \geq \frac{x_j}{x_j+1}\frac{x_k+1}{2x_k+1}\overline{I_1^k} \geq \frac{x_j}{x_j+1}(x_k+1)\max_i \overline{g(I_i^k)}.$$

We finish by applying Theorem 33. □

2.4 Comparison with Hlavka

We will now show that our premises are weaker than those of Hlavka, which lead to the same conclusion. We start with definitions Hlavka uses, and continue by showing that our premises follow from both Hlavka's versions: for the sum of two Cantor sets, and for an arbitrary number of Cantor sets.

Definition 35. *Let G_H be the relative biggest gap of a Cantor set:*

$$G_H((I,g)) = \max_i \frac{\overline{g(I_i)}}{\overline{I_i}}.$$

Definition 36. *Let H_H be the relative smallest remainder of a Cantor set:*

$$H_H((I,g)) = \min\left(\min_i \frac{\overline{I_{2i}}}{\overline{I_i}}, \min_i \frac{\overline{I_{2i+1}}}{\overline{I_i}}\right).$$

From this follows that for the density ratio of C we have $\mathrm{dr}(C) \geq \frac{H_H(C)}{G_H(C)}$.

Theorem 37. [4, Theorem 3] *If there exist Cantor sets (I^1, g^1) and (I^2, g^2) such that:*

$$G_H((I^1,g^1)) \cdot G_H((I^2,g^2)) \leq H_H((I^1,g^1)) \cdot H_H((I^2,g^2)); \tag{2.8}$$

$$G_H((I^1,g^1)) \cdot \overline{I^1} \leq \overline{I^2}; \tag{2.9}$$

$$G_H((I^2,g^2)) \cdot \overline{I^2} \leq \overline{I^1}. \tag{2.10}$$

Then

$$(I^1,g^1) + (I^2,g^2) = I^1 + I^2.$$

We will show that our premises follow from Hlavka's premises.

Lemma 38. *Given Cantor sets $(I^1,g^1),(I^2,g^2)$, let us write G^i for $G_H((I^i,g^i))$ and H^i for $H_H((I^i,g^i))$ If Hlavka's premises (2.8), (2.9) and (2.10) hold, then also (2.4) and (2.5) hold, i.e., there exists x_1 and x_2 smaller than or equal to the density ratios of respectively (I^1,g^1) and (I^2,g^2) such that:*

1. $\sum_{i=1}^{2} \frac{x_i}{x_i+1} \geq 1$;

2. $\forall_{j,k}\overline{I^j} \geq \frac{x_j}{x_j+1}(x_k+1)\max_i \overline{g^k(I_i^k)}$.

Proof. Let

$$x_1 = \sqrt{\frac{H^1 \cdot G^2}{G^1 \cdot H^2}};$$

$$x_2 = \sqrt{\frac{H^2 \cdot G^1}{G^2 \cdot H^1}}.$$

Notice that

$$x_1 \cdot x_2 = 1. \tag{2.11}$$

Then it follows that:

- $x_1 \le \mathrm{dr}((I^1, g^1))$ and $x_2 \le \mathrm{dr}((I^2, g^2))$:

  As $G^1 \cdot G^2 \le H^1 \cdot H^2$ (2.8), we know $\sqrt{\frac{G^1 \cdot G^2}{H^1 \cdot H^2}} \le 1$. So:

$$x_1 = \sqrt{\frac{H^1 \cdot G^2}{G^1 \cdot H^2}} = \frac{H^1}{G^1} \cdot \sqrt{\frac{G^1 \cdot G^2}{H^1 \cdot H^2}} \le \frac{H^1}{G^1} \le \mathrm{dr}((I^1, g^1));$$

$$x_2 = \sqrt{\frac{H^2 \cdot G^1}{G^2 \cdot H^1}} = \frac{H^2}{G^2} \cdot \sqrt{\frac{G^1 \cdot G^2}{H^1 \cdot H^2}} \le \frac{H^2}{G^2} \le \mathrm{dr}((I^2, g^2)).$$

- $\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} \ge 1$: Hence in fact

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} = \frac{x_1 x_2 + x_1 + x_1 x_2 + x_2}{x_1 x_2 + x_1 + x_2 + 1} \overset{2.11}{=} 1.$$

- $\overline{I^2} \ge \frac{x_2}{x_2+1}(x_1+1)\max_i \overline{g^1(I^1_i)}$ and $\overline{I^1} \ge \frac{x_1}{x_1+1}(x_2+1)\max_i \overline{g^2(I^2_i)}$:

  Since $\max_i \overline{g^1(I^1_i)} \le G^1 \cdot \overline{I^1}$ and $\max_i \overline{g^2(I^2_i)} \le G^2 \cdot \overline{I^2}$, we have:

$$\frac{x_2}{x_2+1}(x_1+1)\max_i \overline{g^1(I^1_i)} \le \frac{x_2 + x_1 x_2}{x_2+1} G^1 \cdot \overline{I^1} \overset{2.11}{=} G^1 \cdot \overline{I^1} \overset{2.9}{\le} \overline{I^2},$$

  and

$$\frac{x_1}{x_1+1}(x_2+1)\max_i \overline{g^2(I^2_i)} \le \frac{x_1 + x_1 x_2}{x_1+1} G^2 \cdot \overline{I^2} \overset{2.11}{=} G^2 \cdot \overline{I^2} \overset{2.10}{\le} \overline{I^1}.$$

- For $j = 1, 2$: $\overline{I^j} \ge \frac{x_j}{x_j+1}(x_j+1)\max_i \overline{g^j(I^j_i)}$:

  This follows directly from the definition of the density ratio of (I^j, g^j) as for every i we have

$$\overline{I^j} = \overline{I^j_1} \ge \overline{I^j_i} \ge x_j \cdot \overline{g^j(I^j_i)}.$$

□

Hence, Theorem 37 follows from Theorem 33.

Theorem 39. [4, Theorem 10] *If there exist Cantor sets* $(I^1, g^1), (I^2, g^2), \ldots, (I^n, g^n)$ *such that for all* $i, j \leq n$*:*

$$G_H((I^i, g^i)) + H_H((I^i, g^i)) \leq \sum_k H_H((I^k, g^k)) \tag{2.12}$$

$$H_H((I^i, g^i)) \cdot \overline{I^j} \leq \overline{I^i}; \tag{2.13}$$

Then

$$\sum_{i=1}^{n} (I^i, g^i) = \sum_{i=1}^{n} I^i.$$

Lemma 40. *Given Cantor sets* $(I^1, g^1), (I^2, g^2), \ldots, (I^n, g^n)$*, let us write* G^i *for* $G_H((I^i, g^i))$ *and* H^i *for* $H_H((I^i, g^i))$*. If Hlavka's premises (2.12) and (2.13) hold, then then also (2.4) and (2.5) hold, i.e., there exist* $x_1, x_2, \ldots, x_n$ *such that* $\forall i : x_i \leq \mathrm{dr}((I^i, g^i))$ *and:*

1. $\sum_{i=1}^{n} \frac{x_i}{x_i+1} \geq 1$*;*
2. $\forall_{j,k} \overline{I^j} \geq \frac{x_j}{x_j+1}(x_k + 1) \max_i \overline{g^k(I^k_i)}$*.*

Proof. For each i, let

$$x_i = \frac{H^i}{(\sum_k H^k) - H^i}.$$

Then for every i and j:

- $x_i \leq \mathrm{dr}((I^i, g^i))$:
  As for each i, $G^i \leq (\sum_k H^k) - H^i$, we have:
  $$x_i = \frac{H^i}{(\sum_k H^k) - H^i} \leq \frac{H^i}{G^i} \leq \mathrm{dr}((I^i, g^i)).$$
- $\sum_i \frac{x_i}{x_i+1} \geq 1$:
  Notice that:
  $$\frac{x_i}{x_i + 1} = \frac{\left(\frac{H^i}{(\sum_k H^k) - H^i}\right)}{\left(\frac{\sum_k H^k}{(\sum_k H^k) - H^i}\right)} = \frac{H^i}{\sum_k H^k}.$$
  So:
  $$\sum_i \frac{x_i}{x_i + 1} = \sum_i \frac{H^i}{\sum_k H^k} = 1.$$
- For all $i \leq n$: $\overline{I^i} \geq \frac{x_i}{x_i+1}(x_j + 1) \max_k \overline{g^j(I^j_k)}$:
  Using $\max_k \overline{g^j(I^j_k)} \leq G^j \cdot \overline{I^j}$, and $G^j \leq (\sum_k H^k) - H^j$:
  $$\frac{x_i}{x_i + 1}(x_j + 1) \max_k \overline{g^j(I^j_k)} \leq \frac{H^i}{\sum_k H^k} \frac{\sum_k H^k}{(\sum_k H^k) - H^j} G^j \cdot \overline{I^j}$$
  $$\leq \frac{H^i}{(\sum_k H^k) - H^j} G^j \cdot \overline{I^j} \leq H^i \cdot \overline{I^j} \leq \overline{I^i}.$$ □

Hence, Theorem 39 follows from Theorem 33 as well.

Chapter 3

$\text{NICF}_5 + \text{NICF}_5 = \mathbb{R}$

In this chapter we are going to prove that every real number can be represented as a sum of two nearest integer continued fractions with coefficients that have absolute value less than or equal to five. First, recall Definition 9 with $r = 5$.

Definition. NICF_5 *is a subset of* $\mathbb{R}$*, containing only the numbers representable by a Nearest Integer Continued Fraction where every coefficient except possibly the first has absolute value less than or equal to 5:*

$$\text{NICF}_5 = \{x : x \in \mathbb{R} \mid x = [a_0; a_1, a_2, \ldots] \in \text{NICF} \text{ and } \forall_{i \geq 1} |a_i| \leq 5\}$$

We create a Cantor set, C_{NICF}, and prove it is equal to $\text{NICF}_5^* \setminus \mathbb{Q}$, with NICF_5^* a subset of NICF_5. With the results of the previous chapter, we are able to show $C_{\text{NICF}} + C_{\text{NICF}} \supseteq [\frac{1}{2}, \frac{3}{2}]$, which will lead to:

Theorem 41. $\text{NICF}_5 + \text{NICF}_5 = \mathbb{R}$*, that is, for each* $x \in \mathbb{R}$ *there exist* $a, b \in \text{NICF}_5$ *such that* $a + b = x$*.*

NICF_5 has the following rules (Definition 8):
For every $x = [a_0; a_1, a_2, \ldots] \in \text{NICF}$ we have $x \in \text{NICF}_5$ if and only if for all $i \geq 1$:

- $a_i \in \{-5, -4, -3, -2, 2, 3, 4, 5\}$;
- if $a_i = 2$, then $a_{i+1} \in \{2, 3, 4, 5\}$;
- if $a_i = -2$, then $a_{i+1} \in \{-2, -3, -4, -5\}$.

We will look at a subset of NICF_5:

Definition 42. NICF_5^* *is the subset of* $\text{NICF}_5 \cap [0, 1]$ *containing only the numbers representable by a Nearest Integer Continued Fraction where every coefficient except possibly the first with absolute value 5 is not followed by a coefficient with the same sign.*

NICF_5^* has five extra rules in addition to the ones of NICF_5:
For every $x = [a_0; a_1, a_2, \dots] \in \mathrm{NICF}_5$ we have $x \in \mathrm{NICF}_5^*$ if and only if for all $i \geq 1$:

- if $a_i = 5$, then $a_{i+1} \in \{-2, -3, -4, -5\}$;
- if $a_i = -5$, then $a_{i+1} \in \{2, 3, 4, 5\}$;
- $a_0 \in \{0, 1\}$;
- if $a_0 = 0$, then $a_1 \in \{2, 3, 4, 5\}$;
- if $a_0 = 1$, then $a_1 \in \{-2, -3, -4, -5\}$.

It easily follows that for all $z \in \mathbb{Z}$, and all $x \in \mathrm{NICF}_5^*$, we have $z + x \in \mathrm{NICF}_5$.
We will show that $\mathrm{NICF}_5^* \setminus \mathbb{Q}$ can be described as a Cantor Set, with density ratio greater than 1, and an initial interval with size greater than $\frac{1}{2}$.

Definition 43. *We define* $\mu \in \mathrm{NICF}_5$ *as* $[5; \overline{-5, 5}] = 5 - \frac{1}{\mu} = \frac{5+\sqrt{21}}{2} > 4.79128$.

We have $-\mu = -1 \cdot [5; \overline{-5, 5}] = [-5; \overline{5, -5}]$. This satisfies all the rules of NICF_5, and therefore $-\mu \in \mathrm{NICF}_5$.

Lemma 44. *The smallest value in* NICF_5^* *is* $[0 : \mu] = \frac{5-\sqrt{21}}{2} \leq 0.20872$*, and the largest value in* NICF_5^* *is* $[1 : -\mu] = \frac{\sqrt{21}-5}{2} > 0.79128$.

Proof. Both $[0 : \mu]$ and $[1 : -\mu]$ correspond to all the rules of NICF_5^*. We are going to prove $y = [0 : \mu] = [y_0; y_1, y_2, \dots]$ is the smallest value in NICF_5^*. Let $x = [x_0; x_1, x_2, \dots]$ be the smallest value in NICF_5^*. Then $x < \frac{1}{2}$, so $x_0 = 0$. Suppose $x \neq y$, let n be smallest integer such that $x_n \neq y_n$. This leads to the following contradictions:

- If n is odd, $y_n = 5$. If $x_n < 5$, then
$$x > [x_0; x_1, x_2, \dots, x_{n-1} : x_n + \frac{1}{2}] \geq [x_0; x_1, x_2, \dots, x_{n-1} : 5 - \frac{1}{2}] \geq y.$$
If $x_n > 5$, then $x \notin \mathrm{NICF}_5^*$.
- If n is odd, $y_n = -5$. If $x_n > -5$, then
$$x \geq [x_0; x_1, x_2, \dots, x_{n-1} : x_n - \frac{1}{2}] \geq [x_0; x_1, x_2, \dots, x_{n-1} : -5 + \frac{1}{2}] > y.$$
If $x_n < -5$, then $x \notin \mathrm{NICF}_5^*$.

The proof that $[1 : -\mu]$ is the largest value in NICF_5^* follows a similar pattern. □

Definition 45. *Let* $a_0, a_1, \dots, a_n \in \mathbb{Z}$,
if $k \in \{-5, -4, -3, 2, 3, 4\}$ *such that* $[a_0; a_1, \dots, a_n, k] \in \mathrm{NICF}_5^*$*, we define*
$$P_{k^+}([a_0, a_1, \dots, a_n]) := [a_0; a_1, \dots, a_n, k : \mu] \ (\in \mathrm{NICF}_5^*)$$
and if $k \in \{-4, -3, -2, 3, 4, 5\}$ *such that* $[a_0; a_1, \dots, a_n, k] \in \mathrm{NICF}_5^*$*, we define*
$$P_{k^-}([a_0, a_1, \dots, a_n]) := [a_0; a_1, \dots, a_n, k : -\mu] \ (\in \mathrm{NICF}_5^*).$$

With these, we can define intervals:

Definition 46. *For all* $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ *and all* $u \in \{-5, -4, -3, 2, 3, 4\}$ *and* $v \in \{-4, -3, -2, 3, 4, 5\}$ *such that* $u < v$ *and* $[a_0; a_1, \ldots, a_n, u]$, $[a_0; a_1, \ldots, a_n, v] \in \mathrm{NICF}_5^*$*, we define* $T_{u,v}([a_0, a_1, \ldots a_n])$ *as the interval with endpoints* $P_{u^+}([a_0, a_1, \ldots, a_n])$ *and* $P_{v^-}([a_0, a_1, \ldots, a_n])$*.*

If n is odd, then $P_{u^+}([a_0, a_1, \ldots, a_n]) < P_{v^-}([a_0, a_1, \ldots, a_n])$,
while if n is even, then $P_{u^+}([a_0, a_1, \ldots, a_n]) > P_{v^-}([a_0, a_1, \ldots, a_n])$.

3.1 Creating a gap function

With the intervals defined in Definition 46, we are going to create a Cantor set. The initial interval will be $[[0; \overline{5, -5}], [1; \overline{-5, 5}]]$, which we will call $T_{0,1}$. Our gap function g will create remaining intervals of the following types:

- $T_{0,1}$;
- $T_{b,b+1}([a_0; a_1, \ldots, a_n])$, with $b \in \{-5, -4, -3, 2, 3, 4\}$;
- $T_{2,5}([a_0; a_1, \ldots, a_n])$;
- $T_{3,5}([a_0; a_1, \ldots, a_n])$;
- $T_{-5,-2}([a_0; a_1, \ldots, a_n])$;
- $T_{-5,-3}([a_0; a_1, \ldots, a_n])$.

For each of these types of interval we describe the function g, and will show that the remainders again are of the described types. We will also calculate the lower bound of the ratio between the remainders and the size of the gap. Later, we will use this to derive a lower bound for the density ratio of the Cantor set we are creating.

3.1.1 Ratio calculation

Let $[a_0; a_1, \ldots, a_n : I^-]$ and $[a_0; a_1, \ldots, a_n : I^+]$ be the endpoints of an interval T, and let $[a_0; a_1, \ldots, a_n : C^-]$ and $[a_0; a_1, \ldots, a_n : C^+]$ be the endpoints of the corresponding gap, such that $I^- < C^- < C^+ < I^+$. With the theorem about the approximation of rationals (Theorem 7), and with ω defined as $\omega = \frac{q_{n-1}}{q_n}$, we know that the sizes of the remainders are

$$\frac{|I^- - C^-|}{q_n^2(I^- + \omega)(C^- + \omega)} \qquad \text{and} \qquad \frac{|I^+ - C^+|}{q_n^2(I^+ + \omega)(C^+ + \omega)},$$

while the gap has size

$$\frac{|C^- - C^+|}{q_n^2(C^- + \omega)(C^+ + \omega)}.$$

So the density ratio of this particular interval is the minimum of

$$\frac{|I^- - C^-|}{|C^- - C^+|}\frac{(C^+ + \omega)}{(I^- + \omega)} \qquad \text{and} \qquad \frac{|I^+ - C^+|}{|C^- - C^+|}\frac{(C^- + \omega)}{(I^+ + \omega)}.$$

Recall that, because all endpoints are elements of $\mathrm{NICF}_5 \subset \mathrm{NICF}$, we know that $|\omega| \leq \frac{\sqrt{5}-1}{2}$ according to (1.1).

$T_{0,1}$

Our first type consists of one interval, which can not be described in the fashion of Definition 46, but for consistency will be called $T_{0,1}$. It has size $[1;\overline{-5,5}]-[0;\overline{5,-5}] = \frac{\sqrt{21}-3}{2} - \frac{5-\sqrt{21}}{2} = \sqrt{21}-4 > 0.58256$.
The corresponding gap will be $g(T_{0,1}) = ([0;2:\mu],[1;-2:-\mu])$, which has size: $0.54725-0.45275 < 0.09451$.
The remaining intervals are $T_{2,5}([0])$ and $T_{-5,-2}([1])$, which both have size > 0.24403.
The density ratio corresponding with our initial interval is $\frac{0.24403}{0.09451} > 2.58205$.

I_1
$T_{0,1}$
/ \
I_2 $T_{2,5}([0])$ I_3 $T_{-5,-2}([1])$

$T_{b,b+1}([a_0;a_1,\ldots,a_n])$, $b \in \{-5,-4,-3,2,3,4\}$

For $T_{b,b+1}([a_0;a_1,\ldots,a_n])$, the gap $g(T_{b,b+1}([a_0;a_1,\ldots,a_n]))$ is defined as the open interval with endpoints $P_{2^+}([a_0;a_1,\ldots,a_n,b])$ and $P_{-2^-}([a_0;a_1,\ldots,a_n,b+1])$. With the remaining intervals are $T_{2,5}([a_0;a_1,\ldots,a_n,b])$ and $T_{-5,-2}([a_0;a_1,\ldots,a_n,b+1])$.
Here, the density ratio can be calculated using $I^- = [b:\mu]$, $C^- = [b;2:\mu]$, $C^+ = [b+1;-2:-\mu]$, and $I^+ = [b+1:-\mu]$.
We will now calculate bounds for the corresponding density ratio, for each b:

- When b equals 2:

$$\frac{|T_{2,5}([a_0;a_1,\ldots,a_n,2])|}{|g(T_{2,3}([a_0;a_1,\ldots,a_n]))|} = \frac{|[2:\mu]-[2;2:\mu]|}{|[2;2:\mu]-[3;-2:-\mu]|}\frac{([3;-2:-\mu]+\omega)}{([2:\mu]+\omega)} > 2.89191$$

  and

$$\frac{|T_{-5,-2}([a_0;a_1,\ldots,a_n,3])|}{|g(T_{2,3}([a_0;a_1,\ldots,a_n]))|} = \frac{|[3:-\mu]-[3;-2:-\mu]|}{|[2;2:\mu]-[3;-2:-\mu]|}\frac{([2;2:\mu]+\omega)}{([3:-\mu]+\omega)} > 2.18031;$$

- When b equals 3:

$$\frac{|T_{2,5}([a_0;a_1,\ldots,a_n,3])|}{|g(T_{3,4}([a_0;a_1,\ldots,a_n]))|} = \frac{|[3:\mu]-[3;2:\mu]|}{|[3;2:\mu]-[4;-2:-\mu]|}\frac{([4;-2:-\mu]+\omega)}{([3:\mu]+\omega)} > 2.81108$$

  and

$$\frac{|T_{-5,-2}([a_0;a_1,\ldots,a_n,4])|}{|g(T_{3,4}([a_0;a_1,\ldots,a_n]))|} = \frac{|[4:-\mu]-[4;-2:-\mu]|}{|[3;2:\mu]-[4;-2:-\mu]|}\frac{([3;2:\mu]+\omega)}{([4:-\mu]+\omega)} > 2.30709;$$

- When b equals 4:

$$\frac{|T_{2,5}([a_0;a_1,\ldots,a_n,4])|}{|g(T_{4,5}([a_0;a_1,\ldots,a_n]))|} = \frac{|[4:\mu]-[4;2:\mu]|}{|[4;2:\mu]-[5;-2:-\mu]|}\frac{([5;-2:-\mu]+\omega)}{([4:\mu]+\omega)} > 2.76375$$

and

$$\frac{|T_{-5,-2}([a_0; a_1, \ldots, a_n, 5])|}{|g(T_{4,5}([a_0; a_1, \ldots, a_n]))|} = \frac{|[5 : -\mu] - [5; -2 : -\mu]|}{|[4; 2 : \mu] - [5; -2 : -\mu]|} \frac{([4; 2 : \mu] + \omega)}{([5 : -\mu] + \omega)} > 2.37311;$$

- When b equals -3:

$$\frac{|T_{2,5}([a_0; a_1, \ldots, a_n, -3])|}{|g(T_{-3,-2}([a_0; a_1, \ldots, a_n]))|} = \frac{|[-3 : \mu] - [-3; 2 : \mu]|}{|[-3; 2 : \mu] - [-2; -2 : -\mu]|} \frac{([-2; -2 : -\mu] + \omega)}{([-3 : \mu] + \omega)} > 2.18031$$

and

$$\frac{|T_{-5,-2}([a_0; a_1, \ldots, a_n, -2])|}{|g(T_{-3,-2}([a_0; a_1, \ldots, a_n]))|} = \frac{|[-2 : -\mu] - [-2; -2 : -\mu]|}{|[-3; 2 : \mu] - [-2; -2 : -\mu]|} \frac{([-3; 2 : \mu] + \omega)}{([-2 : -\mu] + \omega)} > 2.89191;$$

- When b equals -4:

$$\frac{|T_{2,5}([a_0; a_1, \ldots, a_n, -4])|}{|g(T_{-4,-3}([a_0; a_1, \ldots, a_n]))|} = \frac{|[-4 : \mu] - [-4; 2 : \mu]|}{|[-4; 2 : \mu] - [-3; -2 : -\mu]|} \frac{([-3; -2 : -\mu] + \omega)}{([-4 : \mu] + \omega)} > 2.30709$$

and

$$\frac{|T_{-5,-2}([a_0; a_1, \ldots, a_n, 4])|}{|g(T_{3,4}([a_0; a_1, \ldots, a_n]))|} = \frac{|[-3 : -\mu] - [-3; -2 : -\mu]|}{|[-4; 2 : \mu] - [-3; -2 : -\mu]|} \frac{([-4; 2 : \mu] + \omega)}{([-3 : -\mu] + \omega)} > 2.81108;$$

- When b equals -5:

$$\frac{|T_{2,5}([a_0; a_1, \ldots, a_n, -5])|}{|g(T_{-5,-4}([a_0; a_1, \ldots, a_n]))|} = \frac{|[-5 : \mu] - [-5; 2 : \mu]|}{|[-5; 2 : \mu] - [-4; -2 : -\mu]|} \frac{([-4; -2 : -\mu] + \omega)}{([-5 : \mu] + \omega)} > 2.37311$$

and

$$\frac{|T_{-5,-2}([a_0; a_1, \ldots, a_n, -4])|}{|g(T_{-5,-4}([a_0; a_1, \ldots, a_n]))|} = \frac{|[-4 : -\mu] - [-4; -2 : -\mu]|}{|[-5; 2 : \mu] - [-4; -2 : -\mu]|} \frac{([-5; 2 : \mu] + \omega)}{([-4 : -\mu] + \omega)} > 2.76375.$$

When n is odd:

I_i

$T_{b,b+1}([a_0; a_1, \ldots, a_n])$

I_{2i} $\quad$ I_{2i+1}

$T_{2,5}([a_0; a_1, \ldots, a_n, b])$ $\quad$ $T_{-5,-2}([a_0; a_1, \ldots, a_n, b+1])$

When n is even:

I_i

$T_{b,b+1}([a_0; a_1, \ldots, a_n])$

I_{2i} $\quad$ I_{2i+1}

$T_{-5,-2}([a_0; a_1, \ldots, a_n, b+1])$ $\quad$ $T_{2,5}([a_0; a_1, \ldots, a_n, b])$

$T_{2,5}([a_0; a_1, \ldots, a_n])$

For $T_{2,5}([a_0; a_1, \ldots, a_n])$, the gap $g(T_{2,5}([a_0; a_1, \ldots, a_n]))$ is defined as the open interval with endpoints $P_{3^-}([a_0; a_1, \ldots, a_n])$ and $P_{3^+}([a_0; a_1, \ldots, a_n])$, with the remaining intervals $T_{2,3}([a_0; a_1, \ldots, a_n])$ and $T_{3,5}([a_0; a_1, \ldots, a_n])$.
Now the density ratio can be calculated using $I^- = [2 : \mu]$, $C^- = [3 : -\mu]$, $C^+ = [3 : \mu]$, and $I^+ = [5 : -\mu]$

$$\frac{|T_{2,3}([a_0; a_1, \ldots, a_n])|}{|g(T_{2,5}([a_0; a_1, \ldots, a_n]))|} = \frac{|[2 : \mu] - [3 : -\mu]|}{|[3 : -\mu] - [3 : \mu]|} \frac{([3 : \mu] + \omega)}{([2 : \mu] + \omega)} > 1.88937$$

and

$$\frac{|T_{3,5}([a_0; a_1, \ldots, a_n])|}{|g(T_{2,5}([a_0; a_1, \ldots, a_n]))|} = \frac{|[5 : -\mu] - [3 : \mu]|}{|[3 : -\mu] - [3 : \mu]|} \frac{([3 : -\mu] + \omega)}{([5 : -\mu] + \omega)} > 1.97434$$

When n is odd:

$$\begin{array}{ccc} & I_i & \\ & T_{2,5}([a_0; a_1, \ldots, a_n]) & \\ & \swarrow \quad \searrow & \\ I_{2i} & & I_{2i+1} \\ T_{2,3}([a_0; a_1, \ldots, a_n]) & & T_{3,5}([a_0; a_1, \ldots, a_n]) \end{array}$$

When n is even:

$$\begin{array}{ccc} & I_i & \\ & T_{2,5}([a_0; a_1, \ldots, a_n]) & \\ & \swarrow \quad \searrow & \\ I_{2i} & & I_{2i+1} \\ T_{3,5}([a_0; a_1, \ldots, a_n]) & & T_{2,3}([a_0; a_1, \ldots, a_n]) \end{array}$$

$T_{3,5}([a_0; a_1, \ldots, a_n])$

For $T_{3,5}([a_0; a_1, \ldots, a_n])$, the gap $g(T_{3,5}([a_0; a_1, \ldots, a_n]))$ is defined as the open interval with endpoints $P_{4^-}([a_0; a_1, \ldots, a_n])$ and $P_{4^+}([a_0; a_1, \ldots, a_n])$, with the remaining intervals $T_{3,4}([a_0; a_1, \ldots, a_n])$ and $T_{4,5}([a_0; a_1, \ldots, a_n])$.
In this case, the density ratio can be calculated using $I^- = [3 : \mu]$, $C^- = [4 : -\mu]$, $C^+ = [4 : \mu]$, and $I^+ = [5 : -\mu]$

$$\frac{|T_{3,4}([a_0; a_1, \ldots, a_n])|}{|g(T_{3,5}([a_0; a_1, \ldots, a_n]))|} = \frac{|[3 : \mu] - [4 : -\mu]|}{|[4 : -\mu] - [4 : \mu]|} \frac{([4 : \mu] + \omega)}{([3 : \mu] + \omega)} > 1.76035$$

and

$$\frac{|T_{4,5}([a_0; a_1, \ldots, a_n])|}{|g(T_{3,5}([a_0; a_1, \ldots, a_n]))|} = \frac{|[5 : -\mu] - [4 : \mu]|}{|[4 : -\mu] - [4 : \mu]|} \frac{([4 : -\mu] + \omega)}{([5 : -\mu] + \omega)} > 1.06122$$

When n is odd:

$$\begin{array}{ccc} & I_i & \\ & T_{3,5}([a_0; a_1, \ldots, a_n]) & \\ \swarrow & & \searrow \\ I_{2i} & & I_{2i+1} \\ T_{3,4}([a_0; a_1, \ldots, a_n]) & & T_{4,5}([a_0; a_1, \ldots, a_n]) \end{array}$$

When n is even:

$$\begin{array}{ccc} & I_i & \\ & T_{3,5}([a_0; a_1, \ldots, a_n]) & \\ \swarrow & & \searrow \\ I_{2i} & & I_{2i+1} \\ T_{4,5}([a_0; a_1, \ldots, a_n]) & & T_{3,4}([a_0; a_1, \ldots, a_n]) \end{array}$$

$T_{-5,-2}([a_0; a_1, \ldots, a_n])$

For $T_{-5,-2}([a_0; a_1, \ldots, a_n])$, the gap $g(T_{-5,-2}([a_0; a_1, \ldots, a_n]))$ is defined as the open interval with endpoints $P_{-3^-}([a_0; a_1, \ldots, a_n])$ and $P_{-3^+}([a_0; a_1, \ldots, a_n])$.
The remaining intervals are $T_{-5,-3}([a_0; a_1, \ldots, a_n])$ and $T_{-3,-2}([a_0; a_1, \ldots, a_n])$.

$$\frac{|T_{-5,-3}([a_0; a_1, \ldots, a_n])|}{|g(T_{-5,-2}([a_0; a_1, \ldots, a_n]))|} = \frac{|T_{3,5}([-a_0; -a_1, \ldots, -a_n])|}{|g(T_{2,5}([-a_0; -a_1, \ldots, -a_n]))|} > 1.97434$$

and

$$\frac{|T_{-3,-2}([a_0; a_1, \ldots, a_n])|}{|g(T_{-5,-2}([a_0; a_1, \ldots, a_n]))|} = \frac{|T_{2,3}([-a_0; -a_1, \ldots, -a_n])|}{|g(T_{2,5}([-a_0; -a_1, \ldots, -a_n]))|} > 1.88937$$

When n is odd:

$$\begin{array}{ccc} & I_i & \\ & T_{-5,-2}([a_0; a_1, \ldots, a_n]) & \\ \swarrow & & \searrow \\ I_{2i} & & I_{2i+1} \\ T_{-5,-3}([a_0; a_1, \ldots, a_n]) & & T_{-3,-2}([a_0; a_1, \ldots, a_n]) \end{array}$$

When n is even:

$$\begin{array}{ccc} & I_i & \\ & T_{-5,-2}([a_0; a_1, \ldots, a_n]) & \\ \swarrow & & \searrow \\ I_{2i} & & I_{2i+1} \\ T_{-3,-2}([a_0; a_1, \ldots, a_n]) & & T_{-5,-3}([a_0; a_1, \ldots, a_n]) \end{array}$$

$T_{-5,-3}([a_0; a_1, \dots, a_n])$

For $T_{-5,-3}([a_0; a_1 \dots, a_n])$, the gap $g(T_{-5,-3}([a_0; a_1, \dots, a_n]))$ is defined as the open interval with endpoints $P_{-4^-}([a_0; a_1, \dots, a_n])$ and $P_{-4^+}([a_0; a_1, \dots, a_n])$.
The remaining intervals are $T_{-5,-4}([a_0; a_1, \dots, a_n])$ and $T_{-4,-3}([a_0; a_1, \dots, a_n])$.

$$\frac{|T_{-5,-4}([a_0; a_1, \dots, a_n])|}{|g(T_{-5,-3}([a_0; a_1, \dots, a_n]))|} = \frac{|T_{4,5}([-a_0; -a_1, \dots, -a_n])|}{|g(T_{3,5}([-a_0; -a_1, \dots, -a_n]))|} > 1.06122$$

and

$$\frac{|T_{-4,-3}([a_0; a_1, \dots, a_n])|}{|g(T_{-5,-3}([a_0; a_1, \dots, a_n]))|} = \frac{|T_{3,4}([-a_0; -a_1, \dots, -a_n])|}{|g(T_{3,5}([-a_0; -a_1, \dots, -a_n]))|} > 1.76035$$

When n is odd:

I_i: $T_{-5,-3}([a_0; a_1 \dots, a_n])$

I_{2i}: $T_{-5,-4}([a_0; a_1, \dots, a_n])$ — I_{2i+1}: $T_{-4,-3}([a_0; a_1, \dots, a_n])$

When n is even:

I_i: $T_{-5,-3}([a_0; a_1 \dots, a_n])$

I_{2i}: $T_{-4,-3}([a_0; a_1, \dots, a_n])$ — I_{2i+1}: $T_{-5,-4}([a_0; a_1, \dots, a_n])$

3.2 Construction of the Cantor Set

By describing the interval types and the corresponding gaps, we can now create a Cantor set:

Definition 47. *We define the Cantor set $C_{\mathrm{NICF}} = (T_{0,1}, g)$. The size of the initial interval $T_{0,1}$ is at least 0.58256 and the density ratio is at least 1.06122.*

We will prove $C_{\mathrm{NICF}} = \mathrm{NICF}_5^* \setminus \mathbb{Q}$, which will take some lemmas.

Lemma 48. *For the intervals I_i in the construction of C_{NICF}, for every $n \geq 1$, if $[a_0; a_1, \dots, a_n] \in \mathrm{NICF}_5^*$ with $a_n \in \{-5, -4, -3, 2, 3, 4\}$, then there exists some $i \geq 4^n$ such that*

$$I_i = T_{a_n, a_n+1}([a_0; a_1, \dots, a_{n-1}]).$$

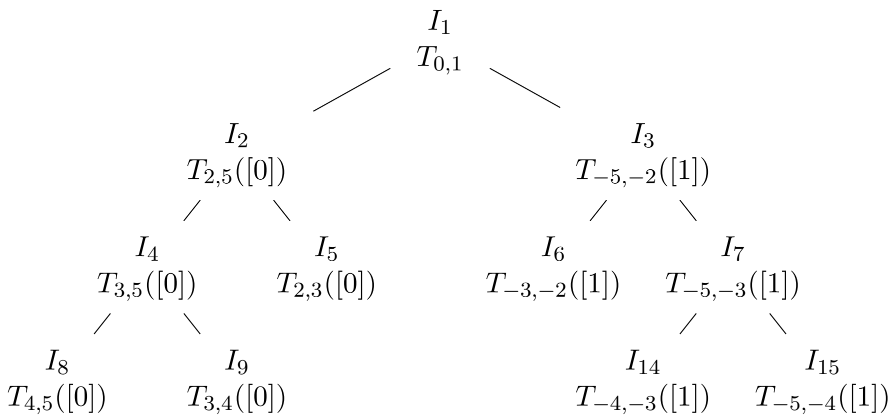
We also have the following lemma:

Lemma 49. *For the intervals I_i in the construction of C_{NICF}, for every $n \geq 1$, if $[a_0; a_1, \dots, a_n] \in \mathrm{NICF}_5^*$ with $a_n \in \{-4, -3, -2, 3, 4, 5\}$, then there exists some $i \geq 4^n$ such that*

$$I_i = T_{a_n-1, a_n}([a_0; a_1, \dots, a_{n-1}]).$$

The proofs of Lemma 48 and Lemma 49 are intertwined. We prove them using one induction argument and when proving the $(n+1)$-case, we use both induction hypotheses. Since they show such similarity, this is a nice way to prove these lemmas.

Proof of Lemma 48 and Lemma 49. By induction on n.
Base case: $n = 1$:



If $a_1 \in \{-5,-4,-3,-2\}$ then $a_0 = 1$ and:

- $I_{15} = T_{-5,-4}([1])$;
- $I_{14} = T_{-4,-3}([1])$;
- $I_6 = T_{-3,-2}([1])$.

If $a_1 \in \{2,3,4,5\}$ then $a_0 = 0$ and:

- $I_5 = T_{2,3}([0])$;
- $I_9 = T_{3,4}([0])$;
- $I_8 = T_{4,5}([0])$.

Induction Hypothesis: Given $n \geq 1$, for all $[a_0; a_1, ..., a_n] \in \text{NICF}_5^*$ with $a_n \in \{-5,-4,-3,2,3,4\}$ there exists an $i \geq 4^n$ such that $I_i = T_{a_n, a_n+1}([a_0; a_1, ..., a_{n-1}])$. And if $a_n \in \{-4,-3,-2,3,4,5\}$ there exists an $i \geq 4^n$ such that $I_i = T_{a_n-1, a_n}([a_0; a_1, ..., a_{n-1}])$.
Induction step, we need to prove: for all $[a_0; a_1, ..., a_n, a_{n+1}] \in \text{NICF}_5^*$, if $a_{n+1} \in \{-5,-4,-3,2,3,4\}$ then there exists an $i \geq 4^{n+1}$ such that $I_i = T_{a_{n+1}, a_{n+1}+1}([a_0; a_1, ..., a_n])$, and if $a_{n+1} \in \{-4,-3,-2,3,4,5\}$ then there exists an $i \geq 4^{n+1}$ such that $I_i = T_{a_{n+1}-1, a_{n+1}}([a_0; a_1, ..., a_n])$.
Let $[a_0; a_1, ..., a_n, a_{n+1}] \in \text{NICF}_5^*$, so $a_{n+1} \in \{-5,-4,-3,-2,2,3,4,5\}$.
We make case distinctions whether $a_{n+1} \in \{-5,-4,-3,-2\}$ or $a_{n+1} \in \{2,3,4,5\}$, and whether n is odd or even.
Case 1: If $a_{n+1} \in \{-5,-4,-3,-2\}$, then $a_n \in \{-4,-3,-2,3,4,5\}$, so with the Induction Hypothesis there exists $j \geq 4^n$ such that $I_j = T_{a_n-1, a_n}([a_0; a_1, ..., a_{n-1}])$.

Case 1a: If n is even:

$$I_j \quad T_{a_n-1,a_n}([a_0;a_1,...,a_{n-1}])$$

$$I_{2j} \quad T_{2,5}([a_0;a_1,...,a_n-1]) \qquad I_{2j+1} \quad T_{-5,-2}([a_0;a_1,...,a_n])$$

$$I_{4j+2} \quad T_{-3,-2}([a_0;a_1,...,a_n]) \qquad I_{4j+3} \quad T_{-5,-3}([a_0;a_1,...,a_n])$$

$$I_{8j+6} \quad T_{-4,-3}([a_0;a_1,...,a_n]) \qquad I_{8j+7} \quad T_{-5,-4}([a_0;a_1,...,a_n])$$

From this we can conclude:

- $I_{8j+7} = T_{-5,-4}([a_0;a_1,...,a_n])$;
- $I_{8j+6} = T_{-4,-3}([a_0;a_1,...,a_n])$;
- $I_{4j+2} = T_{-3,-2}([a_0;a_1,...,a_n])$.

Case 1b: If n is odd:

$$I_j \quad T_{a_n-1,a_n}([a_0;a_1,...,a_{n-1}])$$

$$I_{2j} \quad T_{-5,-2}([a_0;a_1,...,a_n]) \qquad I_{2j+1} \quad T_{2,5}([a_0;a_1,...,a_n-1])$$

$$I_{4j} \quad T_{-5,-3}([a_0;a_1,...,a_n]) \qquad I_{4j+1} \quad T_{-3,-2}([a_0;a_1,...,a_n])$$

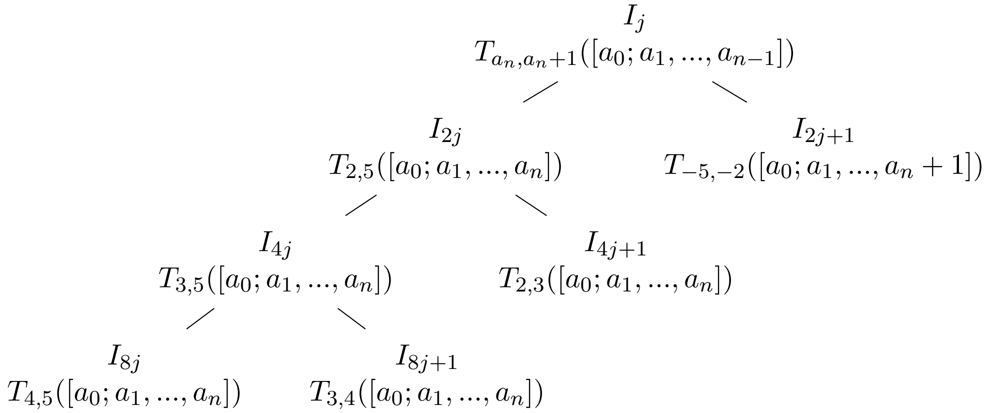
$$I_{8j} \quad T_{-5,-4}([a_0;a_1,...,a_n]) \qquad I_{8j+1} \quad T_{-4,-3}([a_0;a_1,...,a_n])$$

From this we can conclude:

- $I_{8j} = T_{-5,-4}([a_0;a_1,...,a_n])$;
- $I_{8j+1} = T_{-4,-3}([a_0;a_1,...,a_n])$;
- $I_{4j+1} = T_{-3,-2}([a_0;a_1,...,a_n])$.

Case 2: If $a_{n+1} \in \{2,3,4,5\}$, then $a_n \in \{-5,-4,-3,2,3,4\}$, so with the Induction Hypothesis there exists $j \geq 4^n$ such that $I_j = T_{a_n,a_n+1}([a_0;a_1,...,a_{n-1}])$.

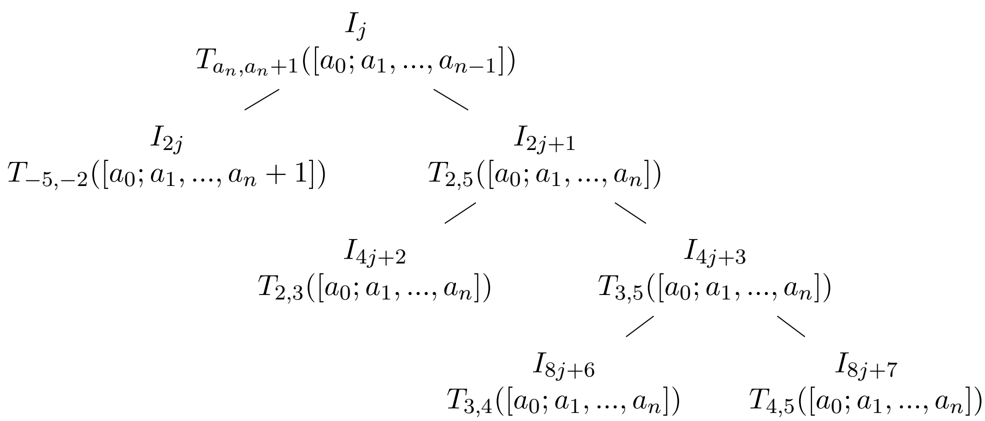
Case 2a: If n is even:



From this we can conclude:

- $I_{4j+1} = T_{2,3}([a_0; a_1, ..., a_n])$;
- $I_{8j} = T_{3,4}([a_0; a_1, ..., a_n])$;
- $I_{8j+1} = T_{4,5}([a_0; a_1, ..., a_n])$.

Case 2b: If n is odd:



From this we can conclude:

- $I_{4j+2} = T_{2,3}([a_0; a_1, ..., a_n])$;
- $I_{8j+6} = T_{3,4}([a_0; a_1, ..., a_n])$;
- $I_{8j+7} = T_{4,5}([a_0; a_1, ..., a_n])$.

This concludes our proof. □

Theorem 50. $\text{NICF}_5^* \setminus \mathbb{Q} \subseteq C_{\text{NICF}}$

Proof. For every $x = [a_0; a_1, ...] \in \text{NICF}_5^* \setminus \mathbb{Q}$ we have $[0 : \mu] \leq x \leq [1 : -\mu]$, so $x \in T_{0,1}$.
We continue by showing that there exists no i such that $x \in g(I_i)$.
Proof by contradiction: Suppose that there exists an i such that $x \in g(I_i)$. Take $n \geq 2$ such that $4^n > i$. We know that $[a_0; a_1, ..., a_{n+1}] \in \text{NICF}_5^*$. We make a case distinction whether $a_{n+1} \in \{2, 3, 4, 5\}$ or $a_{n+1} \in \{-5, -4, -3, -2\}$.
Case 1: If $a_{n+1} \in \{2, 3, 4, 5\}$, then $a_n \in \{-5, -4, -3, 2, 3, 4\}$. So by Lemma 48 there exists some $j \geq 4^n > i$ such that $I_j = T_{a_n, a_n+1}([a_0; a_1, ..., a_n - 1])$, with $x \in I_j$. Because $j > i$, we have $g(I_i) \cap I_j = \emptyset$. Contradiction.
Case 2: If $a_{n+1} \in \{-5, -4, -3, -2\}$, then $a_n \in \{-4, -3, -2, 3, 4, 5\}$. So by Lemma 49 there exists some $j \geq 4^n > i$ such that $I_j = T_{a_n - 1, a_n}([a_0; a_1, ..., a_n - 1])$, with $x \in I_j$. Because $j > i$, we have $g(I_i) \cap I_j = \emptyset$. Contradiction. □

A first step towards proving $C_{\text{NICF}} \subseteq \text{NICF}_5^* \setminus \mathbb{Q}$ (Theorem 54) is proving that for every $x \notin \text{NICF}_5^* \setminus \mathbb{Q}$ we have $x \notin T_{0,1}$ or there exists an i such that $x \in g(I_i)$.
For $x \notin \text{NICF}_5^*$ at least one of the following rules must be true (logical negation of $x \in \text{NICF}_5^*$):

1. $a_0 \notin \{0, 1\}$;
2. $a_0 = 0$ and $a_1 < 0$;
3. $a_0 = 1$ and $a_1 > 0$;
4. for some $i \geq 1$, $|a_i| \geq 6$;
5. for some $i \geq 1$, $a_i = 5$ and $a_{i+1} > 0$;
6. for some $i \geq 1$, $a_i = -5$ and $a_{i+1} < 0$.

Theorem 51. *Given* $x = [a_0; a_1, \ldots, a_{n-1} : a_n + r]$, *with* $0 \leq x \leq 1$ *and* $|r| \leq \frac{1}{\mu}$. *If for each* $i \leq n$, $|[a_i; a_{i+1}, a_{i+2}, \ldots : a_n + r]| \leq \mu$, *then:* $[a_0; a_1, \ldots, a_{n-1}, a_n] \in \text{NICF}_5^*$ *and if* $n \geq 1$, *then* $a_n \in \{-4, -3, -2, 2, 3, 4\}$.

Proof. We will write r_i (remainder) as $x_i - a_i$ in the construction of NICF, so $r_i = \frac{1}{[a_{i+1}; a_{i+2}, \ldots]}$ and $x = [a_0; a_1, \ldots, a_{i-1} : a_i + r_i]$. Note that for every i we have $|r_i| \leq \frac{1}{2}$.
Proof by contradiction: suppose that $x \notin \text{NICF}_5^*$. By a case distinction, it then leads to (at least) one of the following contradictions:

1. $a_0 \notin \{0, 1\}$: either $a_0 \leq -1$, such that $x = a_0 + r_0 \leq -\frac{1}{2} < 0$, or $a_0 \geq 2$, such that $x = a_0 + r_0 \geq \frac{3}{2} > 1$;
2. $a_0 = 0$ and $a_1 < 0$: $r_0 < 0$, so $x = a_0 + r_0 < 0$;
3. $a_0 = 1$ and $a_1 > 0$: $r_0 > 0$, so $x = a_0 + r_0 > 1$;
4. $\exists_{1 \leq i \leq n}$ with $|a_i| \geq 6$: $|[a_i; a_{i+1}, \ldots, a_n]| > |a_i| - r_i \geq 6 - \frac{1}{2} > \mu$;
5. $\exists_{1 \leq i < n}$ with $a_i = 5$ and $a_{i+1} > 0$: $r_i > 0$ and $[a_i; a_{i+1}, \ldots, a_n] > 5 > \mu$;
6. $\exists_{1 \leq i < n}$ with $a_i = -5$ and $a_{i+1} < 0$: $r_i < 0$ and $[a_i; a_{i+1}, \ldots, a_n] < -5 < -\mu$.

Furthermore, if $|a_n| = 5$, then (take $i = n$) $|[a_n + r]| > 5 - \frac{1}{\mu} = \mu$. □

Theorem 52. *If $x = [a_0; a_1, \dots, a_n] \in \mathrm{NICF}_5^*$ with $n \geq 1$ and $a_n \in \{-4, -3, -2, 2, 3, 4\}$, then for $y = [a_0; a_1, \dots, a_{n-1} : a_n + r]$ with $|r| < \frac{1}{\mu}$, there exists some i such that $y \in g(I_i)$.*

Proof. Note that, by construction of NICF, $|a_n + r| \geq 2$.

- If $n = 1$ and $a_1 \in \{2, 3, 4\}$, then $a_0 = 0$. Since $[0 : 4 + \frac{1}{2}] \leq y \leq [0; 2]$, we know that $y \in I_1$, and that $y \notin I_3$:
  - $a_1 = 2$: $y > [0; 2 : \mu] = P_{2^+}([0])$, so $y \notin I_2$, thus $y \in g(I_1)$;
  - $a_1 = 3$: $[0; 3 : \mu] < y < [0; 3 : -\mu]$, so $y \in g(I_2)$;
  - $a_1 = 4$: $[0; 4 : \mu] < y < [0; 4 : -\mu]$, so $y \in g(I_4)$.
- If $n = 1$ and $a_1 \in \{-4, -3, -2\}$, then $a_0 = 1$. Since $[1; -2] \leq y \leq [1 : -4 - \frac{1}{2}]$, we know that $y \in I_1$, and that $y \notin I_2$:
  - $a_1 = -2$: $y < [1; -2 : -\mu] = P_{-2^-}([1])$, so $y \notin I_3$, thus $y \in g(I_1)$;
  - $a_1 = -3$: $[1; -3 : \mu] < y < [1; -3 : -\mu]$, so $y \in g(I_3)$;
  - $a_1 = -4$: $[1; -4 : \mu] < y < [1; -4 : -\mu]$, so $y \in g(I_7)$.
- If $n > 1$, n even and $a_1 \in \{2, 3, 4\}$, then $a_{n-1} \in \{-5, -4, -3, 2, 3, 4\}$, thus there exists an i such that $I_i = T_{a_{n-1}, a_{n-1}+1}([a_0; a_1, \dots, a_{n-2}])$. Since $[a_0; a_1, \dots, a_{n-1} : 4 + \frac{1}{2}] < y \leq [a_0; a_1, \dots, a_{n-1}, 2]$, we know that $y \in I_i$ and that $y \notin I_{2i+1}$:
  - $a_1 = 2$: $[a_0; a_1, \dots, a_{n-1}, 2 : \mu] < y < [a_0; a_1, \dots, a_{n-1}, 2]$, so $y \notin I_{2i}$, thus $y \in g(I_i)$;
  - $a_1 = 3$: $[a_0; a_1, \dots, a_{n-1}, 3 : \mu] < y < [a_0; a_1, \dots, a_{n-1}, 3 : -\mu]$, so $y \in g(I_{2i})$;
  - $a_1 = 4$: $[a_0; a_1, \dots, a_{n-1}, 4 : \mu] < y < [a_0; a_1, \dots, a_{n-1}, 4 : -\mu]$, so $y \in g(I_{4i})$.
- If $n > 1$, n odd and $a_1 \in \{2, 3, 4\}$, then $a_{n-1} \in \{-5, -4, -3, 2, 3, 4\}$, thus there exists an i such that $I_i = T_{a_{n-1}, a_{n-1}+1}([a_0; a_1, \dots, a_{n-2}])$. Since $[a_0; a_1, \dots, a_{n-1}, 2] \leq y < [a_0; a_1, \dots, a_{n-1} : 4 + \frac{1}{2}]$, we know that $y \in I_i$ and that $y \notin I_{2i}$:
  - $a_1 = 2$: $[a_0; a_1, \dots, a_{n-1}, 2] \leq y < [a_0; a_1, \dots, a_{n-1}, 2 : \mu]$, so $y \notin I_{2i+1}$, thus $y \in g(I_i)$;
  - $a_1 = 3$: $[a_0; a_1, \dots, a_{n-1}, 3 : -\mu] < y < [a_0; a_1, \dots, a_{n-1}, 3 : \mu]$, so $y \in g(I_{2i+1})$;
  - $a_1 = 4$: $[a_0; a_1, \dots, a_{n-1}, 4 : -\mu] < y < [a_0; a_1, \dots, a_{n-1}, 4 : \mu]$, so $y \in g(I_{4i+3})$.
- If $n > 1$, n even and $a_1 \in \{-4, -3, -2\}$, then $a_{n-1} \in \{-4, -3, -2, 3, 4, 5\}$, thus there exists an i such that $I_i = T_{a_{n-1}-1, a_{n-1}}([a_0; a_1, \dots, a_{n-2}])$. Since $[a_0; a_1, \dots, a_{n-1} : -4 - \frac{1}{2}] < y \leq [a_0; a_1, \dots, a_{n-1}, -2]$, we know that $y \in I_i$ and that $y \notin I_{2i}$:
  - $a_1 = -2$: $[a_0; a_1, \dots, a_{n-1}, -2 : -\mu] < y \leq [a_0; a_1, \dots, a_{n-1}, -2]$, so $y \notin I_{2i+1}$, thus $y \in g(I_i)$;
  - $a_1 = -3$: $[a_0; a_1, \dots, a_{n-1}, -3 : \mu] < y < [a_0; a_1, \dots, a_{n-1}, -3 : -\mu]$, so $y \in g(I_{2i+1})$;
  - $a_1 = -4$: $[a_0; a_1, \dots, a_{n-1}, -4 : \mu] < y < [a_0; a_1, \dots, a_{n-1}, -4 : -\mu]$, so $y \in g(I_{4i+3})$.
- If $n > 1$, n odd and $a_1 \in \{-4, -3, -2\}$, then $a_{n-1} \in \{-4, -3, -2, 3, 4, 5\}$, thus there exists an i such that $I_i = T_{a_{n-1}-1, a_{n-1}}([a_0; a_1, \dots, a_{n-2}])$. Since $[a_0; a_1, \dots, a_{n-1}, -2] \leq y < [a_0; a_1, \dots, a_{n-1} : -4 - \frac{1}{2}]$, we know that $y \in I_i$ and that $y \notin I_{2i+1}$:

- $a_1 = -2$: $[a_0; a_1, \ldots, a_{n-1}, -2] \leq y < [a_0; a_1, \ldots, a_{n-1}, -2 : -\mu]$, so $y \notin I_{2i}$, thus $y \in g(I_i)$;
- $a_1 = -3$: $[a_0; a_1, \ldots, a_{n-1}, -3 : -\mu] < y < [a_0; a_1, \ldots, a_{n-1}, -3 : \mu]$, so $y \in g(I_{2i})$;
- $a_1 = -4$: $[a_0; a_1, \ldots, a_{n-1}, -4 : -\mu] < y < [a_0; a_1, \ldots, a_{n-1}, -4 : \mu]$, so $y \in g(I_{4i})$. □

Theorem 53. *If $x = [a_0; a_1, \ldots] \in \mathbb{R} \setminus (\mathbb{Q} \cap \mathrm{NICF}_5^*)$ then $x < 0$ or $x > 1$ or there exists an $n > 0$ such that $|[a_n; a_{n+1}, a_{n+2}, \ldots]| > \mu$.*

Proof. Take $x = [a_0; a_1, \ldots] \in \mathbb{R} \setminus (\mathbb{Q} \cap \mathrm{NICF}_5^*)$, then, because $x \notin \mathrm{NICF}_5^*$, (at least) one of the following arguments is true:

1. $a_0 \notin \{0, 1\}$: either $a_0 \leq -1$, such that $x = a_0 + r_0 \leq -\frac{1}{2} < 0$, or $a_0 \geq 2$, such that $x = a_0 + r_0 \geq \frac{3}{2} > 1$;
2. $a_0 = 0$ and $a_1 < 0$: $r_0 < 0$, so $x = a_0 + r_0 < 0$;
3. $a_0 = 1$ and $a_1 > 0$: $r_0 > 0$, so $x = a_0 + r_0 > 1$;
4. $\exists_{1 \leq i \leq n}$ with $|a_i| \geq 6$: $|[a_i; a_{i+1}, \ldots, a_n]| > |a_i| - r_i \geq 6 - \frac{1}{2} > \mu$;
5. $\exists_{1 \leq i < n}$ with $a_i = 5$ and $a_{i+1} > 0$: $r_i > 0$ and $[a_i; a_{i+1}, \ldots, a_n] > 5 > \mu$;
6. $\exists_{1 \leq i < n}$ with $a_i = -5$ and $a_{i+1} < 0$: $r_i < 0$ and $[a_i; a_{i+1}, \ldots, a_n] < -5 < -\mu$. □

Theorem 54. $C_{\mathrm{NICF}} \subseteq \mathrm{NICF}_5^* \setminus \mathbb{Q}$

Proof. We will prove that for every $x \in \mathbb{R}$, if $x \notin \mathrm{NICF}_5^* \setminus \mathbb{Q}$, then $x \notin I_1 = T_{0,1}$ or there exists an i such that $x \in g(I_i)$.
If $x < 0$ or $x > 1$ then $x \notin I_1$, so let us assume $0 \leq x \leq 1$. First we are going to show that there exists some n such that $x = [a_0; a_1, \ldots, a_{n-1} : a_n + r]$ and $|r| < \frac{1}{\mu}$:

- Suppose that $x \in \mathbb{Q}$. Then there exists an n such that $x = [a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots : a_n + r]$ where $r = 0$.
- Suppose that $x \notin \mathbb{Q}$, say $x = [a_0; a_1, \ldots]$. So $x \in \mathbb{R} \setminus (\mathbb{Q} \cap \mathrm{NICF}_5^*)$. With Theorem 53, there exists some $n > 0$ such that $|[a_n; a_{n+1}, a_{n+2}, \ldots]| > \mu$. Let $r = \frac{1}{[a_n; a_{n+1}, a_{n+2}, \ldots]}$, then $x = [a_0; a_1, \ldots, a_{n-2} : a_{n-1} + r]$ and $|r| < \frac{1}{\mu}$.

We can define k as the smallest n such that $x = [a_0; a_1, \ldots, a_{n-1} : a_n + r_n]$ with $|r_n| < \frac{1}{\mu}$. Now we know that $x = [a_0; a_1, \ldots, a_{k-1} : a_k + r_k]$ and $|r_k| < \frac{1}{\mu}$ and, when we represent x as $[a_0; a_1, \ldots, a_{l-1} : a_l + r_l]$, because $|r_l| \leq \frac{1}{\mu}$, we have $[a_{l+1}; a_{l+2}, \ldots, a_{k-1} : a_k + r_k] > \mu$.

- Suppose that $k = 0$: We have $x = a_0 + r$, so $x < 0 + \mu$ or $x > 1 - \mu$, so $x \notin I_1$.
- Suppose that $k > 0$, with Theorem 51, we know that $[a_0; a_1, \ldots, a_{k-1}, a_k] \in \mathrm{NICF}_5^*$ and $a_k \in \{-4, -3, -2, 2, 3, 4\}$. Then, with Theorem 52, we know that there exists an i such that $x \in g(I_i)$. □

3.3 $C_{\mathrm{NICF}} + C_{\mathrm{NICF}}$

Theorem 55. *For every* $x \in \mathbb{R} \cap [\frac{2}{\mu}, 2 - \frac{2}{\mu}]$*, there exist* $a, b \in C_{\mathrm{NICF}}$ *such that* $a + b = x$*.*

Proof. C_{NICF} is a Cantor set with initial interval $[\frac{1}{\mu}, 1 - \frac{1}{\mu}]$, and has a density ratio bigger than 1. We can now use Theorem 34, as $\frac{x}{x+1} \geq \frac{1}{2}$ if $x \geq 1$, and $\overline{I_1^1} = \overline{I_1^2}$ □

Theorem 56. *For every* $x \in \mathbb{R} \cap [\frac{1}{2}, \frac{3}{2}]$*, there exist* $a, b \in \mathrm{NICF}_5$ *such that* $a + b = x$*.*

Proof. Because $[\frac{1}{2}, \frac{3}{2}] \subset [\frac{2}{\mu}, 2 - \frac{2}{\mu}]$, we can apply Theorem 55, so there exist $a, b \in C_{\mathrm{NICF}}$ such that $a + b = x$. Because $C_{\mathrm{NICF}} = \mathrm{NICF}_5^* \setminus \mathbb{Q}$ and $(\mathrm{NICF}_5^* \setminus \mathbb{Q}) \subset \mathrm{NICF}_5$, we have $a, b \in \mathrm{NICF}_5$. □

Now we can prove our main result of this chapter:

Proof of Theorem 41. We can write x as $y + n$ with $n \in \mathbb{Z}$ and $y \in [\frac{1}{2}, \frac{3}{2}]$. We know that there exist $a, b \in \mathrm{NICF}_5$ such that $a + b = y$, so $(a + n) + b = x$, with $a + n$ and b in NICF_5. □

Chapter 4

$\mathrm{NICF}_4 + \mathrm{NICF}_4 \neq \mathbb{R}$

In this chapter, we are going to give a counterexample to $\mathrm{NICF}_4 + \mathrm{NICF}_4 = \mathbb{R}$. First, recall Definition 9 with $r = 4$.

Definition. NICF_4 *is a subset of* $\mathbb{R}$*, containing only the numbers representable by a Nearest Integer Continued Fraction where every coefficient except possibly the first has absolute value less than or equal to 4:*

$$\mathrm{NICF}_4 = \{x : x \in \mathbb{R} \mid x = [a_0; a_1, a_2, \ldots] \in \mathrm{NICF} \text{ and } \forall_{i \geq 1} |a_i| \leq 4\}$$

Lemma 57. *For every* $x, y \in \mathrm{NICF}_4$ *there exists* x' *and* y' *in* NICF_4 *with integer part 0 such that* $x' + y' \equiv x + y \mod 1$.

Proof. Let $x, y \in \mathrm{NICF}_4$, set $x' = x - \lfloor x \rceil$ and $y' = y - \lfloor y \rceil$, then $x + y = \lfloor x \rceil + \lfloor y \rceil + x' + y' \equiv x' + y' \mod 1$. As $x, y \in [-\frac{1}{2}, \frac{1}{2})$, their integer part is 0. □

Definition 58. *Let*

$$\mathrm{NICF}_{\backslash\mathbb{Q}} = \mathrm{NICF}_4 \cap [-\frac{1}{2}, \frac{1}{2}) \setminus \mathbb{Q}$$

and

$$\mathrm{NICF}_{\mathbb{Q}} = \mathrm{NICF}_4 \cap [-\frac{1}{2}, \frac{1}{2}) \cap \mathbb{Q}.$$

We will show that:

1. $(\mathrm{NICF}_{\backslash\mathbb{Q}} + \mathrm{NICF}_{\backslash\mathbb{Q}}) \cap ([-0.627705, -0.627695] \cup [0.372295, 0.372305]) = \emptyset$;

2. $\mathrm{NICF}_{\mathbb{Q}} + \mathrm{NICF}_{\mathbb{Q}}$ is a countable set;

3. $\mathrm{NICF}_{\backslash\mathbb{Q}} + \mathrm{NICF}_{\mathbb{Q}}$ has a Lebesgue measure of 0.

Lemma 59. $[0; \overline{4, 2}]$ *is the smallest value above zero in* $\mathrm{NICF}_{\backslash\mathbb{Q}}$*, and* $[0; \overline{2, 4}]$ *is the largest value below* $\frac{1}{2}$ *in* $\mathrm{NICF}_{\backslash\mathbb{Q}}$*.*

Proof. Suppose x and y are respectively the smallest value above zero and the largest value below $\frac{1}{2}$ in $\text{NICF}_{\backslash\mathbb{Q}}$. We know both x and y start with a zero, followed by a positive number. For $[0, a_1, a_2, \ldots] \in \text{NICF}_{\backslash\mathbb{Q}}$ with $a_1 > 0$ we know that

$$\frac{1}{a_1 + \frac{1}{2}} \leq [0, a_1, a_2, \ldots] \leq \frac{1}{a_1 - \frac{1}{2}}.$$

Also

$$[0, a_1, a_2, \ldots] = \frac{1}{a_1 + [0; a_2, a_3, \ldots]},$$

so $x = [0 : 4 + y]$ and $y = [0 : 2 + x]$. □

Definition 60. *The set* $\text{NICF}_4\langle a_1, \ldots, a_n\rangle$ *is the subset of* $\text{NICF}_{\backslash\mathbb{Q}}$ *in which the first* $n+1$ *coefficients are equal to* $0, a_1, \ldots, a_n$*:*

$$\text{NICF}_4\langle a_1, \ldots, a_n\rangle = \{x \in \text{NICF}_{\backslash\mathbb{Q}} \,|\exists y, x = [0; a_1, \ldots, a_n : y]\}$$

The infimum of $\text{NICF}_4\langle a_1, \ldots, a_n\rangle$ is

$$[0; a_1, \ldots, a_n, \overline{4, 2}] \text{ if } a_n = 2;$$
$$[0; a_1, \ldots, a_n, \overline{-2, -4}] \text{ otherwise.}$$

The supremum of $\text{NICF}_4\langle a_1, \ldots, a_n\rangle$ is

$$[0; a_1, \ldots, a_n, \overline{-4, -2}] \text{ if } a_n = -2;$$
$$[0; a_1, \ldots, a_n, \overline{2, 4}] \text{ otherwise.}$$

We can now fill a table with $\text{NICF}_4\langle a_1, \ldots, a_n\rangle$ for different values of $\langle a_1, \ldots, a_n\rangle$, see Table 4.1. Every element of $\text{NICF}_{\backslash\mathbb{Q}}$ is included in one of these sets, and therefore lies in one of the covering intervals.

The sum of each combination of two covering intervals does not overlap with either of the intervals $[-0.627705, -0.627695]$ and $[0.372295, 0.372305]$. We only show the combinations of intervals that are most relevant. For each combination of intervals I, J there exists a sum $I' + J'$ in the list below such that $I' \geq I$ and $J' \geq J$, or $I' \leq I$ and $J' \leq J$.

The sums closest to $[-0.627705, -0.627695]$:

$$\begin{aligned}
\langle -2, -4\rangle + \langle 4\rangle &= [-0.44949, -0.43827] + [0.22474, 0.28165] = [-0.22475, -0.15662]\\
\langle -2, -2\rangle + \langle -4, -2\rangle &= [-0.41524, -0.40824] + [-0.22685, -0.22474] = [-0.66356, -0.64278]\\
\langle -3, 2, 4, 2\rangle + \langle -4, -4, 4\rangle &= [-0.39208, -0.39201] + [-0.23448, -0.23425] = [-0.62656, -0.62626]\\
\langle -3, 2, 4, 4\rangle + \langle -4, -4, -4\rangle &= [-0.39181, -0.39170] + [-0.23621, -0.23603] = [-0.62802, -0.62773]\\
\langle -3, 2, 4, -4\rangle + \langle -4, -4, -2\rangle &= [-0.39086, -0.39073] + [-0.23671, -0.23658] = [-0.62757, -0.62731]\\
\langle -3, 4, -4\rangle + \langle -4, 4, 2\rangle &= [-0.36616, -0.36561] + [-0.26504, -0.26488] = [-0.63120, -0.63049]\\
\langle -3, 4, 4\rangle + \langle -4, 4, 4\rangle &= [-0.36189, -0.36147] + [-0.26573, -0.26550] = [-0.62762, -0.62697]\\
\langle -3, 4, 2\rangle + \langle -4, 4, -4\rangle &= [-0.36061, -0.36032] + [-0.26803, -0.26772] = [-0.62864, -0.62804]\\
\langle -3, -4\rangle + \langle -3, -4\rangle &= [-0.31011, -0.30472] + [-0.31011, -0.30472] = [-0.62022, -0.60944]
\end{aligned}$$

The sums closest to $[0.372295, 0.372305]$:

$$\begin{aligned}
\langle -4, -2\rangle + \langle 2\rangle &= [-0.22685, -0.22474] + [0.40824, 0.44949] = [0.18139, 0.22475]\\
\langle 4\rangle + \langle 4\rangle &= [0.22474, 0.28165] + [0.22474, 0.28165] = [0.44948, 0.56330]
\end{aligned}$$

Lemma 61. $\mathrm{NICF}_{\mathbb{Q}} + \mathrm{NICF}_{\mathbb{Q}} \cap ([-0.627705, -0.627695] \cup [0.372295, 0.372305]) = \emptyset$

Proof. Let I be the union of intervals in the above table. $\mathrm{NICF}_{\mathbb{Q}}$ is a subset of I, and

$$I + I \cap ([-0.627705, -0.627695] \cup [0.372295, 0.372305]) = \emptyset.$$

□

Lemma 62. $\mathrm{NICF}_{\backslash\mathbb{Q}}$ *has Lebesgue-measure 0.*

Proof. There exists an $\epsilon > 0$ such that for each $x, y \in \mathbb{R}$, with $x < y$, there exist x' and y' such that $x \leq x' < y' \leq y$, $[x', y'] \cap \mathrm{NICF}_{\backslash\mathbb{Q}} = \emptyset$ and $\frac{y'-x'}{y-x} \geq \epsilon$. We also know that $\mathrm{NICF}_{\backslash\mathbb{Q}}$ is a subset of the interval $[-\frac{1}{2}, \frac{1}{2}]$. This lets us create a sequence of sets of intervals S, such that the measure of $S_i \leq (1-\epsilon)^i$ and $\mathrm{NICF}_{\backslash\mathbb{Q}} \subseteq S_i$. Let

$$S_0 = \{[-\frac{1}{2}, \frac{1}{2}]\},$$

and

$$S_{i+1} = \{[x, x'], [y', y] : [x, y] \in S_i, x', y' \text{ as above}\}.$$

□

Remark 63. $\mathrm{NICF}_{\mathbb{Q}}$ *is a countable set.*

Lemma 64. *The set* $\{x + y \mid x \in \mathrm{NICF}_{\mathbb{Q}}, y \in \mathrm{NICF}_{\backslash\mathbb{Q}}\}$ *has Lebesgue-measure 0.*

Proof. $\mathrm{NICF}_{\backslash\mathbb{Q}}$ has Lebesgue-measure 0 and there are only a countably infinite number of elements in $\mathrm{NICF}_{\mathbb{Q}}$. Because of the subadditivity of Lebesgue-measure, we have

$$\mu(\mathrm{NICF}_{\mathbb{Q}} + \mathrm{NICF}_{\backslash\mathbb{Q}}) \leq \mu(\bigcup_{x \in \mathrm{NICF}_{\mathbb{Q}}} x + \mathrm{NICF}_{\backslash\mathbb{Q}}) \leq \sum_{x \in \mathrm{NICF}_{\mathbb{Q}}} \mu(\mathrm{NICF}_{\backslash\mathbb{Q}}) = 0.$$

□

Lemma 65. $\mathrm{NICF}_{\mathbb{Q}} + \mathrm{NICF}_{\mathbb{Q}}$ *has Lebesgue-measure 0.*

Proof. $\mathrm{NICF}_{\mathbb{Q}} + \mathrm{NICF}_{\mathbb{Q}}$ is a subset of $\mathbb{Q} + \mathbb{Q} = \mathbb{Q}$, which is a countable set. Every countable set has Lebesque measure 0. □

Theorem 66. $\mathrm{NICF}_4 + \mathrm{NICF}_4 \neq \mathbb{R}$

Proof. Proof by contraposition. Suppose for every $x \in \mathbb{R}$ there exists $a, b \in \mathrm{NICF}_4$ such that $a + b = x$. By Lemma 57, we have that for each $x \in \mathbb{R}$, there exists $a', b' \in \mathrm{NICF}_{\backslash\mathbb{Q}} \cup \mathrm{NICF}_{\mathbb{Q}}$ such that $a' + b' \equiv x \mod 1$. Thus $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\backslash\mathbb{Q}} \cup \mathrm{NICF}_{\mathbb{Q}}\}$ has Lebesgue measure 1. We will split this set in 3 parts:

- $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\backslash\mathbb{Q}}\}$:

  Because $\mathrm{NICF}_{\backslash\mathbb{Q}} \subseteq [-\frac{1}{2}, \frac{1}{2})$, and Lemma 61, we know that $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\backslash\mathbb{Q}}\} \cap [0.372295, 0.372305] = \emptyset$, thus the Lebesgue measure of $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\backslash\mathbb{Q}}\}$ is smaller than or equal to 0.99999.

- $\{a' + b' \mod 1 \mid a' \in \mathrm{NICF}_{\mathbb{Q}}, b' \in \mathrm{NICF}_{\backslash\mathbb{Q}}\}$:

  By Lemma 64, the Lebesgue measure of $\{a' + b' \mod 1 \mid a' \in \mathrm{NICF}_{\mathbb{Q}}, b' \in \mathrm{NICF}_{\backslash\mathbb{Q}}\}$ is 0.

- $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\mathbb{Q}}\}$:

  By Lemma 65, $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\mathbb{Q}}\}$ has Lebesgue measure 0.

The union of these 3 parts has Lebesgue measure at most 0.99999. Which leads to a contradiction with: $\{a' + b' \mod 1 \mid a', b' \in \mathrm{NICF}_{\backslash\mathbb{Q}} \cup \mathrm{NICF}_{\mathbb{Q}}\}$ has Lebesgue measure 1.
Therefore, $\mathrm{NICF}_4 + \mathrm{NICF}_4 \neq \mathbb{R}$. □

fixed coeff.	minimum	maximum	covering interval
$\langle -2,-4\rangle$	$[0;\overline{-2,-4}]$	$[0;-2,-4,\overline{2,4}]$	$[-0.44949,-0.43827]$
$\langle -2,-3\rangle$	$[0;-2,-3,\overline{-2,-4}]$	$[0;-2,-3,\overline{2,4}]$	$[-0.43671,-0.41804]$
$\langle -2,-2\rangle$	$[0;-2,-2,\overline{-2,-4}]$	$[0;-2,\overline{-2,-4}]$	$[-0.41524,-0.40824]$
$\langle -3,2,4,2\rangle$	$[0;-3,\overline{2,4}]$	$[0;-3,2,4,2,\overline{2,4}]$	$[-0.39208,-0.39201]$
$\langle -3,2,4,3\rangle$	$[0;-3,2,4,3,\overline{-2,-4}]$	$[0;-3,2,4,3,\overline{2,4}]$	$[-0.39199,-0.39181]$
$\langle -3,2,4,4\rangle$	$[0;-3,2,4,4,\overline{-2,-4}]$	$[0;-3,2,4,4,\overline{2,4}]$	$[-0.39181,-0.39170]$
$\langle -3,2,4,-4\rangle$	$[0;-3,2,4,-4,\overline{-2,-4}]$	$[0;-3,2,4,-4,\overline{2,4}]$	$[-0.39086,-0.39073]$
$\langle -3,2,4,-3\rangle$	$[0;-3,2,4,-3,\overline{-2,-4}]$	$[0;-3,2,4,-3,\overline{2,4}]$	$[-0.39072,-0.39049]$
$\langle -3,2,4,-2\rangle$	$[0;-3,2,4,-2,\overline{-2,-4}]$	$[0;-3,2,4,\overline{-2,-4}]$	$[-0.39046,-0.39036]$
$\langle -3,2,3\rangle$	$[0;-3,2,3,\overline{2,4}]$	$[0;-3,2,3,\overline{-2,-4}]$	$[-0.39013,-0.38730]$
$\langle -3,2,2\rangle$	$[0;-3,2,2,\overline{2,4}]$	$[0;-3,2,\overline{2,4}]$	$[-0.38689,-0.38583]$
$\langle -3,3\rangle$	$[0;-3,3,\overline{-2,-4}]$	$[0;-3,3,\overline{2,4}]$	$[-0.38345,-0.36898]$
$\langle -3,4,-2\rangle$	$[0;-3,4,\overline{-2,-4}]$	$[0;-3,4,-2,\overline{-2,-4}]$	$[-0.36788,-0.36743]$
$\langle -3,4,-3\rangle$	$[0;-3,4,-3,\overline{2,4}]$	$[0;-3,4,-3,\overline{-2,-4}]$	$[-0.36727,-0.36623]$
$\langle -3,4,-4\rangle$	$[0;-3,4,-4,\overline{2,4}]$	$[0;-3,4,-4,\overline{-2,-4}]$	$[-0.36616,-0.36561]$
$\langle -3,4,4\rangle$	$[0;-3,4,4,\overline{2,4}]$	$[0;-3,4,4,\overline{-2,-4}]$	$[-0.36189,-0.36147]$
$\langle -3,4,3\rangle$	$[0;-3,4,3,\overline{2,4}]$	$[0;-3,4,3,\overline{-2,-4}]$	$[-0.36142,-0.36070]$
$\langle -3,4,2\rangle$	$[0;-3,4,2,\overline{2,4}]$	$[0;-3,4,\overline{2,4}]$	$[-0.36061,-0.36032]$
$\langle -3,-4\rangle$	$[0;-3,-4,\overline{-2,-4}]$	$[0;-3,-4,\overline{2,4}]$	$[-0.31011,-0.30472]$
$\langle -3,-3\rangle$	$[0;-3,-3,\overline{-2,-4}]$	$[0;-3,-3,\overline{2,4}]$	$[-0.30397,-0.29480]$
$\langle -3,-2\rangle$	$[0;-3,-2,\overline{-2,-4}]$	$[0;-3,\overline{-2,-4}]$	$[-0.29341,-0.28989]$
$\langle -4,2\rangle$	$[0;-4,\overline{2,4}]$	$[0;-4,2,\overline{2,4}]$	$[-0.28165,-0.27841]$
$\langle -4,3\rangle$	$[0;-4,3,\overline{-2,-4}]$	$[0;-4,3,\overline{2,4}]$	$[-0.27717,-0.26953]$
$\langle -4,4,-2\rangle$	$[0;-4,4,\overline{-2,-4}]$	$[0;-4,4,-2,\overline{-2,-4}]$	$[-0.26894,-0.26870]$
$\langle -4,4,-3\rangle$	$[0;-4,4,-3,\overline{2,4}]$	$[0;-4,4,-3,\overline{-2,-4}]$	$[-0.26862,-0.26806]$
$\langle -4,4,-4\rangle$	$[0;-4,4,-4,\overline{2,4}]$	$[0;-4,4,-4,\overline{-2,-4}]$	$[-0.26803,-0.26772]$
$\langle -4,4,4\rangle$	$[0;-4,4,4,\overline{2,4}]$	$[0;-4,4,4,\overline{-2,-4}]$	$[-0.26573,-0.26550]$
$\langle -4,4,3\rangle$	$[0;-4,4,3,\overline{2,4}]$	$[0;-4,4,3,\overline{-2,-4}]$	$[-0.26548,-0.26508]$
$\langle -4,4,2\rangle$	$[0;-4,4,2,\overline{2,4}]$	$[0;-4,4,\overline{2,4}]$	$[-0.26504,-0.26488]$
$\langle -4,-4,-2\rangle$	$[0;-4,-4,\overline{-2,-4}]$	$[0;-4,-4,-2,\overline{-2,-4}]$	$[-0.23671,-0.23658]$
$\langle -4,-4,-3\rangle$	$[0;-4,-4,-3,\overline{2,4}]$	$[0;-4,-4,-3,\overline{-2,-4}]$	$[-0.23654,-0.23623]$
$\langle -4,-4,-4\rangle$	$[0;-4,-4,-4,\overline{2,4}]$	$[0;-4,-4,-4,\overline{-2,-4}]$	$[-0.23621,-0.23603]$
$\langle -4,-4,4\rangle$	$[0;-4,-4,4,\overline{2,4}]$	$[0;-4,-4,4,\overline{-2,-4}]$	$[-0.23448,-0.23425]$
$\langle -4,-4,3\rangle$	$[0;-4,-4,3,\overline{2,4}]$	$[0;-4,-4,3,\overline{-2,-4}]$	$[-0.23422,-0.23379]$
$\langle -4,-4,2\rangle$	$[0;-4,-4,2,\overline{2,4}]$	$[0;-4,-4,\overline{2,4}]$	$[-0.23374,-0.23355]$
$\langle -4,-3\rangle$	$[0;-4,-3,\overline{-2,-4}]$	$[0;-4,-3,\overline{2,4}]$	$[-0.23311,-0.22768]$
$\langle -4,-2\rangle$	$[0;-4,-2,\overline{-2,-4}]$	$[0;-4,\overline{-2,-4}]$	$[-0.22685,-0.22474]$
$\langle 4\rangle$	$[0;4,\overline{2,4}]$	$[0;4,\overline{-2,-4}]$	$[0.22474,0.28165]$
$\langle 3\rangle$	$[0;3,\overline{2,4}]$	$[0;3,\overline{-2,-4}]$	$[0.28989,0.39208]$
$\langle 2\rangle$	$[0;2,\overline{2,4}]$	$[0;\overline{2,4}]$	$[0.40824,0.44949]$

Table 4.1: Specific cases of $\mathrm{NICF}_4\langle a_1,\ldots,a_n\rangle$ to include every element of $\mathrm{NICF}_{\setminus\mathbb{Q}}$

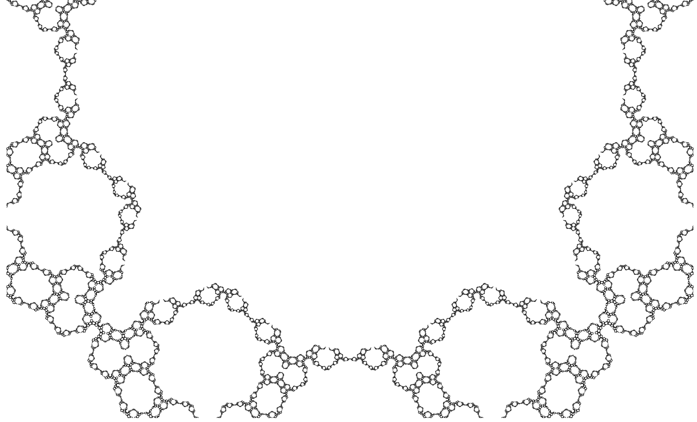
Chapter 5

$\mathrm{HCF}_{\sqrt{5}} + \mathrm{HCF}_{\sqrt{5}} = \mathbb{C}$

In this chapter, we are going to define a simple closed curve in $\mathrm{HCF}_{\sqrt{5}}$, and show that it encloses $\{a + bi \mid a, b \in [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}\}$. Together with the following lemma, this will enable us to show that every element of $\mathbb{C}$ is the sum of two elements of $\mathrm{HCF}_{\sqrt{5}}$. First, recall Definition 12 with $r = \sqrt{5}$.

Definition. $\mathrm{HCF}_{\sqrt{5}}$ *is a subset of* $\mathbb{C}$, *containing only the numbers representable by a Hurwitz complex continued fraction where every coefficient except possibly the first has absolute value less than or equal to* $\sqrt{5}$*:*

$$\mathrm{HCF}_{\sqrt{5}} = \left\{ x : x \in \mathbb{C} \mid x = [a_0; a_1, a_2, \ldots] \in \mathrm{HCF} \text{ and } \forall_{j \geq 1} |a_j| \leq \sqrt{5} \right\}$$



$\mathrm{HCF}_{\sqrt{5}}$ with first coefficient zero.

Lemma 67. *For every simple closed curve $B \in \mathbb{C}$, Let X be the region enclosed by B, then for all $a, b \in X$ there exist $c, d \in B$ such that $a + b$ equals $c + d$.*

Proof. Let X' be the closed region defined as X mirrored in the point $\frac{a+b}{2}$, with boundary B', the simple closed curve defined by the mirror image of B in $\frac{a+b}{2}$. We write $y' \in \mathbb{C}$ as the image of $y \in \mathbb{C}$ while mirrored in $\frac{a+b}{2}$. Note that for all $y \in \mathbb{C}$, $y' + y = a + b$. Also note $a' = b \in X$, so $X \cap X' \neq \emptyset$. Case distinction:

- Case 1, $X' = X$: Then $B = B'$. For every $c \in B$ there exists $d \in B$ such that $d' = c$, now $c + d = d' + d = a + b$.
- Case 2, $X' \neq X$:

  Let B'' be the boundary of $X \cup X'$ with $B'' \subset B \cup B'$. As $X \cap X' \neq \emptyset$, B'' is connected. Because $X' \neq X$, either:

  - there exists $y \in X \setminus X'$, then $y' \in X' \setminus X$; or
  - there exists $z \in X' \setminus X$, then $z' \in X \setminus X'$.

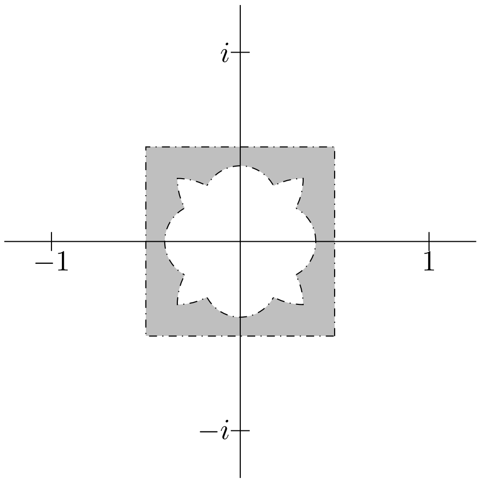
  We can conclude $X \not\subset X'$ and $X' \not\subset X$, thus $B'' \not\subset B$ and $B'' \not\subset B'$.

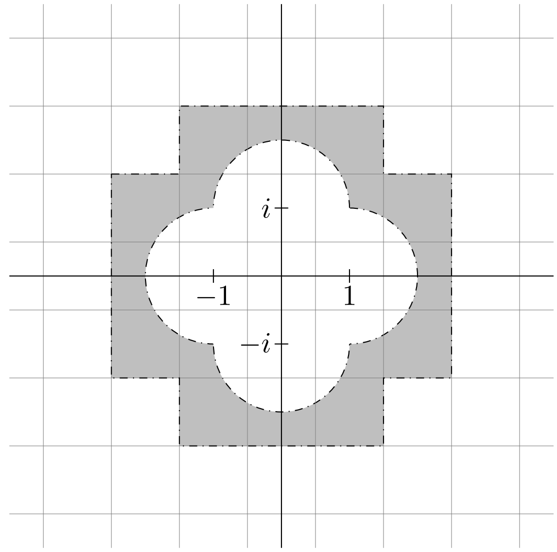
  Because B'' is connected, there exists $c \in B \cup B'$ such that for all $\epsilon > 0$, there exists $\delta \in B$ and $\delta' \in B'$ such that both $|\delta - c| < \epsilon$ and $|\delta' - c| < \epsilon$. By continuity of the curves B and B', we have $c \in B \cap B'$.

  Let $d = c'$, with $c' \in B' \cap B$. Then $c, d \in B$ and $c + d = c + c' = a + b$. □

We start by defining a transition function $\delta_{\sqrt{5}}$ on the known set $\mathbb{S}$, using the same names for Shapes as before in Section 1.5. For every shape, the left picture depicts the shape in HCF, where the first coefficient is bounded by $\sqrt{5}$. The right picture depicts the (pointwise) reciprocal of the left picture.

Square in $\mathrm{HCF}_{\sqrt{5}}$

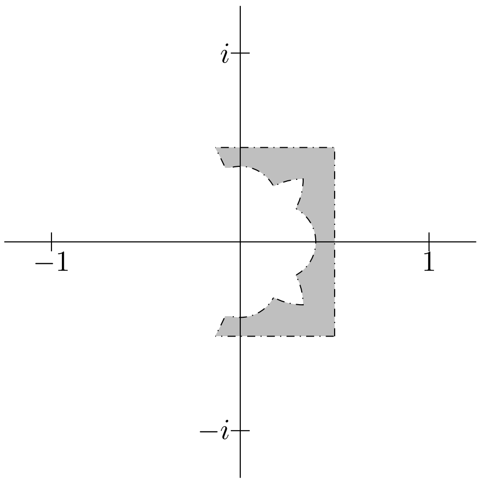


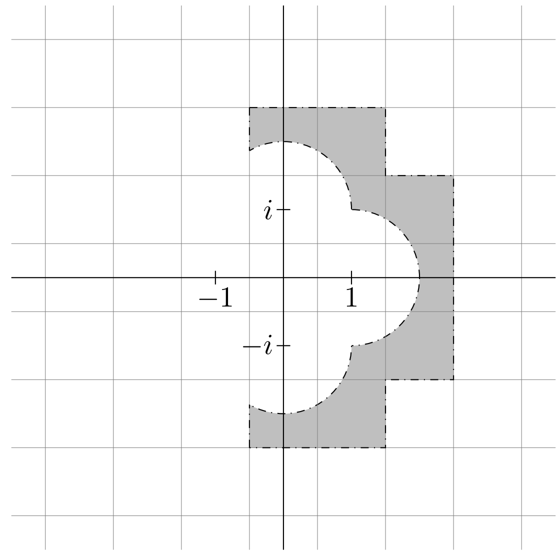


Given the coefficient $a_j + b_j \cdot i$, the transition function $\delta_{\sqrt{5}}$ on S is defined by:
$\delta_{\sqrt{5}}(S, a_j + b_j \cdot i) =$

$$
\begin{aligned}
M_{\mathrm{HCF}} &\text{ if } a_j = 2 \text{ and } b_j = 0;\\
\rho(M_{\mathrm{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = 2;\\
\rho^2(M_{\mathrm{HCF}}) &\text{ if } a_j = -2 \text{ and } b_j = 0;\\
\rho^3(M_{\mathrm{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = -2;\\
W_{\mathrm{HCF}} &\text{ if } a_j = 1 \text{ and } b_j = 2,\\
&\text{ or } a_j = 2 \text{ and } b_j = 1;\\
\rho(W_{\mathrm{HCF}}) &\text{ if } a_j = -1 \text{ and } b_j = 2,\\
&\text{ or } a_j = -2 \text{ and } b_j = 1;\\
\rho^2(W_{\mathrm{HCF}}) &\text{ if } a_j = -1 \text{ and } b_j = -2,\\
&\text{ or } a_j = -2 \text{ and } b_j = -1;\\
\rho^3(W_{\mathrm{HCF}}) &\text{ if } a_j = 1 \text{ and } b_j = -2,\\
&\text{ or } a_j = 2 \text{ and } b_j = -1;\\
J_{\mathrm{HCF}} &\text{ if } a_j = 1 \text{ and } b_j = 1;\\
\rho(J_{\mathrm{HCF}}) &\text{ if } a_j = -1 \text{ and } b_j = 1;\\
\rho^2(J_{\mathrm{HCF}}) &\text{ if } a_j = -1 \text{ and } b_j = -1;\\
\rho^3(J_{\mathrm{HCF}}) &\text{ if } a_j = 1 \text{ and } b_j = -1.
\end{aligned}
$$

Moonshaped in $\mathrm{HCF}_{\sqrt{5}}$





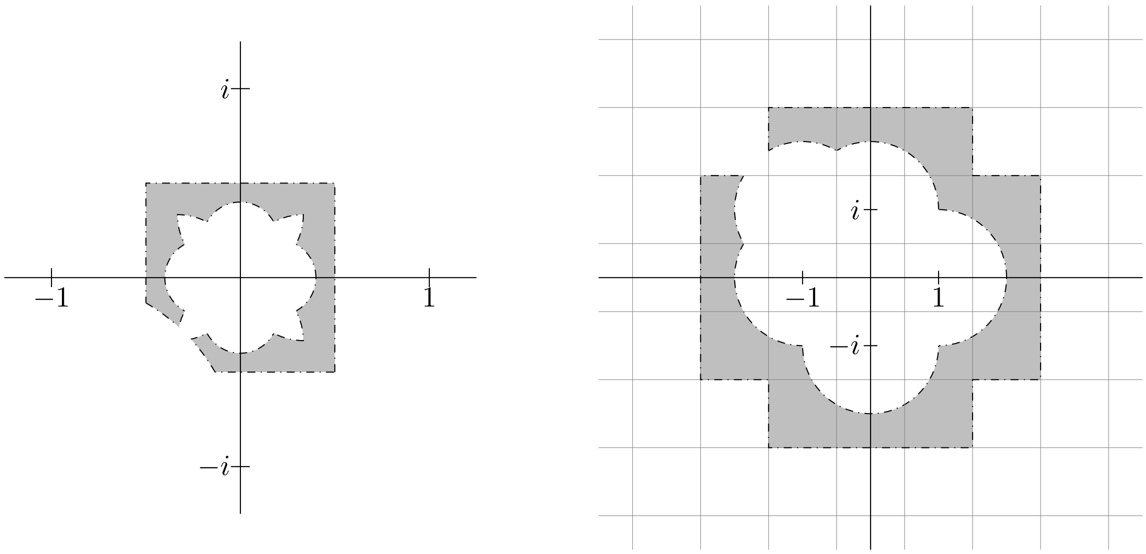
Given the coefficient $a_j + b_j \cdot i$, the transition function $\delta_{\sqrt{5}}$ on M is defined by:
$\delta_{\sqrt{5}}(M, a_j + b_j \cdot i) =$

$$\begin{aligned}
M_{\mathrm{HCF}} &\text{ if } a_j = 2 \text{ and } b_j = 0;\\
\rho(M_{\mathrm{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = 2;\\
\rho^3(M_{\mathrm{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = -2;\\
W_{\mathrm{HCF}} &\text{ if } a_j = 1 \text{ and } b_j = 2,\\
&\text{ or } a_j = 2 \text{ and } b_j = 1;\\
\rho^3(W_{\mathrm{HCF}}) &\text{ if } a_j = 1 \text{ and } b_j = -2,\\
&\text{ or } a_j = 2 \text{ and } b_j = -1;\\
J_{\mathrm{HCF}} &\text{ if } a_j = 1 \text{ and } b_j = 1;\\
\rho^3(J_{\mathrm{HCF}}) &\text{ if } a_j = 1 \text{ and } b_j = -1.
\end{aligned}$$

Without-a-corner in $\mathrm{HCF}_{\sqrt{5}}$

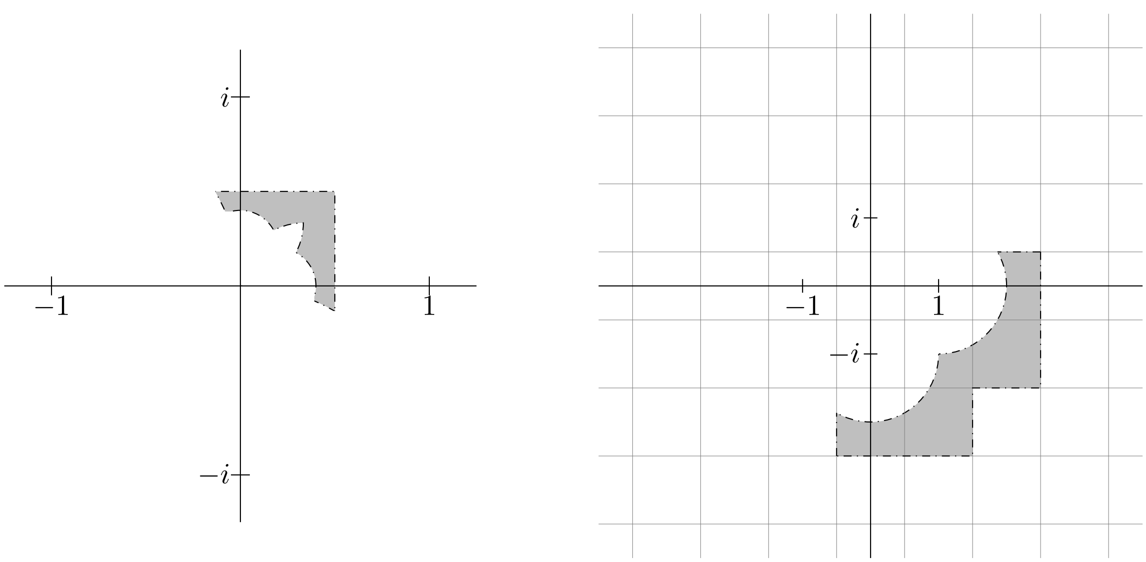
Given the coefficient $a_j + b_j \cdot i$, the transition function $\delta_{\sqrt{5}}$ on W is defined by:
$\delta_{\sqrt{5}}(W, a_j + b_j \cdot i) =$

$$\begin{aligned}
M_{\mathrm{HCF}} &\text{ if } a_j = 2 \text{ and } b_j = 0;\\
\rho(M_{\mathrm{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = 2;\\
&\text{ or } a_j = -1 \text{ and } b_j = 2;\\
\rho^2(M_{\mathrm{HCF}}) &\text{ if } a_j = -2 \text{ and } b_j = 0;\\
&\text{ or } a_j = -2 \text{ and } b_j = 1;\\
\rho^3(M_{\mathrm{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = -2;\\
W_{\mathrm{HCF}} &\text{ if } a_j = 1 \text{ and } b_j = 2,
\end{aligned}$$



$$
\begin{aligned}
& \text{or } a_j = 2 \text{ and } b_j = 1; \\
\rho^2(W_{\mathrm{HCF}}) \ & \text{if } a_j = -1 \text{ and } b_j = -2, \\
& \text{or } a_j = -2 \text{ and } b_j = -1; \\
\rho^3(W_{\mathrm{HCF}}) \ & \text{if } a_j = 1 \text{ and } b_j = -2, \\
& \text{or } a_j = 2 \text{ and } b_j = -1; \\
J_{\mathrm{HCF}} \ & \text{if } a_j = 1 \text{ and } b_j = 1; \\
\rho^2(J_{\mathrm{HCF}}) \ & \text{if } a_j = -1 \text{ and } b_j = -1; \\
\rho^3(J_{\mathrm{HCF}}) \ & \text{if } a_j = 1 \text{ and } b_j = -1.
\end{aligned}
$$

Just-a-corner in $\mathrm{HCF}_{\sqrt{5}}$



Given the coefficient $a_j + b_j \cdot i$, the transition function $\delta_{\sqrt{5}}$ on J is defined by:
$\delta_{\sqrt{5}}(J, a_j + b_j \cdot i) =$

$$\begin{aligned} M_{\text{HCF}} &\text{ if } a_j = 2 \text{ and } b_j = 0; \\ \rho^3(M_{\text{HCF}}) &\text{ if } a_j = 0 \text{ and } b_j = -2; \\ \rho^3(W_{\text{HCF}}) &\text{ if } a_j = 1 \text{ and } b_j = -2, \\ &\text{ or } a_j = 2 \text{ and } b_j = -1; \\ \rho^3(J_{\text{HCF}}) &\text{ if } a_j = 1 \text{ and } b_j = -1. \end{aligned}$$

Just like for HCF, we define

$$\delta_{\sqrt{5}}(\rho(X, a+bi) = (\rho^3 \circ \delta_{\sqrt{5}})(X, \rho(a+bi)).$$

For all other values, let $\delta_{\sqrt{5}}(X, a+bi)$ be the empty shape. In particular for all $a + bi \in \mathbb{Z}[i]$:

$$\delta_{\sqrt{5}}(E_{\text{HCF}}, a+bi) = E_{\text{HCF}}.$$

We can now define a function $\text{Shape}_{\sqrt{5}}$ which gives the shape corresponding to a sequence of Gaussian integers.

Definition 68. *Let* $\text{Shape}_{\sqrt{5}} : \mathbb{Z}[i]^* \to \mathbb{S}$ *be inductively defined by:*

$$\text{Shape}_{\sqrt{5}}(\lambda) = S_{\text{HCF}}$$

and

$$\text{Shape}_{\sqrt{5}}(a_1 \dots a_n) = \delta_{\sqrt{5}}(\text{Shape}_{\sqrt{5}}(a_0 a_1 \dots a_{n-1}), a_n).$$

We have for all $a_0, a_1, \dots, a_n \in \mathbb{Z}[i]$:

$$\text{Shape}_{\sqrt{5}}(a_1 \dots a_n) \neq E_{\text{HCF}} \iff [a_0; a_1, \dots, a_{n-1}] \in \text{HCF}_{\sqrt{5}}.$$

And for all infinite sequences $a_0, a_1, \dots \in \mathbb{Z}[i]$:

$$[a_0; a_1, \dots] \in \text{HCF}_{\sqrt{5}} \iff \forall_n [a_0; a_1, \dots, a_n] \in \text{HCF}_{\sqrt{5}}.$$

5.1 Initial segments

In this section, we are going to look at initial segments of HCF.

Definition 69. *Given* $[x_0; x_1, \dots] \in \text{HCF}$*, we define* $x \upharpoonright n$ *to be the initial segment of length* $n+1$*:*

$$[x_0; x_1, \dots] \upharpoonright n := [x_0; x_1, \dots, x_n].$$

With this, we can define partial bounds on HCF.

Definition 70. *We call* $[x_0; x_1, \dots] \in \text{HCF}$ *n-bounded if* $[x_0; x_1, \dots] \upharpoonright n \in \text{HCF}_{\sqrt{5}}$.

Trivially, if x is n-bounded, then for all $m < n$ we have that x is m-bounded. Also, $x \in \text{HCF}_{\sqrt{5}}$ if and only if x is n-bounded for every n.
We can also define equivalence relations between points in HCF.

Definition 71. *We call* $x = [x_0; x_1, \dots]$ *and* $y = [y_0; y_1, \dots] \in \mathrm{HCF}$ *n-equal if and only if* $x \restriction n = y \restriction n$*, i.e., when*

$$[x_0; x_1, \dots, x_n] = [y_0; y_1, \dots, y_n].$$

We will write this as

$$x \equiv_{\restriction n} y.$$

With these definitions we are going to create rules for our simple closed curves.

Definition 72. *A* curve *(in* $\mathbb{C}$*) is a continuous function from* $[0, 1]$ *to* $\mathbb{C}$*.*

Definition 73. *A curve* f *is a simple closed curve if* $f(0) = f(1)$ *and* f *is injective on* $[0, 1)$*.*

Definition 74. *We call a curve* f n-bounded *if for every* $t \in [0, 1]$ *we have that* $f(t)$ *is* n*-bounded, when seen as an element of* $\mathrm{HCF} = \mathbb{C}$*.*

We will start by defining transition points.

Definition 75. *A point* $x \in \mathrm{HCF}_{\sqrt{5}}$ *is called an* n-transition *point if for each* ϵ *there exists* $y \in \mathbb{C}$ *with* $|y| < \epsilon$ *such that*

$$x + y \not\equiv_{\restriction n} x.$$

As we can see, n-transition points lie on the edge of the range of two different initial segments. But as with every point in $\mathbb{C}$, n-transition points have a value that has only one representation in HCF. We therefore would like a non unique representation for these transition points so we can define them to be n-equal to both adjacent constant intervals.

Definition 76. *Given* $a_0, a_1, a_2, \dots \in \mathbb{Z}[i]$*, such that for each* n*,* $[a_0; a_1, \dots, a_n] \in \mathrm{HCF}$ *we write* $\lfloor a_0; a_1, a_2, \dots \rceil$ *with*

$$\lfloor a_0; a_1, a_2, \dots \rceil := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

As intended, this representation has many similarities with continued fractions. Instead of having a construction from a complex number to a sequence of Gaussian integers, we accept limit points of finite HCF representations. Just like with continued fractions, we will also define a shorthand version.

Definition 77. *For all* n *and every* $x = \lfloor a_0; a_1, \dots \rceil$*, if* $y = \lfloor a_{n+1}; a_{n+2}, \dots \rceil$

$$\lfloor a_0; a_1, a_2, \dots, a_n : y \rceil := x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{\vdots}{a_n + \cfrac{1}{y}}}}}$$

Of course, we have that

$$\text{if } x = [a_0; a_1, \dots] \in \mathrm{HCF}, \text{ then } \lfloor a_0; a_1, \dots \rceil = x.$$

5.2 Transition points in $\mathrm{HCF}_{\sqrt{5}}$

We are going to create transition points in $\mathrm{HCF}_{\sqrt{5}}$. We start by defining four *edge* points, these are on the boundary of the range.

$$
\begin{aligned}
E_L &= \frac{2-\sqrt{7}}{2} - \frac{1}{2}i && =[0; -1+i : 1-2i + E_D] \\
&&& =[0; \overline{-1+i, 1-2i, -1+i, -2+i}]; \\
E_D &= -\frac{1}{2} + \frac{2-\sqrt{7}}{2}i && =[0; -1+i : -2+i + E_L] \\
&&& =[0; \overline{-1+i, -2+i, -1+i, 1-2i}]; \\
E_R &= \frac{\sqrt{7}-2}{2} - \frac{1}{2}i && =[0; 1+i : -2i + E_D] \\
&&& =[0; 1+i, -2i, \overline{-1+i, -2+i, -1+i, 1-2i}]; \\
E_U &= -\frac{1}{2} + \frac{\sqrt{7}-2}{2}i && =[0; -1-i : -2 + E_L] \\
&&& =[0; -1-i, -2, \overline{-1+i, 1-2i, -1+i, -2+i}].
\end{aligned}
$$

To remove the need for quotation marks, we will write $a \| b$ for the concatenation of the Gaussian integers a and b.

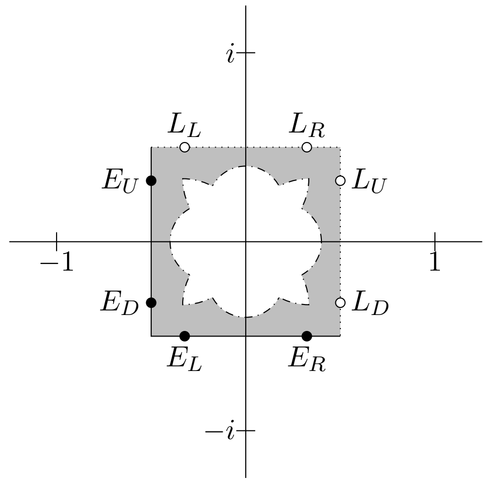
Remark 78. *We ignored edges in our* $\mathrm{Shape}_{\sqrt{5}}$ *function. So for the first coefficients of* E_L *we have:*

$$
\begin{aligned}
& \mathrm{Shape}_{\sqrt{5}}(-1+i \| 1-2i) \\
=& \, \delta_{\sqrt{5}}(\mathrm{Shape}_{\sqrt{5}}(-1+i), 1-2i) \\
=& \, \delta_{\sqrt{5}}(\rho(J_{\mathrm{HCF}}), 1-2i) \\
=& \, (\rho \circ \delta_{\sqrt{5}})(J_{\mathrm{HCF}}, -2-i) \\
=& \, E.
\end{aligned}
$$

The reason for this is that all the numbers whose $\mathrm{HCF}_{\sqrt{5}}$*-presentation start with* $[0; -1+i, 1-2i]$ *are on the line* $-\frac{1}{2} + bi$*, which has an empty internal part.*

When rotating these edge points, we find four more points. Here, we notice that the $\mathrm{HCF}_{\sqrt{5}}$ representation doesn't start with a zero, but can be described as a limit of points represented in $\mathrm{HCF}_{\sqrt{5}}$ starting with a zero. Therefore, we call these points *limit* points:

$$
\begin{aligned}
L_L &= \frac{2-\sqrt{7}}{2} + \frac{1}{2}i && = i + E_L = && [i; -1+i : 1-2i + E_D]; \\
L_D &= \frac{1}{2} + \frac{2-\sqrt{7}}{2}i && = 1 + E_D = && [1; -1+i : -2+i + E_L]; \\
L_R &= \frac{\sqrt{7}-2}{2} + \frac{1}{2}i && = i + E_R = && [i; 1+i : -2i + E_D]; \\
L_U &= \frac{1}{2} + \frac{\sqrt{7}-2}{2}i && = 1 + E_U = && [1; -1-i : -2 + E_L].
\end{aligned}
$$



For these points, we have the following rotations:

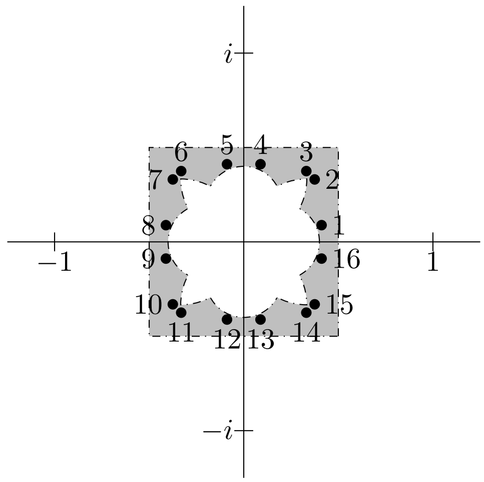
$$
\begin{aligned}
\rho(E_L) &= L_D & \rho(E_R) &= L_U \\
\rho(L_D) &= L_R & \rho(L_U) &= L_L \\
\rho(L_R) &= E_U & \rho(L_L) &= E_D \\
\rho(E_U) &= E_L & \rho(E_D) &= E_R
\end{aligned}
$$

These rotations also show other ways to describe the limit points, in our extended non-unique way. This is done by rotating per coefficient.

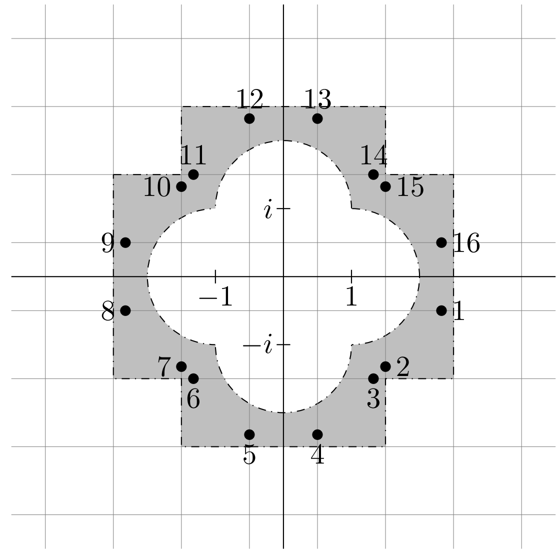
$$
\begin{aligned}
L_L &= \lfloor 0; -1-i, 2i, \overline{1-i, 2-i, 1-i, -1+2i} \rceil; \\
L_D &= \lfloor 0; 1+i, 2, \overline{1-i, -1+2i, 1-i, 2-i} \rceil; \\
L_R &= \lfloor 0; \overline{1-i, -1+2i, 1-i, 2-i} \rceil; \\
L_U &= \lfloor 0; \overline{1-i, 2-i, 1-i, -1+2i} \rceil.
\end{aligned}
$$

and we can also describe the edge points in the same way:

$$
\begin{aligned}
E_L &= \lfloor -i; -1-i, 2i, \overline{1-i, 2-i, 1-i, -1+2i} \rceil; \\
E_D &= \lfloor -1; 1+i, 2, \overline{1-i, -1+2i, 1-i, 2-i} \rceil; \\
E_R &= \lfloor -i; \overline{1-i, -1+2i, 1-i, 2-i} \rceil; \\
E_U &= \lfloor -1; \overline{1-i, 2-i, 1-i, -1+2i} \rceil.
\end{aligned}
$$



Circle points



Reciprocal

With these edge-points, we can define 16 *Circle* points:

$$
\begin{aligned}
C_1 &= [0 : 2 + E_R] && = [0 : 2 - i + L_R] && = \rho(C_{13});\\
C_2 &= [0 : 2 - i + E_D] && = [0 : 1 - i + L_D] && = \rho(C_{14});\\
C_3 &= [0 : 1 - i + E_R] && = [0 : 1 - 2i + L_R] && = \rho(C_{15});\\
C_4 &= [0 : 1 - 2i + E_D] && = [0 : -2i + L_D] && = \rho(C_{16});\\
C_5 &= [0 : -2i + E_D] && = [0 : -1 - 2i + L_D] && = \rho(C_1);\\
C_6 &= [0 : -1 - i + E_L] && = [0 : -1 - 2i + L_L] && = \rho(C_2);\\
C_7 &= [0 : -1 - i + E_D] && = [0 : -2 - i + L_D] && = \rho(C_3);\\
C_8 &= [0 : -2 + E_L] && = [0 : -2 - i + L_L] && = \rho(C_4);\\
C_9 &= [0 : -2 + i + E_L] && = [0 : -2 + L_L] && = \rho(C_5);\\
C_{10} &= [0 : -1 + i + E_U] && = [0 : -2 + i + L_U] && = \rho(C_6);\\
C_{11} &= [0 : -1 + 2i + E_L] && = [0 : -1 + i + L_L] && = \rho(C_7);\\
C_{12} &= [0 : 2i + E_U] && = [0 : -1 + 2i + L_D] && = \rho(C_8);\\
C_{13} &= [0 : 1 + 2i + E_U] && = [0 : 2i + L_U] && = \rho(C_9);\\
C_{14} &= [0 : 1 + 2i + E_R] && = [0 : 1 + i + L_L] && = \rho(C_{10});\\
C_{15} &= [0 : 2 + i + E_U] && = [0 : 1 + i + L_U] && = \rho(C_{11});\\
C_{16} &= [0 : 2 + i + E_R] && = [0 : 2 + L_R] && = \rho(C_{12}).
\end{aligned}
$$

We will make use of the fact that the following points lie in the interior of the corresponding shapes:

- $C_1, \ldots, C_{16} \in S$;
- C_1, C_2, C_{15}, and $C_{16} \in M$;
- $C_1, \ldots, C_6, C_{15}$, and $C_{16} \in W$;

- $C_1, \ldots, C_4 \in J$.

With these, we can describe the edge points in terms of circle points:

$$E_L = [0 : -1 + i + C_4] \qquad E_R = [0 : 1 + i + C_5]$$
$$E_U = [0 : -1 - i + C_8] \qquad E_D = [0 : -1 + i + C_9]$$

We can use the rotation rules to describe the limit points in terms of circle points.

$$L_L = \lfloor 0 : -1 - i + C_{13} \rceil \qquad L_R = \lfloor 0 : 1 - i + C_{12} \rceil$$
$$L_U = \lfloor 0 : 1 - i + C_1 \rceil \qquad L_D = \lfloor 0 : 1 + i + C_{16} \rceil$$

Example 79. We start with $[c_0; c_1, \ldots, c_n] \in \mathrm{HCF}_{\sqrt{5}}$ with $\mathrm{Shape}_{\sqrt{5}}(c_0 c_1 \ldots c_n) = \rho(M_{\mathrm{HCF}})$.
We know C_5 lies in $\rho(M_{\mathrm{HCF}})$, for C_1 lies in M_{HCF}, and $\rho(C_1) = C_5$.
We therefore have two ways to describe this point, as an edge point starting with $[c_0; c_1, \ldots, c_n, -2i]$ or as a limit point starting with $[c_0; c_1, \ldots, c_n, -1 - 2i]$. Both will have one way to describe them as limit of finite sequences in $\mathrm{HCF}_{\sqrt{5}}$, as they lie on the edge.

$$
\begin{aligned}
&[c_0; c_1, \ldots : c_n + C_5] \in \mathrm{HCF}_{\sqrt{5}} \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n) = \rho(M_{\mathrm{HCF}}) \\
=&[c_0; c_1, \ldots, c_n : -2i + E_D] \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i) = \rho^3(M_{\mathrm{HCF}}) \\
=&[c_0; c_1, \ldots, c_n, -2i : -1 + i + C_9] \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i \| -1 + i) = \rho(J_{\mathrm{HCF}}) \\
=&[c_0; c_1, \ldots, c_n, -2i, -1 + i : -2 + i + E_L] \\
=&[c_0; c_1, \ldots, c_n, -2i, -1 + i : -2 + L_L] \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i \| -1 + i \| -2) = \rho^2(M_{\mathrm{HCF}}) \\
=&\lfloor c_0; c_1, \ldots, c_n, -2i, -1 + i, -2 + i : -1 - i + C_{13} \rceil \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i \| -1 + i \| -2 \| -1 - i) = \rho^2(J_{\mathrm{HCF}}) \\
=&\lfloor c_0; c_1, \ldots, c_n, -2i, -1 + i, -2 + i, -1 - i : 1 + 2i + E_U \rceil \\
=&\lfloor c_0; c_1, \ldots, c_n, -2i, -1 + i, -2 + i, -1 - i : 2i + L_U \rceil \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i \| -1 + i \| -2 \| -1 - i \| 2i) = \rho(M_{\mathrm{HCF}}) \\
=&\lfloor c_0; c_1, \ldots, c_n, -2i, -1 + i, -2 + i, -1 - i, 2i : 1 - i + C_1 \rceil \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i \| -1 + i \| -2 \| -1 - i \| 2i \| 1 - i) = \rho^3(J_{\mathrm{HCF}}) \\
=&\lfloor c_0; c_1, \ldots, c_n, -2i, -1 + i, -2 + i, -1 - i, 2i, 1 - i : 2 + E_R \rceil \\
&\quad \text{with } \mathrm{Shape}_{\sqrt{5}}(c_1 \ldots c_n \| -2i \| -1 + i \| -2 \| -1 - i \| 2i \| 1 - i \| 2) = M_{\mathrm{HCF}} \\
=&\lfloor c_0; c_1, \ldots, c_n, -2i, -1 + i, -2 + i, -1 - i, 2i, 1 - i, 2 : 1 + i + C_5 \rceil
\end{aligned}
$$

$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -2i \| -1+i \| -2 \| -1-i \| 2i \| 1-i \| 2 \| 1+i) = J_{\text{HCF}}$$
$$= \lfloor c_0; c_1, \dots, c_n, -2i, -1+i, -2+i, -1-i, 2i, 1-i, 2, 1+i : -2i + E_D \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -2i \| -1+i \| -2 \| -1-i \| 2i \| 1-i \| 2 \| 1+i \| -2i) = \rho^3(M_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, \overline{-2i, -1+i, -2+i, -1-i, 2i, 1-i, 2, 1+i} \rceil$$

and:

$$[c_0; c_1, \dots : c_n + C_5] \in \text{HCF}_{\sqrt{5}}$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n) = \rho(M_{\text{HCF}})$$
$$= [c_0; c_1, \dots, c_n : -1-2i + L_D]$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i) = \rho^2(W_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i : 1+i + C_{16} \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i) = J_{\text{HCF}}$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i : 2+i + E_R \rceil$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i : 2 + L_R \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2) = M_{\text{HCF}}$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2 : 1-i + C_{12} \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i) = \rho^3(J_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i : 2i + E_U \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i \| 2i) = \rho(M_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i, 2i : -1-i + C_8 \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i \| 2i \| -1-i) = \rho^2(J_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i, 2i, -1-i : -2 + E_L \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i \| 2i \| -1-i \| -2) = \rho^2(M_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i, 2i, -1-i, -2 : -1+i + C_4 \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i \| 2i \| -1-i \| -2 \| -1+i) = \rho(J_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i, 2i, -1-i, -2, -1+i : 1-2i + E_D \rceil$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i, 2i, -1-i, -2, -1+i : -2i + L_D \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i \| 2i \| -1-i \| -2 \| -1+i \| -2i) = \rho^3(M_{\text{HCF}})$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, 1+i, 2, 1-i, 2i, -1-i, -2, -1+i, -2i : 1+i + C_{16} \rceil$$
$$\text{with Shape}_{\sqrt{5}}(c_1 \dots c_n \| -1-2i \| 1+i \| 2 \| 1-i \| 2i \| -1-i \| -2 \| -1+i \| -2i \| 1+i) = J_{\text{HCF}}$$
$$= \lfloor c_0; c_1, \dots, c_n, -1-2i, \overline{1+i, 2, 1-i, 2i, -1-i, -2, -1+i, -2i} \rceil$$

5.3 Construction of building blocks

Definition 80. *We call a quadruple of*

- *a base* $[c_0; c_1, c_2, \ldots, c_n] \in \mathrm{HCF} \cap \mathbb{Q}$*;*
- *an interval* $[p, q]$ *with* $p, q \in \mathbb{R}$*;*
- *a starting point* $T_p \in \mathbb{C}$*;*
- *an endpoint* $T_q \in \mathbb{C}$*,*

with $\lfloor c_0; c_1, c_2, \ldots, c_{n-1} : c_n + T_p \rceil$ *and* $\lfloor c_0; c_1, c_2, \ldots, c_{n-1} : c_n + T_q \rceil$ *n-transition points, a* building block*, written as* $\langle [c_0; c_1, c_2, \ldots, c_n], [p, q], T_p, T_q \rangle$*.*

Definition 81. *The* type *of a building block* $\langle [c_0; c_1, c_2, \ldots, c_n], [p, q], T_p, T_q \rangle$ *is:*

$$\ll \mathrm{Shape}_{\sqrt{5}}(c_0 c_1 \ldots c_n), T_p, T_q \gg$$

We will describe six types of building blocks, These types can be rotated and reversed, for a total of 48 subtypes.

- $\ll M_{\mathrm{HCF}}, L_R, E_R \gg$;
- $\ll W_{\mathrm{HCF}}, E_U, E_R \gg$;
- $\ll J_{\mathrm{HCF}}, L_R, L_U \gg$;
- $\ll J_{\mathrm{HCF}}, C_5, C_{16} \gg$;
- $\ll J_{\mathrm{HCF}}, C_5, L_U \gg$;
- $\ll J_{\mathrm{HCF}}, L_R, C_{16} \gg$.

The aforementioned rotations and reversion are defined as follows.

Definition 82. *The reverse of a building block type* $\ll X, T_p, T_q \gg$ *is defined as:*

$$\ll X, T_p, T_q \gg^{-1} = \ll X, T_q, T_p \gg .$$

Definition 83. *The rotation of a building block type* $\ll X, T_p, T_q \gg$ *is defined as:*

$$\rho(\ll X, T_p, T_q \gg) = \ll \rho(X), \rho(T_q), \rho(T_p) \gg .$$

It is easy to see that

- $(\ll X, T_p, T_q \gg^{-1})^{-1} = \ll X, T_p, T_q \gg$;
- $\rho^4(\ll X, T_p, T_q \gg) = \ll X, T_p, T_q \gg$;
- $(\rho(\ll X, T_p, T_q \gg))^{-1} = \rho(\ll X, T_p, T_q \gg^{-1})$.

Definition 84. *We call a sequence of building blocks $[X_0, X_1, \dots, X_m]$ (where each X_g is of the form $\langle [c_0^g; c_1^g, \dots, c_{n^g}^g], [p^g, q^g], T_p^g, T_q^g)$ a* chain *of depth n if:*

1. *for all $g \leq m$: $n^g = n$;*
2. *for all $g, h \geq m$: if $g \neq h$, then $[c_0^g; c_1^g, \dots, c_{n^g}^g] \neq [c_0^h; c_1^h, \dots : c_{n^h}^h]$;*
3. *for all $g < m$: $q^g = p^{g+1}$;*
4. *for all $g < m$: $\lfloor c_0^g; c_1^g, \dots : c_{n^g}^g + T_q^g \rceil = \lfloor c_0^{g+1}; c_1^{g+1}, \dots : c_{n^{g+1}}^{g+1} + T_p^{g+1} \rceil$.*

Definition 85. *We define the function* curve *from a building block to a function $(p, q) \to \mathbb{C}$ as*

$$\mathrm{curve}(\langle [c_0; c_1, c_2, \dots, c_n], [p, q], T_p, T_q \rangle)(t) = [c_0; c_1, c_2, \dots, c_{n-1} : c_n + (\frac{q-t}{q-p} T_p + \frac{t-p}{q-p} T_q)].$$

Definition 86. *There exists a function* mesh *from a building block $X = \langle [c_0; c_1, \dots, c_n], [p, q], T_p, T_q \rangle$ of a described type to a finite sequence of building blocks $[Y_0, \dots, Y_m]$ (with each Y_g of the form $\langle [c_0^g; c_1^g, \dots, c_{n^g}^g], [p^g, q^g], T_p^g, T_q^g \rangle$) such that*

- *$[Y_0, \dots, Y_m]$ is a chain of depth $n + 1$;*
- *For each $g \leq m$ and $k \leq n$ we have $c_k^g = c_g$;*
- *$p^0 = p$ and $q^m = q$;*
- *For each $g < m$ there exists $c \in \{1, \dots, 16\}$ such that*

  $$[c_0; c_1, \dots, c_n : C_c] = \lfloor c_0; c_1, \dots, c_n : c_{n+1}^g + T_q^g \rceil (= \lfloor c_0; c_1, \dots, c_n : c_{n+1}^{g+1} + T_p^{g+1} \rceil),$$

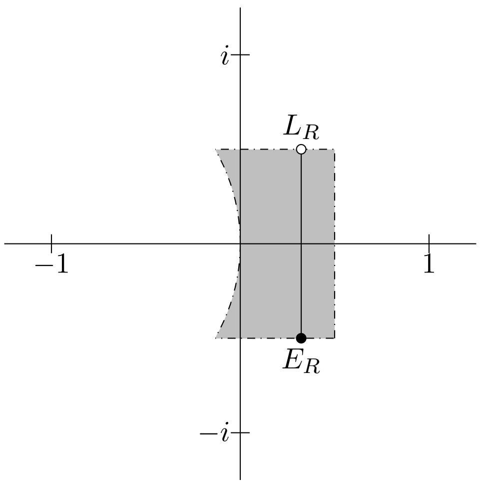
  where C_c is one of the Circle points.
- *$\lfloor c_0; c_1, \dots, c_n : c_{n+1}^0 + T_p^0 \rceil = \lfloor c_0; c_1, \dots : c_n + T_p \rceil$;*
- *$\lfloor c_0; c_1, \dots, c_n : c_{n+1}^m + T_q^m \rceil = \lfloor c_0; c_1, \dots : c_n + T_q \rceil$.*

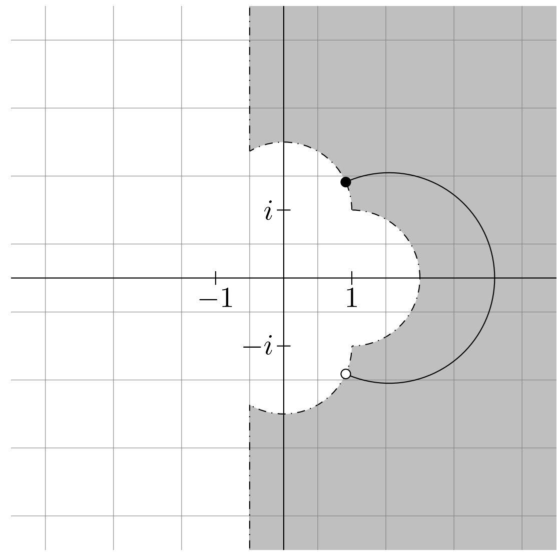
For each of the types of building blocks, we will show curve is well defined, and give the definition for the function mesh for that specific type.

First type: $\ll M_{\text{HCF}}, L_R, E_R \gg$

Given building block $B = \langle [c_0; c_1, c_2, \ldots, c_n], [p, q], L_R, E_R \rangle$ of the type $\ll M_{\text{HCF}}, L_R, E_R \gg$, then curve$(B)$ is defined as

$$\text{curve}(B)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} L_R + \frac{t-p}{q-p} E_R)].$$





By definition $\lfloor c_0; c_1, \ldots : c_n + L_R \rceil$ and $\lfloor c_0; c_1, \ldots : c_n + E_R \rceil$ are transition points. Notice that

$$\lfloor c_0; c_1, \ldots : c_n + L_R \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 - i + C_{12} \rceil$$

and

$$\lfloor c_0; c_1, \ldots : c_n + E_R \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 + i + C_5 \rceil.$$

As C_2, C_1, C_{16} and C_{15} are elements of M_{HCF}, the following points are $(n+1)$-transition points:

$$\begin{aligned}
[c_0; c_1, \ldots : c_n + C_2] &= [c_0; c_1, \ldots, c_n : 1 - i + L_D] &&= [c_0; c_1, \ldots, c_n : 2 - i + E_D] \\
[c_0; c_1, \ldots : c_n + C_1] &= [c_0; c_1, \ldots, c_n : 2 - i + L_R] &&= [c_0; c_1, \ldots, c_n : 2 + E_R] \\
[c_0; c_1, \ldots : c_n + C_{16}] &= [c_0; c_1, \ldots, c_n : 2 + L_R] &&= [c_0; c_1, \ldots, c_n : 2 + i + E_R] \\
[c_0; c_1, \ldots : c_n + C_{15}] &= [c_0; c_1, \ldots, c_n : 2 + i + E_U] &&= [c_0; c_1, \ldots, c_n : 1 + i + L_U]
\end{aligned}$$

Let us define mesh(B) when B has type $\ll M_{\text{HCF}}, L_R, E_R \gg$ as

$$[\langle [c_0; c_1, \ldots, c_n, 1 - i], [p, \frac{4p+q}{5}], C_{12}, L_D \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 2 - i], [\frac{4p+q}{5}, \frac{3p+2q}{5}], E_D, L_R \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 2], [\frac{3p+2q}{5}, \frac{2p+3q}{5}], E_R, L_R \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 2+i], [\frac{2p+3q}{5}, \frac{p+4q}{5}], E_R, E_U \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 1+i], [\frac{p+4q}{5}, q], L_U, C_5 \rangle].$$

With the types:

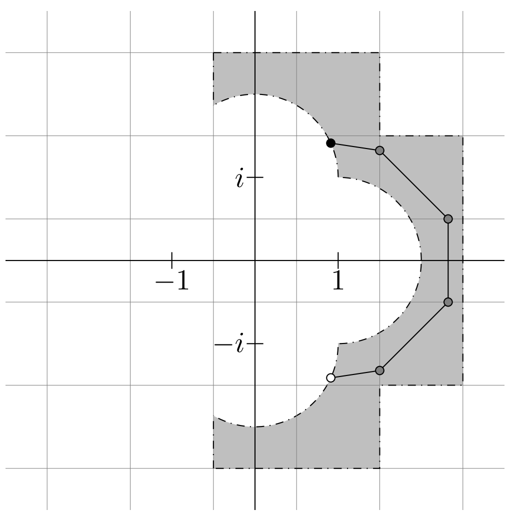
$$[\rho^3(\ll J_{\mathrm{HCF}}, L_R, C_{16} \gg)^{-1},$$

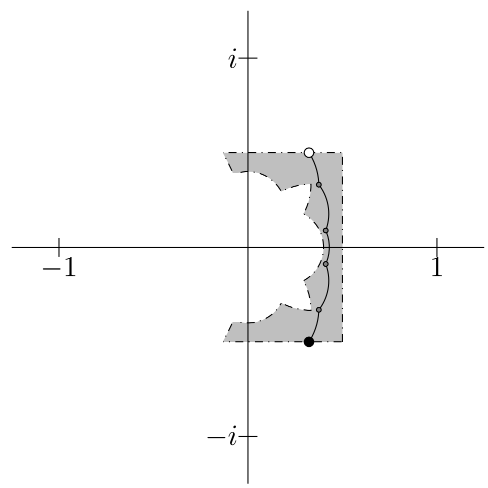
$$\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1},$$

$$\ll M_{\mathrm{HCF}}, L_R, E_R \gg^{-1},$$

$$\ll W_{\mathrm{HCF}}, E_U, E_R \gg^{-1},$$

$$\ll J_{\mathrm{HCF}}, C_5, L_U \gg^{-1}].$$





Second type: $\ll W_{\mathrm{HCF}}, E_U, E_R \gg$

Given building block $B = \langle [c_0; c_1, c_2, \ldots, c_n], [p, q], L_R, E_R \rangle$ of the type $\ll W_{\mathrm{HCF}}, E_U, E_R \gg$, then curve$(B)$ is defined as

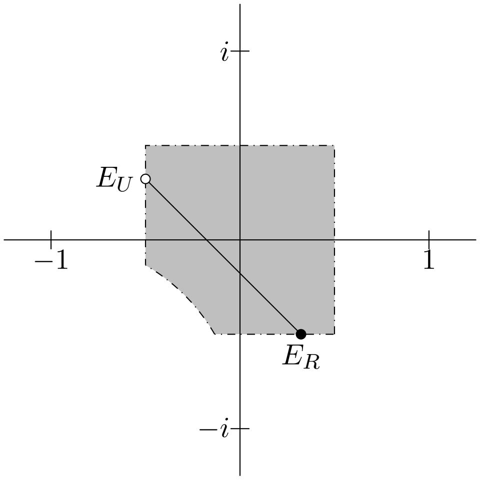
$$\mathrm{curve}(B)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} E_U + \frac{t-p}{q-p} E_R)].$$

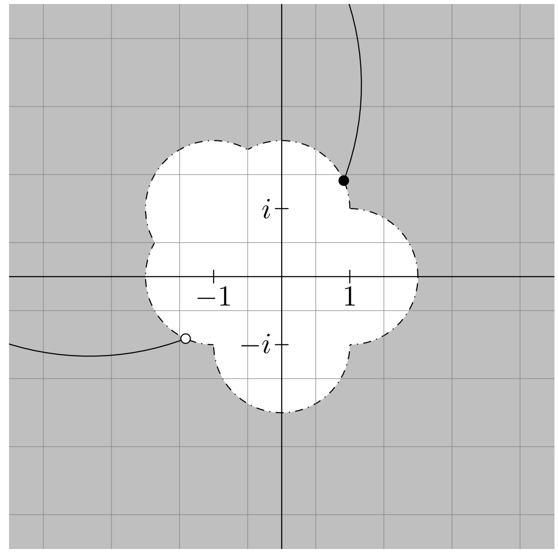
By definition $\lfloor c_0; c_1, \ldots : c_n + E_U \rceil$ and $\lfloor c_0; c_1, \ldots : c_n + E_R \rceil$ are transition points. Notice that

$$\lfloor c_0; c_1, \ldots : c_n + L_R \rceil = \lfloor c_0; c_1, \ldots, c_n : -1 - i + C_8 \rceil$$

and

$$\lfloor c_0; c_1, \ldots : c_n + E_R \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 + i + C_5 \rceil.$$





As $C_6, C_5, C_4, C_3, C_2, C_1, C_{16}$ and C_{15} are elements of W_{HCF}, the following points are $(n+1)$-transition points:

$$
\begin{aligned}
[c_0; c_1, \ldots : c_n + C_6] &= [c_0; c_1, \ldots, c_n : -1 - i + E_L] = [c_0; c_1, \ldots, c_n : -1 - 2i + L_L] \\
[c_0; c_1, \ldots : c_n + C_5] &= [c_0; c_1, \ldots, c_n : -1 - 2i + L_D] = [c_0; c_1, \ldots, c_n : -2i + E_D] \\
[c_0; c_1, \ldots : c_n + C_4] &= [c_0; c_1, \ldots, c_n : -2i + L_D] = [c_0; c_1, \ldots, c_n : 1 - 2i + E_D] \\
[c_0; c_1, \ldots : c_n + C_3] &= [c_0; c_1, \ldots, c_n : 1 - 2i + L_R] = [c_0; c_1, \ldots, c_n : 1 - i + E_R] \\
[c_0; c_1, \ldots : c_n + C_2] &= [c_0; c_1, \ldots, c_n : 1 - i + L_D] = [c_0; c_1, \ldots, c_n : 2 - i + E_D] \\
[c_0; c_1, \ldots : c_n + C_1] &= [c_0; c_1, \ldots, c_n : 2 - i + L_R] = [c_0; c_1, \ldots, c_n : 2 + E_R] \\
[c_0; c_1, \ldots : c_n + C_{16}] &= [c_0; c_1, \ldots, c_n : 2 + L_R] = [c_0; c_1, \ldots, c_n : 2 + i + E_R] \\
[c_0; c_1, \ldots : c_n + C_{15}] &= [c_0; c_1, \ldots, c_n : 2 + i + E_U] = [c_0; c_1, \ldots, c_n : 1 + i + L_U]
\end{aligned}
$$

Let us define mesh(B) when B has type $\ll W_{\text{HCF}}, E_U, E_R \gg$ as

$$
[\langle [c_0; c_1, \ldots, c_n, -1 - i], [p, \frac{8p+q}{9}], C_8, E_L \rangle,
$$

$$
\langle [c_0; c_1, \ldots, c_n, 1 - 2i], [\frac{8p+q}{9}, \frac{7p+2q}{9}], L_L, L_D \rangle,
$$

$$
\langle [c_0; c_1, \ldots, c_n, -2i], [\frac{7p+2q}{9}, \frac{6p+3q}{9}], E_D, L_D \rangle,
$$

$$
\langle [c_0; c_1, \ldots, c_n, 1 - 2i], [\frac{6p+3q}{9}, \frac{5p+4q}{9}], E_D, L_R \rangle,
$$

$$
\langle [c_0; c_1, \ldots, c_n, 1 - i], [\frac{5p+4q}{9}, \frac{4p+5q}{9}], E_R, L_D \rangle,
$$

$$
\langle [c_0; c_1, \ldots, c_n, 2 - i], [\frac{4p+5q}{9}, \frac{3p+6q}{9}], E_D, L_R \rangle,
$$

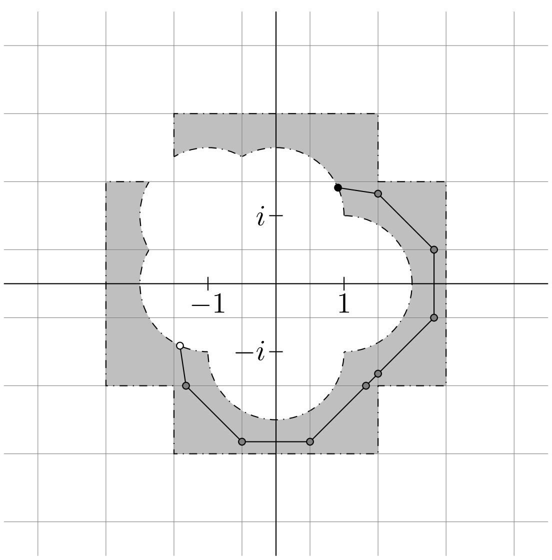
$$
\langle [c_0; c_1, \ldots, c_n, 2], [\frac{3p+6q}{9}, \frac{2p+7q}{9}], E_R, L_R \rangle,
$$

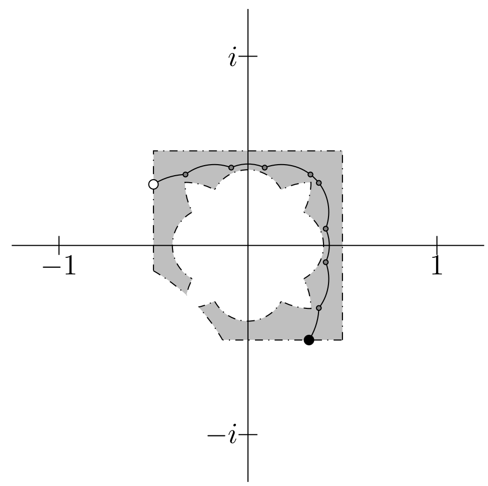
$$\langle [c_0; c_1, \ldots, c_n, 2+i], [\frac{2p+7q}{9}, \frac{p+8q}{9}], E_R, E_U \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 1+i], [\frac{p+8q}{9}, q] L_U, C_5 \rangle].$$

With the types:

$$[\rho^2(\ll J_{\mathrm{HCF}}, L_R, C_{16} \gg)^{-1},$$
$$\rho^2(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1},$$
$$\rho^3(\ll M_{\mathrm{HCF}}, L_R, E_R \gg)^{-1},$$
$$\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1},$$
$$\rho^3(\ll J_{\mathrm{HCF}}, L_R, L_U \gg)^{-1},$$
$$\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1},$$
$$\ll M_{\mathrm{HCF}}, L_R, E_R \gg^{-1},$$
$$\ll W_{\mathrm{HCF}}, E_U, E_R \gg^{-1},$$
$$\ll J_{\mathrm{HCF}}, C_5, L_U \gg^{-1}].$$

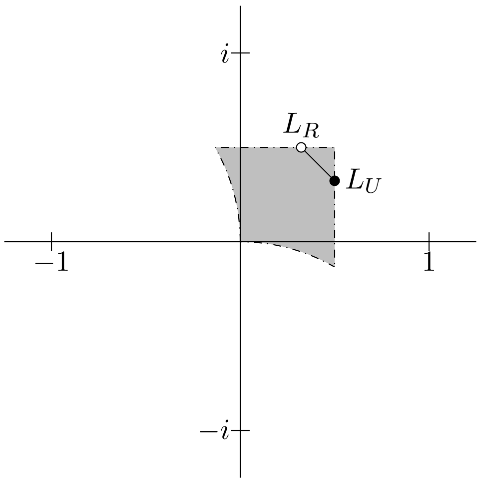


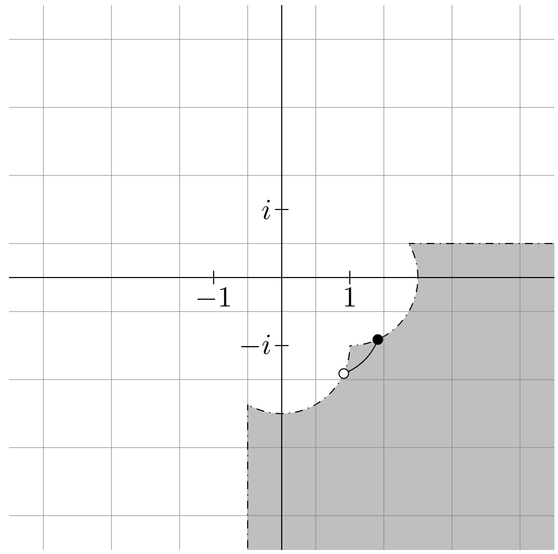


Third type: $\ll J_{\text{HCF}}, L_R, L_U \gg$

Given building block $B = \langle [c_0; c_1, c_2, \ldots, c_n], [p, q], L_R, E_R \rangle$ of the type $\ll J_{\text{HCF}}, L_R, L_U \gg$, then curve$(B)$ is defined as

$$\text{curve}(B)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} L_R + \frac{t-p}{q-p} L_U)].$$





By definition $\lfloor c_0; c_1, \ldots : c_n + L_R \rceil$ and $\lfloor c_0; c_1, \ldots : c_n + L_U \rceil$ are transition points. Notice that

$$\lfloor c_0; c_1, \ldots : c_n + L_R \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 - i + C_{12} \rceil$$

and

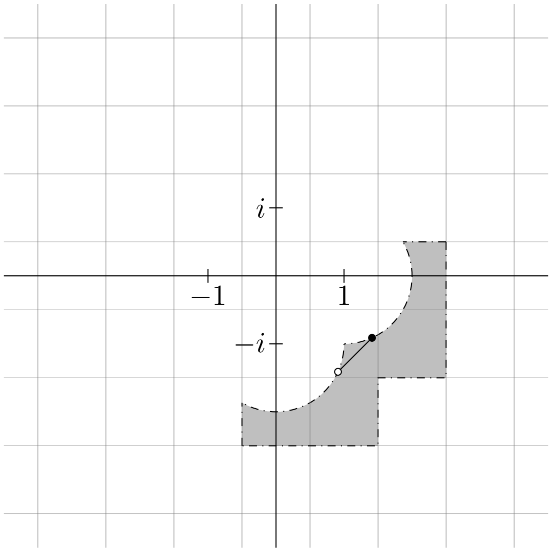
$$\lfloor c_0; c_1, \ldots : c_n + L_U \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 + i + C_1 \rceil.$$

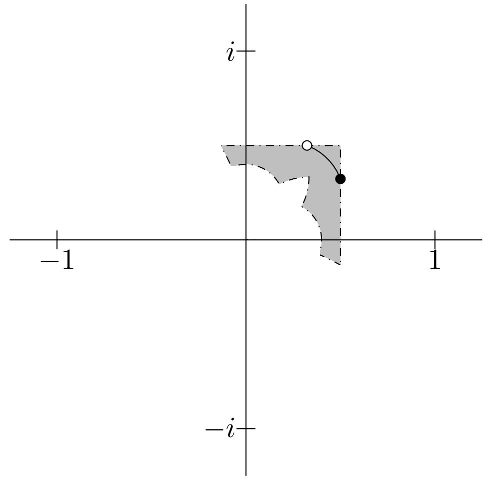
Let us define mesh(B) when B has type $\ll J_{\text{HCF}}, L_R, L_U \gg$ as

$$[\langle [c_0; c_1, \ldots, c_n, 1 - i], [p, q], C_{12}, C1 \rangle]$$

With the type:

$$[\rho^3 (\ll J_{\text{HCF}}, C_5, C_{16} \gg)^{-1}]$$

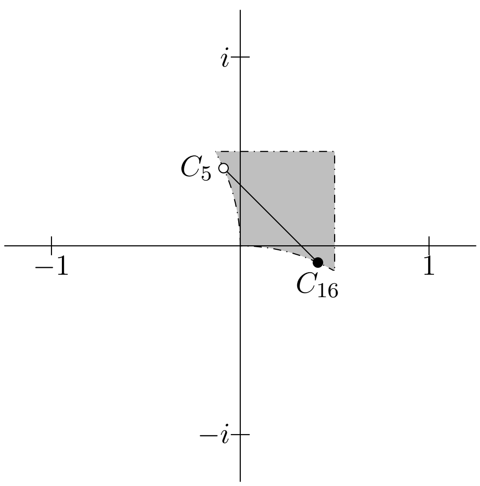


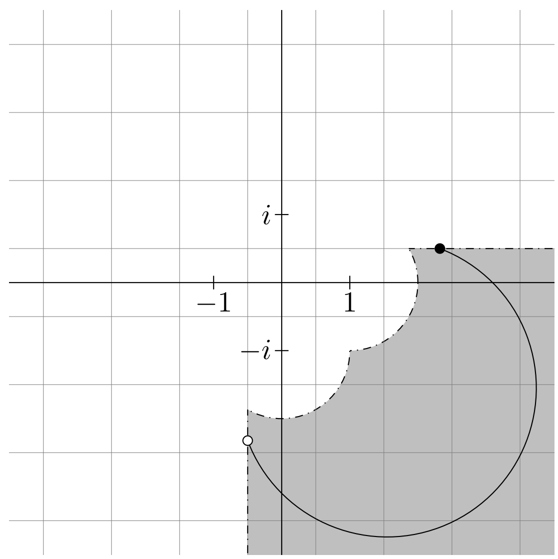


Fourth type: $\ll J_{\mathrm{HCF}}, C_5, C_{16} \gg$

Given building block $B = \langle [c_0; c_1, c_2, \ldots, c_n], [p, q], L_R, E_R \rangle$ of the type $\ll J_{\mathrm{HCF}}, C_5, C_{16} \gg$, then curve($B$) is defined as

$$\mathrm{curve}(B)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} C_5 + \frac{t-p}{q-p} C_{16})].$$





By definition $\lfloor c_0; c_1, \ldots : c_n + C_5 \rceil$ and $\lfloor c_0; c_1, \ldots : c_n + C_{16} \rceil$ are transition points. Notice that

$$\lfloor c_0; c_1, \ldots : c_n + C_5 \rceil = \lfloor c_0; c_1, \ldots, c_n : -2i + E_D \rceil$$

and

$$\lfloor c_0; c_1, \ldots : c_n + C_{16} \rceil = \lfloor c_0; c_1, \ldots, c_n : 2 + L_R \rceil.$$

As C_4, C_3, C_2 and C_1 are elements of J_{HCF}, the following points are $(n+1)$-transition points:

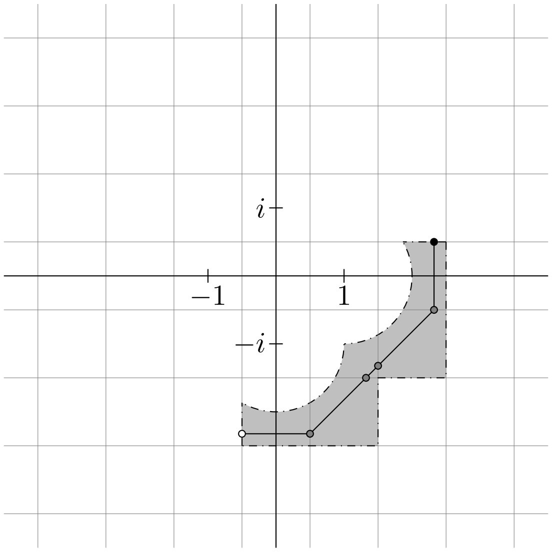
$$\begin{aligned}
[c_0; c_1, \ldots : c_n + C_4] &= [c_0; c_1, \ldots, c_n : -2i + L_D] = [c_0; c_1, \ldots, c_n : 1 - 2i + E_D] \\
[c_0; c_1, \ldots : c_n + C_3] &= [c_0; c_1, \ldots, c_n : 1 - 2i + L_R] = [c_0; c_1, \ldots, c_n : 1 - i + E_R] \\
[c_0; c_1, \ldots : c_n + C_2] &= [c_0; c_1, \ldots, c_n : 1 - i + L_D] = [c_0; c_1, \ldots, c_n : 2 - i + E_D] \\
[c_0; c_1, \ldots : c_n + C_1] &= [c_0; c_1, \ldots, c_n : 2 - i + L_R] = [c_0; c_1, \ldots, c_n : 2 + E_R]
\end{aligned}$$

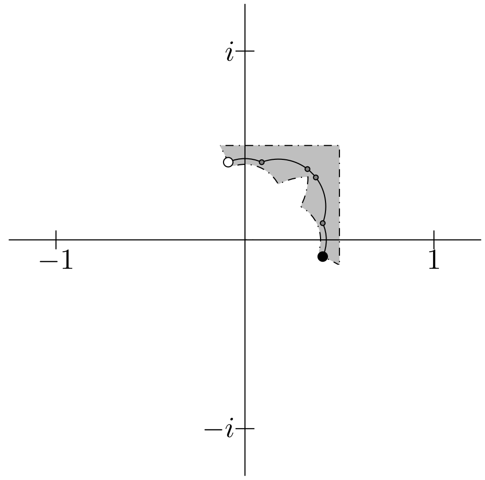
Let us define mesh(B) when B has type $\ll J_{\mathrm{HCF}}, C_5, C_{16} \gg$ as

$$\begin{gathered}
[\langle [c_0; c_1, \ldots, c_n, -2i], [p, \frac{4p+q}{5}], E_D, L_D \rangle, \\
\langle [c_0; c_1, \ldots, c_n, 1-2i], [\frac{4p+q}{5}, \frac{3p+2q}{5}], E_D, L_R \rangle, \\
\langle [c_0; c_1, \ldots, c_n, 1-i], [\frac{3p+2q}{5}, \frac{2p+3q}{5}], E_R, L_D \rangle, \\
\langle [c_0; c_1, \ldots, c_n, 2-i], [\frac{2p+3q}{5}, \frac{p+4q}{5}], E_D, L_R \rangle, \\
\langle [c_0; c_1, \ldots, c_n, 2], [\frac{p+4q}{5}, q], E_R, L_R \rangle].
\end{gathered}$$

With the types:

$$\begin{gathered}
[\rho^3(\ll M_{\mathrm{HCF}}, L_R, E_R \gg)^{-1}, \\
\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1}, \\
\rho^3(\ll J_{\mathrm{HCF}}, L_R, L_U \gg)^{-1}, \\
\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1}, \\
\ll M_{\mathrm{HCF}}, L_R, E_R \gg^{-1}].
\end{gathered}$$

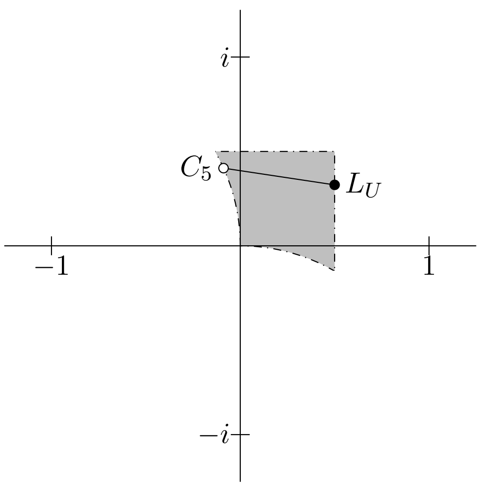


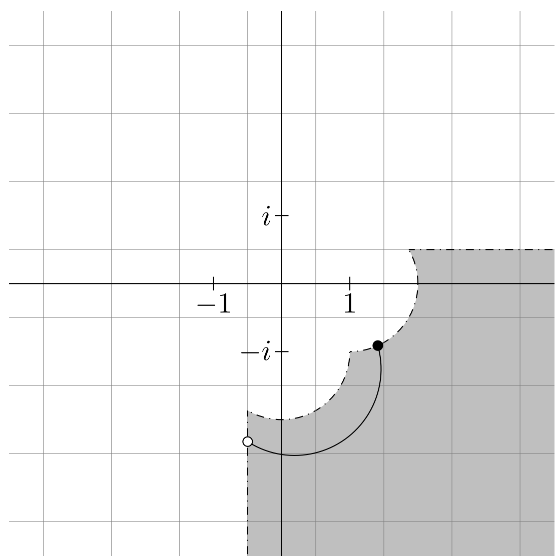


Fifth type: $\ll J_{\mathrm{HCF}}, C_5, L_U \gg$

Given building block $B = \langle [c_0; c_1, c_2, \ldots, c_n], [p, q], L_R, E_R \rangle$ of the type $\ll J_{\mathrm{HCF}}, C_5, L_U \gg$, then curve$(B)$ is defined as

$$\mathrm{curve}(B)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} C_5 + \frac{t-p}{q-p} L_U)].$$





By definition $\lfloor c_0; c_1, \ldots : c_n + C_5 \rceil$ and $\lfloor c_0; c_1, \ldots : c_n + L_U \rceil$ are transition points. Notice that

$$\lfloor c_0; c_1, \ldots : c_n + C_5 \rceil = \lfloor c_0; c_1, \ldots, c_n : -2i + E_D \rceil$$

and

$$\lfloor c_0; c_1, \ldots : c_n + L_U \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 - i + C_1 \rceil.$$

As C_4 and C_3 are elements of J_{HCF}, the following points are $(n+1)$-transition points:

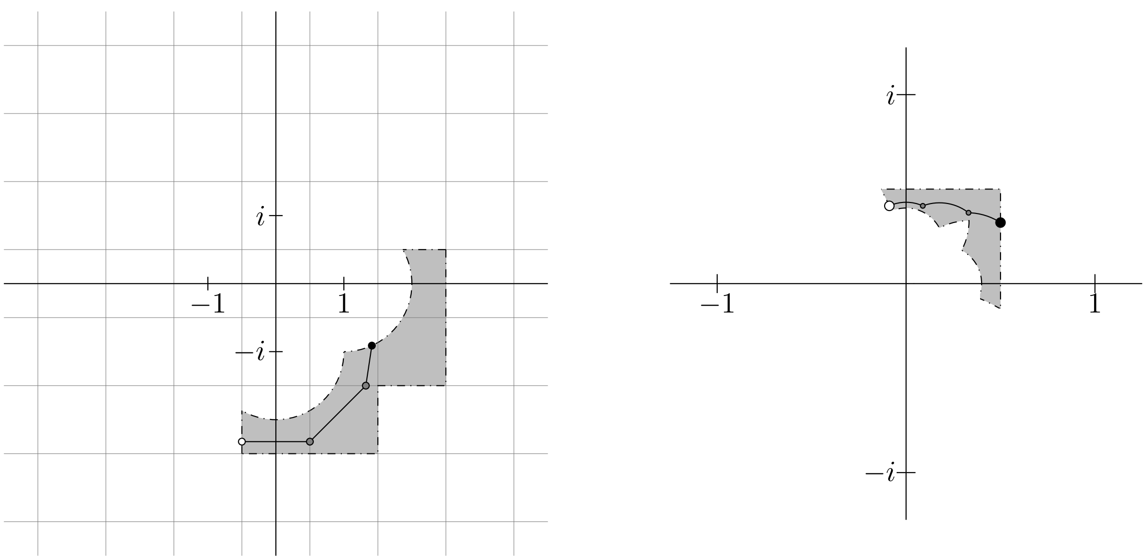
$$[c_0; c_1, \ldots : c_n + C_4] = [c_0; c_1, \ldots, c_n : -2i + L_D] = [c_0; c_1, \ldots, c_n : 1 - 2i + E_D]$$
$$[c_0; c_1, \ldots : c_n + C_3] = [c_0; c_1, \ldots, c_n : 1 - 2i + L_R] = [c_0; c_1, \ldots, c_n : 1 - i + E_R]$$

Let us define mesh(B) when B has type $\ll J_{\mathrm{HCF}}, C_5, L_U \gg$ as

$$[\langle [c_0; c_1, \ldots, c_n, -2i], [p, \frac{2p+q}{3}], E_D, L_D \rangle,$$
$$\langle [c_0; c_1, \ldots, c_n, 1-2i], [\frac{2p+q}{3}, \frac{p+2q}{3}], E_D, L_R \rangle,$$
$$\langle [c_0; c_1, \ldots, c_n, 1-i], [\frac{p+2q}{3}, q], E_R, C_1 \rangle]$$

With the types:

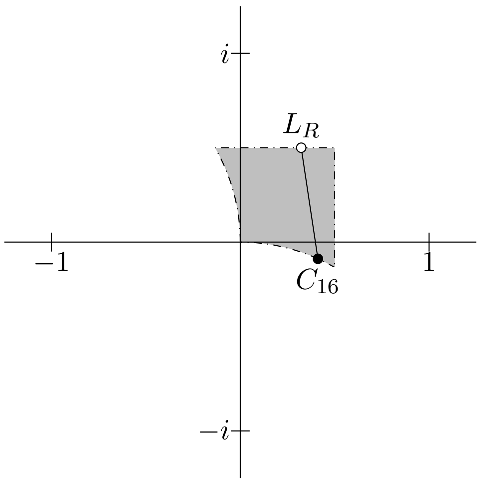
$$[\rho^3(\ll M_{\mathrm{HCF}}, L_R, E_R \gg)^{-1},$$
$$\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1},$$
$$\rho^3(\ll J_{\mathrm{HCF}}, C_5, L_U \gg)^{-1}]$$

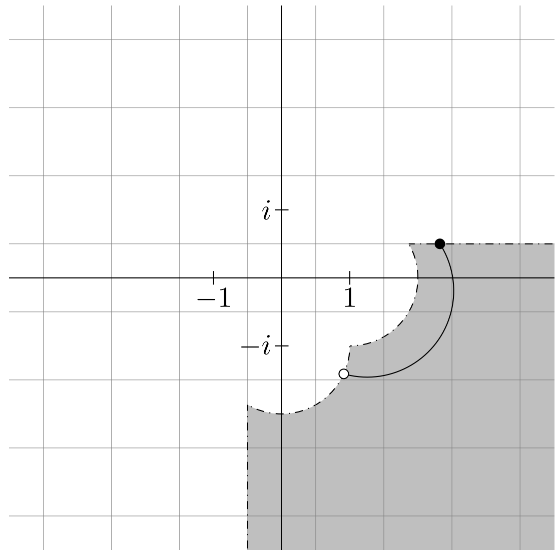


Sixth type: $\ll J_{\mathrm{HCF}}, L_R, C_{16} \gg$

Given building block $B = \langle [c_0; c_1, c_2, \ldots, c_n], [p, q], L_R, E_R \rangle$ of the type $\ll J_{\mathrm{HCF}}, L_R, C_{16} \gg$, then curve$(B)$ is defined as

$$\text{curve}(B)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} L_R + \frac{t-p}{q-p} C_{16})].$$





By definition $\lfloor c_0; c_1, \ldots : c_n + L_R \rceil$ and $\lfloor c_0; c_1, \ldots : c_n + C_{16} \rceil$ are transition points. Notice that

$$\lfloor c_0; c_1, \ldots : c_n + L_R \rceil = \lfloor c_0; c_1, \ldots, c_n : 1 - i + C_{12} \rceil$$

and

$$\lfloor c_0; c_1, \ldots : c_n + C_{16} \rceil = \lfloor c_0; c_1, \ldots, c_n : 2 + L_R \rceil.$$

As C_2 and $C1$ are elements of J_{HCF}, the following points are $(n+1)$-transition points:

$$[c_0; c_1, \ldots : c_n + C_2] = [c_0; c_1, \ldots, c_n : 1 - i + L_D] = [c_0; c_1, \ldots, c_n : 2 - i + E_D]$$
$$[c_0; c_1, \ldots : c_n + C_1] = [c_0; c_1, \ldots, c_n : 2 - i + L_R] = [c_0; c_1, \ldots, c_n : 2 + E_R]$$

Let us define mesh(B) when B has type $\ll J_{\mathrm{HCF}}, L_R C_{16} \gg$ as

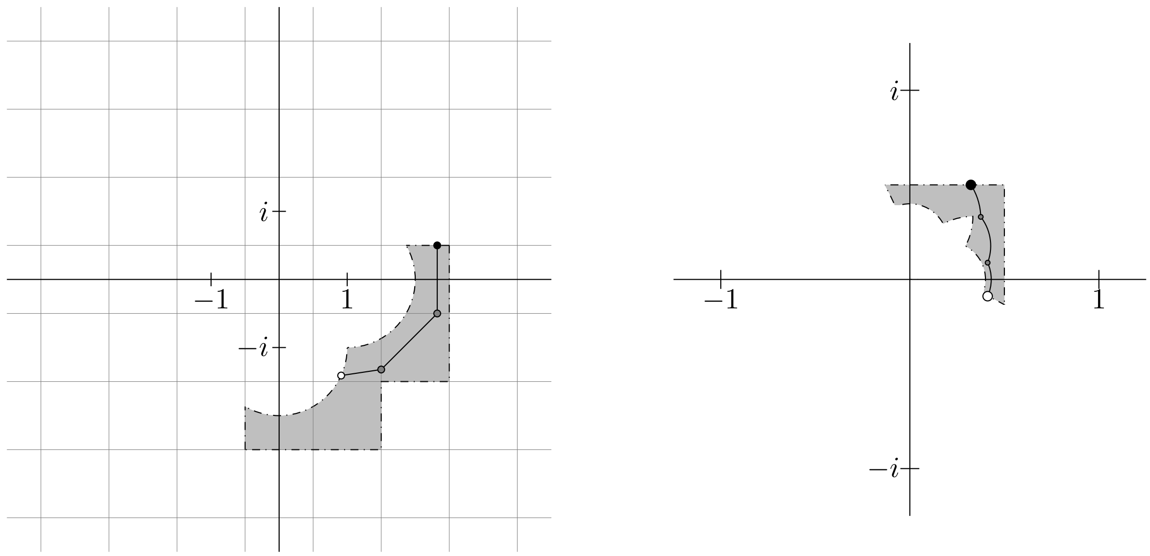
$$[\langle [c_0; c_1, \ldots, c_n, 1-i], [p, \frac{2p+q}{3}], C_{12}, L_D \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 2-i], [\frac{2p+q}{3}, \frac{p+2q}{3}], E_D, L_R \rangle,$$

$$\langle [c_0; c_1, \ldots, c_n, 2], [\frac{p+2q}{3}, q], E_R, L_R \rangle].$$

With the types:

$$[\rho^3(\ll J_{\mathrm{HCF}}, L_R, C_{16} \gg)^{-1},$$
$$\rho^3(\ll W_{\mathrm{HCF}}, E_U, E_R \gg)^{-1},$$
$$\ll M_{\mathrm{HCF}}, L_R, E_R \gg^{-1}].$$



5.3.1 Rotating

We are going to extend the rotation function ρ to building blocks:

Definition 87. *Given a building block $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$, of type $\ll S, T_p, T_q \gg$ we define the rotation of X depending on n.*

$$\rho(X) = \begin{cases} \langle \rho^3([c_0; c_1, \ldots, c_n]), [p, q], \rho(T_p), \rho(T_q) \rangle & \text{if } n \text{ is odd;} \\ \langle \rho([c_0; c_1, \ldots, c_n]), [p, q], \rho(T_p), \rho(T_q) \rangle & \text{if } n \text{ is even.} \end{cases}$$

When n is odd, we have

$$\rho([c_0; c_1, \ldots, c_n]) = [\rho(c_0); \rho^3(c_1), \rho(c_2), \ldots, \rho^3(c_n)]$$

and by Lemma 17:

$$\mathrm{Shape}_{\sqrt{5}}(\rho^3(c_1)\rho(c_2)\rho^3(c_3) \ldots \rho^3(c_n)) = \rho^3(\mathrm{Shape}_{\sqrt{5}}(c_1 c_2 c_3 \ldots c_n)) = \rho^3(S).$$

When n is even, we have

$$\rho([c_0; c_1, \ldots, c_n]) = [\rho(c_0); \rho^3(c_1), \rho(c_2), \ldots, \rho(c_n)]$$

and by Lemma 17:

$$\mathrm{Shape}_{\sqrt{5}}(\rho^3(c_1)\rho(c_2)\rho^3(c_3) \ldots \rho(c_n)) = \rho(\mathrm{Shape}_{\sqrt{5}}(c_1 c_2 c_3 \ldots c_n)) = \rho(S).$$

In both cases the type of $\rho(X)$ is $\ll \rho(S), \rho(T_p), \rho(T_q) \gg = \rho(\ll S, T_p, T_q \gg)$.

Lemma 88. *Let $X = \langle [c_0; c_1, \ldots, c_n], [p,q], T_p, T_q \rangle$, when n is odd we have:*

$$\text{curve}(\rho(X)) = (\rho^3 \circ \text{curve})(X),$$

and when n is even:

$$\text{curve}(\rho(X)) = (\rho \circ \text{curve})(X).$$

Proof. When n is odd, we have:

$$\text{curve}(\rho(\langle [c_0; c_1, \ldots, c_n], [p,q], T_p, T_q \rangle)) = \text{curve}(\langle \rho^3([c_0; c_1, \ldots, c_n]), [p,q], \rho(T_p), \rho(T_q) \rangle =$$

$$\begin{aligned} t \to \left[\rho^3(c_0); \rho(c_1), \ldots : \rho(c_n) + \frac{q-t}{q-p}\rho(T_p) + \frac{t-p}{q-p}\rho(T_q)\right] &= \\ t \to \left[\rho^3(c_0); \rho(c_1), \ldots : \rho(c_n) + \rho\left(\frac{q-t}{q-p}T_p\right) + \rho\left(\frac{t-p}{q-p}T_q\right)\right] &= \\ t \to \left[\rho^3(c_0); \rho(c_1), \ldots : \rho\left(c_n + \frac{q-t}{q-p}T_p + \frac{t-p}{q-p}T_q\right)\right] &= \\ t \to \rho^3\left(\left[c_0; c_1, \ldots : c_n + \frac{q-t}{q-p}T_p + \frac{t-p}{q-p}T_q\right]\right) &= \\ t \to \rho^3(\text{curve}(X)). \end{aligned}$$

When n is even, we have:

$$\text{curve}(\rho(\langle [c_0; c_1, \ldots, c_n], [p,q], T_p, T_q \rangle)) = \text{curve}(\langle \rho([c_0; c_1, \ldots, c_n]), [p,q], \rho(T_p), \rho(T_q) \rangle =$$

$$\begin{aligned} t \to \left[\rho(c_0); \rho^3(c_1), \ldots : \rho(c_n) + \frac{q-t}{q-p}\rho(T_p) + \frac{t-p}{q-p}\rho(T_q)\right] &= \\ t \to \left[\rho(c_0); \rho^3(c_1), \ldots : \rho(c_n) + \rho\left(\frac{q-t}{q-p}T_p\right)\right] + \rho\left(\frac{t-p}{q-p}T_q\right) &= \\ t \to \left[\rho(c_0); \rho^3(c_1), \ldots : \rho\left(c_n + \frac{q-t}{q-p}T_p + \frac{t-p}{q-p}T_q\right)\right] &= \\ t \to \rho\left(\left[c_0; c_1, \ldots : c_n + \frac{q-t}{q-p}T_p + \frac{t-p}{q-p}T_q\right]\right) &= \\ t \to \rho(\text{curve}(X)). \end{aligned}$$

□

Definition 89. *We extend the definition of the rotation ρ to chains. Let $[X_0, X_1, \ldots, X_m]$ be a chain of depth n, then:*

$$\rho([X_0, X_1, \ldots, X_m]) = [\rho(X_0), \rho(X_1), \ldots, \rho(X_m)]$$

Lemma 90. *Let $[X_0, X_1, \ldots, X_m]$ be a chain of depth n, then $\rho([X_0, X_1, \ldots, X_m])$ is a chain of depth n.*

Proof. Let $X_g = \langle [c_0^g; c_1^g, \ldots, c_{n^g}^g], [p^g, q^g], T_p^g, T_q^g \rangle$. Because for all g: $n = n^g$, we have that $\rho(X_g)$ equals

- $$\langle \rho^3([c_0^g; c_1^g, c_2^g, \ldots, c_n^g]), [p^g, q^g], \rho(T_p^g), \rho(T_q^g)\rangle$$

  where
  $$\rho^3([c_0^g; c_1^g, c_2^g, \ldots, c_n^g]) = [\rho^3(c_0^g); \rho(c_1^g), \rho^3(c_2^g), \ldots, \rho(c_n^g)]$$
  when n is odd.

- $$\langle \rho([c_0^g; c_1^g, c_2^g, \ldots, c_n^g]), [p^g, q^g], \rho(T_p^g), \rho(T_q^g)\rangle$$

  where
  $$\rho([c_0^g; c_1^g, c_2^g, \ldots, c_n^g]) = [\rho(c_0^g); \rho^3(c_1^g), \rho(c_2^g), \ldots, \rho(c_n^g)]$$
  when n is even.

We will check each property of a chain:

1. The number of coefficients does not change with rotations.
2. For all $g, h \geq m$ with $g \neq h$ and $s \in \{1, 3\}$, we have $\rho^s([c_0^g; c_1^g, c_2^g, \ldots, c_n^g]) \neq \rho^s([c_0^h; c_1^h, c_2^h, \ldots, c_n^h])$, because $\rho^{4-s}(\rho^s([c_0^g; c_1^g, c_2^g, \ldots, c_n^g])) \neq \rho^{4-s}(\rho^s([c_0^h; c_1^g, c_2^h, \ldots, c_n^h]))$.
3. For each $g < m$ we have $q^g = p^{g+1}$, so $\rho(q^g) = \rho(p^{g+1})$.
4. For each $g < m$,
   - When n is odd, we have:
   $$\begin{aligned}
   &\lfloor \rho^3(c_0^g); \rho(c_1^g), \rho^3(c_2^g), \ldots : \rho(c_{n^g}^g) + \rho(T_q^g)\rceil \\
   =&\lfloor \rho^3(c_0^g); \rho(c_1^g), \rho^3(c_2^g), \ldots : \rho(c_{n^g}^g + T_q^g)\rceil \\
   =&\rho^3(\lfloor c_0^g; c_1^g, c_2^g, \ldots : c_{n^g}^g + T_q^g\rceil) \\
   =&\rho^3(\lfloor c_0^{g+1}; c_1^{g+1}, c_2^{g+1}, \ldots : c_{n^{g+1}}^{g+1} + T_p^{g+1}\rceil) \\
   =&\lfloor \rho^3(c_0^{g+1}); \rho(c_1^{g+1}), \rho^3(c_2^{g+1}), \ldots : \rho(c_{n^{g+1}}^{g+1} + T_p^{g+1})\rceil \\
   =&\lfloor \rho^3(c_0^{g+1}); \rho(c_1^{g+1}), \rho^3(c_2^{g+1}), \ldots : \rho(c_{n^{g+1}}^{g+1}) + \rho(T_p^{g+1})\rceil
   \end{aligned}$$
   - When n is even, we have:
   $$\begin{aligned}
   &\lfloor \rho(c_0^g); \rho^3(c_1^g), \rho(c_2^g), \ldots : \rho(c_{n^g}^g) + \rho(T_q^g)\rceil \\
   =&\lfloor \rho(c_0^g); \rho^3(c_1^g), \rho(c_2^g), \ldots : \rho(c_{n^g}^g + T_q^g)\rceil \\
   =&\rho(\lfloor c_0^g; c_1^g, c_2^g, \ldots : c_{n^g}^g + T_q^g\rceil) \\
   =&\rho(\lfloor c_0^{g+1}; c_1^{g+1}, c_2^{g+1}, \ldots : c_{n^{g+1}}^{g+1} + T_p^{g+1}\rceil) \\
   =&\lfloor \rho(c_0^{g+1}); \rho^3(c_1^{g+1}), \rho(c_2^{g+1}), \ldots : \rho(c_{n^{g+1}}^{g+1} + T_p^{g+1})\rceil \\
   =&\lfloor \rho(c_0^{g+1}); \rho^3(c_1^{g+1}), \rho(c_2^{g+1}), \ldots : \rho(c_{n^{g+1}}^{g+1}) + \rho(T_p^{g+1})\rceil
   \end{aligned}$$ □

Definition 91. *Let $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q\rangle$, with type $X_T = \ll X, T_p, T_q \gg$. We define:*

$$\text{mesh}(\rho(X)) = \rho^3(\text{mesh}(X)).$$

We will prove this definition is sound, by checking the properties of mesh:
Let mesh$(X) = [Y_0, Y_1, \ldots, Y_m]$ where $Y_g = \langle [c_0^g; c_1^g, \ldots, c_n^g, c_{n+1}^g], [p^g, q^g], T_p^g, T_q^g \rangle$.
By the properties of mesh (Definition 86) we know:

1. $[Y_0, Y_1, \ldots, Y_m]$ is a chain of depth $n+1$;
2. For each $g \leq m$ and $k \leq n$, $c_k^g = c_k$;
3. $p^0 = p$ and $q^m = q$;
4. For each $g < m$ there exists an $s \in \{1, \ldots, 16\}$ such that

$$C_s = \lfloor 0 : c_{n+1}^g + T_q^g \rceil (= \lfloor 0 : c_{n+1}^{g+1} + T_p^{g+1} \rceil);$$

5.
   - $\lfloor c_0^0; c_1^0, \ldots, c_n^0 : c_{n+1}^0 + T_p^0 \rceil = \lfloor c_0; c_1, \ldots : c_n + T_p \rceil$, so $T_p = \lfloor 0 : c_{n+1}^0 + T_p^0 \rceil$,
   - $\lfloor c_0^m; c_1^m, \ldots, c_n^m : c_{n+1}^m + T_q^m \rceil = \lfloor c_0; c_1, \ldots : c_n + T_q \rceil$, so $T_q = \lfloor 0 : c_{n+1}^m + T_q^m \rceil$,

and by definition

$$\rho^3([Y_0, Y_1, \ldots, Y_m]) = [\rho^3(Y_0), \rho^3(Y_1), \ldots, \rho^3(Y_m)].$$

We will distinguish between whether n is odd or n is even.

- When n is odd, we have

  $$\rho(X) = \langle \rho^3([c_0; c_1, \ldots, c_n]), [p, q], \rho(T_p), \rho(T_q) \rangle$$

  with

  $$\rho^3([c_0; c_1, \ldots, c_n]) = [\rho^3(c_0); \rho(c_1), \ldots, \rho(c_n)].$$

  And

  $$\rho^3(Y_g) = \langle \rho^3([c_0^g; c_1^g, \ldots, c_n^g, c_{n+1}^g]), [p^g, q^g], \rho^3(T_p^g), \rho^3(T_q^g) \rangle$$

  with

  $$\rho^3([c_0^g; c_1^g, \ldots, c_{n+1}^g]) = [\rho^3(c_0^g); \rho(c_1^g), \ldots, \rho(c_n^g), \rho^3(c_{n+1}^g)].$$

  We check the properties of mesh

  1. By Lemma 90, $\rho^3([Y_0, Y_1, \ldots, Y_m])$ is a chain of depth $n+1$.
  2. For each $g \leq m$ and $k \leq n$ we have
     - when k is odd: $\rho(c_k^g) = \rho(c_k)$;
     - when k is even: $\rho^3(c_k^g) = \rho^3(c_k)$.
  3. $p^0 = p$ and $q^m = q$.
  4. For each $g < m$ we have that there exists C_s such that

     $$C_s = \lfloor 0 : c_{n+1}^g + T_q^g \rceil.$$

     There exists an s' such that $C_{s'} = \rho(C_s)$, then

     $$C_{s'} = \rho(C_s) = \rho(\lfloor 0 : c_{n+1}^g + T_q^g \rceil) = \lfloor 0 : \rho^3(c_{n+1}^g + T_q^g) \rceil = \lfloor 0 : \rho^3(c_{n+1}^g) + \rho^3(T_q^g) \rceil.$$

5. – Because $T_p = \lfloor 0 : c^0_{n+1} + T^0_p \rceil$ we have $\rho(T_p) = \lfloor 0 : \rho^3(c^0_{n+1} + T^0_p)\rceil$. So

$$\lfloor \rho^3(c^0_0); \rho(c^0_1), \ldots, \rho(c^0_n) : \rho^3(c^0_{n+1}) + \rho^3(T^0_p)\rceil =$$

$$\lfloor \rho^3(c^0_0); \rho(c^0_1), \ldots, \rho(c^0_n) : \rho^3(c^0_{n+1} + T^0_p)\rceil =$$

$$\lfloor \rho^3(c_0); \rho(c_1), \ldots, \rho(c_n) : \rho^3(c^0_{n+1} + T^0_p)\rceil =$$

$$\lfloor \rho^3(c_0); \rho(c_1), \ldots : \rho(c_n) + \rho(T_p)\rceil.$$

– Because $T_q = \lfloor 0 : c^m_{n+1} + T^m_q \rceil$ we have $\rho(T_q) = \lfloor 0 : \rho^3(c^m_{n+1} + T^m_q)\rceil$. So

$$\lfloor \rho^3(c^m_0); \rho(c^m_1), \ldots, \rho(c^m_n) : \rho^3(c^m_{n+1}) + \rho^3(T^m_q)\rceil =$$

$$\lfloor \rho^3(c^m_0); \rho(c^m_1), \ldots, \rho(c^m_n) : \rho^3(c^m_{n+1} + T^m_q)\rceil =$$

$$\lfloor \rho^3(c_0); \rho(c_1), \ldots, \rho(c_n) : \rho^3(c^m_{n+1} + T^m_q)\rceil =$$

$$\lfloor \rho^3(c_0); \rho(c_1), \ldots : \rho(c_n) + \rho(T_q)\rceil.$$

- When n is even, we have

$$\rho(X) = \langle \rho([c_0; c_1, \ldots, c_n]), [p, q], \rho(T_p), \rho(T_q)\rangle$$

with

$$\rho([c_0; c_1, \ldots, c_n]) = [\rho(c_0); \rho^3(c_1), \ldots, \rho(c_n)].$$

And

$$\rho^3(Y_g) = \langle \rho^9([c^g_0; c^g_1, \ldots, c^g_n, c^g_{n+1}]), [p^g, q^g], \rho^3(T^g_p), \rho^3(T^g_q)\rangle$$

with

$$\rho^9([c^g_0; c^g_1, \ldots, c^g_{n+1}]) = \rho([c^g_0; c^g_1, \ldots, c^g_{n+1}]) = [\rho(c^g_0); \rho^3(c^g_1), \ldots, \rho(c^g_n), \rho^3(c^g_{n+1})].$$

We check the properties of mesh

1. By Lemma 90, $\rho^3([Y_0, Y_1, \ldots, Y_m])$ is a chain of depth $n + 1$.
2. For each $g \leq m$ and $k \leq n$ we have
   - when k is odd: $\rho^3(c^g_k) = \rho^3(c_k)$;
   - when k is even: $\rho(c^g_k) = \rho(c_k)$.
3. $p^0 = p$ and $q^m = q$.
4. For each $g < m$ we have that there exists C_s such that

$$C_s = \lfloor 0 : c^g_{n+1} + T^g_q\rceil.$$

   There exists an s' such that $C_{s'} = \rho(C_s)$, then

$$C_{s'} = \rho(C_s) = \rho(\lfloor 0 : c^g_{n+1} + T^g_q\rceil) = \lfloor 0 : \rho^3(c^g_{n+1} + T^g_q)\rceil = \lfloor 0 : \rho^3(c^g_{n+1}) + \rho^3(T^g_q)\rceil.$$

5. – Because $T_p = \lfloor 0 : c^0_{n+1} + T^0_p \rceil$ we have $\rho(T_p) = \lfloor 0 : \rho^3(c^0_{n+1} + T^0_p) \rceil$. So

$$\lfloor \rho(c^0_0); \rho^3(c^0_1), \ldots, \rho(c^0_n) : \rho^3(c^0_{n+1}) + \rho^3(T^0_p) \rceil =$$

$$\lfloor \rho(c^0_0); \rho^3(c^0_1), \ldots, \rho(c^0_n) : \rho^3(c^0_{n+1} + T^0_p) \rceil =$$

$$\lfloor \rho(c_0); \rho^3(c_1), \ldots, \rho(c_n) : \rho^3(c^0_{n+1} + T^0_p) \rceil =$$

$$\lfloor \rho(c_0); \rho^3(c_1), \ldots : \rho(c_n) + \rho(T_p) \rceil.$$

– Because $T_q = \lfloor 0 : c^m_{n+1} + T^m_q \rceil$ we have $\rho(T_q) = \lfloor 0 : \rho^3(c^m_{n+1} + T^m_q) \rceil$. So

$$\lfloor \rho(c^m_0); \rho^3(c^m_1), \ldots, \rho(c^m_n) : \rho^3(c^m_{n+1}) + \rho^3(T^m_q) \rceil =$$

$$\lfloor \rho(c^m_0); \rho^3(c^m_1), \ldots, \rho(c^m_n) : \rho^3(c^m_{n+1} + T^m_q) \rceil =$$

$$\lfloor \rho(c_0); \rho^3(c_1), \ldots, \rho(c_n) : \rho^3(c^m_{n+1} + T^m_q) \rceil =$$

$$\lfloor \rho(c_0); \rho^3(c_1), \ldots : \rho(c_n) + \rho(T_q) \rceil.$$

So the definition is sound.

5.3.2 Reversing

Lemma 92. *Let $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$, we have*

$$\text{curve}(X^{-1})(t) = \text{curve}(X)(q + p - t)$$

Proof. We know $X^{-1} = \langle [c_0; c_1, \ldots, c_n], [p, q], T_q, T_p \rangle$, so

$$\text{curve}(X^{-1})(t) = \left[c_0; c_1, \ldots : c_n + \frac{q-t}{q-p} T_q + \frac{t-p}{q-p} T_p \right] =$$

$$\left[c_0; c_1, \ldots : c_n + \frac{q+p-t-p}{q-p} T_q + \frac{q-q-p+t}{q-p} T_p \right] =$$

$$\left[c_0; c_1, \ldots : c_n + \frac{(q+p-t)-p}{q-p} T_q + \frac{q-(q+p-t)}{q-p} T_p \right] =$$

$$\text{curve}(X)(q + p - t).$$

□

Definition 93. *We extend the definition of* reversed *to chains: Let $[Y_0, Y_1, \ldots, Y_m]$ be a chain, then*

$$[Y_0, Y_1, \ldots, Y_m]^{-1} = [Y^{-1}_m, Y^{-1}_{m-1}, \ldots, Y^{-1}_0].$$

Lemma 94. *If $[X_0, X_1, \ldots, X_m]$ is a chain of depth n, where $X_g = \langle [c^g_0; c^g_1, \ldots, c^g_n], [p^g, q^g], T^g_p, T^g_q \rangle$ of type $\ll S^g, T^g_p, T^g_q \gg$. Let*

$$Y_g = \langle [c^{m-g}_0; c^{m-g}_1, \ldots, c^{m-g}_n], [p^g, q^g], T^{m-g}_q, T^{m-g}_p \rangle.$$

which has type $\ll S^{m-g}, T^{m-g}_q, T^{m-g}_p \gg = \ll S^{m-g}, T^{m-g}_p, T^{m-g}_q \gg^{-1}$. Then $[X_0, X_1, \ldots, X_m]^{-1} = [Y_0, Y_1, \ldots, Y_m]$ is a chain of depth n.

Proof. We will show that the properties of a chain (Definition 84) hold.

1. For each $g \leq m$, Y_g has a base with the same number of coefficients as X_{m-g}.

2. For all $g, h \leq m$, if $g \neq h$, we have $m - g \neq m - h$. Because $[X_0, X_1, \ldots, X_m]$ is a chain there exists a $k \leq n$ such that $c_k^{m-g} \neq c_k^{m-h}$. So the base of Y_g is not equal to the base of Y_h.

3. For all $g \leq m$, the interval of Y_g equals the interval of Y_h.

4. We know for all $h < m$ that we have

$$\lfloor c_0^h; c_1^h, \ldots : c_n^h + T_q^h \rceil = \lfloor c_0^{h+1}; c_1^{h+1}, \ldots : c_n^{h+1} + T_p^{h+1} \rceil.$$

   Substituting $g = m - h + 1$, we get what we need: For all $g < m =$

$$\lfloor c_0^{m-g}; c_1^{m-g}, \ldots : c_n^{m-g} + T_p^{m-g} \rceil = \lfloor c_0^{m-(g+1)}; c_1^{m-(g+1)}, \ldots : c_n^{m-(g+1)} + T_q^{m-(g+1)} \rceil. \qquad \square$$

Definition 95. *Let* $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$*, with type* $X_T = \ll X, T_p, T_q \gg$*. We define:*

$$\text{mesh}(X^{-1}) = (\text{mesh}(X))^{-1}.$$

We will prove that this definition is sound, by checking the properties of mesh:
Let $\text{mesh}(X) = [Y_0, Y_1, \ldots, Y_m]$ where $Y_g = \langle [c_0^g; c_1^g, \ldots, c_n^g, c_{n+1}^g], [p^g, q^g], T_p^g, T_q^g \rangle$.
By the properties of mesh we know:

1. $[Y_0, Y_1, \ldots, Y_m]$ is a chain of depth $n + 1$.

2. For each $g \leq m$ and $k \leq n$, $c_k^g = c_k$.

3. $p^0 = p$ and $q^m = q$

4. For each $g < m$ there exists $s \in \{1, \ldots, 16\}$ such that

$$C_s = \lfloor 0 : c_{n+1}^g + T_q^g \rceil (= \lfloor 0 : c_{n+1}^{g+1} + T_p^{g+1} \rceil).$$

5.
   - $\lfloor c_0^0; c_1^0, \ldots, c_n^0 : c_{n+1}^0 + T_p^0 \rceil = \lfloor c_0; c_1, \ldots : c_n + T_p \rceil$, so $T_p = \lfloor 0 : c_{n+1}^0 + T_p^0 \rceil$.
   - $\lfloor c_0^m; c_1^m, \ldots, c_n^m : c_{n+1}^m + T_q^m \rceil = \lfloor c_0; c_1, \ldots : c_n + T_q \rceil$, so $T_q = \lfloor 0 : c_{n+1}^m + T_q^m \rceil$.

Recall that

$$X^{-1} = \langle [c_0; c_1, \ldots, c_n], [p, q], T_q, T_p \rangle.$$

and $[Y_0, Y_1, \ldots, Y_m]^{-1} = [Z_0, Z_1, \ldots, Z_m]$ with

$$Z_g = \langle [c_0^{m-g}; c_1^{m-g}, \ldots, c_{n+1}^{m-g}], [p^g, q^g], T_q^{m-g}, T_p^{m-g} \rangle.$$

We check the properties of mesh

1. By Lemma 93, $[Y_0, Y_1, \ldots, Y_m]^{-1}$ is a chain of depth $n + 1$.

2. For each $h \leq m$ and $k \leq n$ we have, by substituting $m - h = g$, that $c_k^{m-g} = c_k^h = c_k$.

3. $q^{m-0} = q$ and $p^{m-m} = p$.

4. For each $g < m$, we have $m - (g+1) < m$, so there exists an s such that

$$C_s = \lfloor 0 : c_{n+1}^{m-(g+1)+1} + T_p^{m-(g+1)+1} \rceil = \lfloor 0 : c_{n+1}^{m-g} + T_p^{m-g} \rceil$$

5. • Because $T_q = \lfloor 0 : c_{n+1}^{m} + T_q^{m} \rceil$, we have

$$\lfloor c_0^{m-0}; c_1^{m-0}, \ldots, c_n^{m-0} : c_{n+1}^{m-0} + T_q^{m-0} \rceil =$$

$$\lfloor c_0^{m-0}; c_1^{m-0}, \ldots : c_n^{m-0} + T_q \rceil =$$

$$\lfloor c_0; c_1, \ldots : c_n + T_q \rceil$$

   • Because $T_p = \lfloor 0 : c_{n+1}^{0} + T_p^{0} \rceil$, we have

$$\lfloor c_0^{m-m}; c_1^{m-m}, \ldots, c_n^{m-m} : c_{n+1}^{m-m} + T_p^{m-m} \rceil =$$

$$\lfloor c_0^{m-m}; c_1^{m-m}, \ldots : c_n^{m-m} + T_p \rceil =$$

$$\lfloor c_0; c_1, \ldots : c_n + T_p \rceil$$

So the definition is sound.

5.3.3 Chaining mesh functions

Lemma 96. *Let* $[X_0, X_1, \ldots, X_m]$ *be a chain of depth* n*, and let* $\mathrm{mesh}(X_g) = [Y_0^g, Y_1^g, \ldots, Y_{m^g}^g]$*. Then* $[Y_0^0, Y_1^0, \ldots, Y_{m^0}^0, Y_0^1, Y_1^1, \ldots, Y_{m^1}^1, Y_0^2, \ldots, Y_{m^m}^m]$ *is a chain of depth* $n+1$*.*

Proof. Let Let $g, h \leq m$ and $k \leq m_g$, $l \leq m_h$. We will write

$$X_g = \langle [c_0^g; c_1^g, \ldots, c_n^g], [p^g, q^g], T_p^g, T_q^g \rangle$$

and

$$X_h = \langle [c_0^h; c_1^h, \ldots, c_n^h], [p^h, q^h], T_p^h, T_q^h \rangle.$$

With $Y_k^g \in \mathrm{mesh}(X_g)$ and $Y_l^h \in \mathrm{mesh}(X_h)$, which we will write as:

$$Y_k^g = \langle [c_0^u; c_1^u, \ldots, c_{n+1}^u], [p^u, q^u], T_p^u, T_q^u \rangle$$

and

$$Y_l^h = \langle [c_0^w; c_1^w, \ldots, c_{n+1}^w], [p^w, q^w], T_p^w, T_q^w \rangle.$$

We will prove all the properties of a chain:

1. As $[X_0, X_1, \ldots, X_m]$ is a chain of depth n, for each g, we have that $\mathrm{mesh}(X_g)$ is a chain of depth $n+1$, and so every building block Y has a base that has quotients up to $n+1$.

2. We want to prove that if $g \neq h$ or $k \neq l$, then the base of $[c_0^u; c_1^u, \ldots, c_{n+1}^u] \neq [c_0^w; c_1^w, \ldots, c_{n+1}^w]$.

   • Let $g = h$. Then we have that Y_k^g and Y_l^h are an element of $\mathrm{mesh}(X_g)$. Because $\mathrm{mesh}(X_g)$ is a chain, we know that when $k \neq l$, we have $[c_0^u; c_1^u, \ldots, c_{n+1}^u] \neq [c_0^w; c_1^w, \ldots, c_{n+1}^w]$.

- Let $g \neq h$. $[X_0, X_1, \ldots, X_m]$ is a chain, so we have that $[c_0^g; c_1^g, \ldots, c_n^g] \neq [c_0^h; c_1^h, \ldots, c_n^h]$. Thus, there exists an $r \leq n$ such that $c_r^g \neq c_r^h$. From the properties of mesh, we know $c_r^u = c_r^g \neq c_r^h = c_r^w$. So $[c_0^u; c_1^u, \ldots, c_{n+1}^u] \neq [c_0^w; c_1^w, \ldots, c_{n+1}^w]$.

3. We have three possibilities:

   - $g = m$ and $k = m_m$. We don't have to prove anything in this situation.
   - $g < m$ and $k = m_g$. Let $h = g + 1$ and $l = 0$, then Y_k^g is followed by Y_0^h. We will have to prove $q^u = p^w$.

     Because $[X_0, X_1, \ldots, X_m]$ is a chain, we know $q^g = p^h$. Combined with the properties of the mesh function, we have $q^u = q^g = p^h = p^w$.
   - $g < m$ and $k < m_g$. Let $h = g$ and $l = k + 1$, then Y_k^g is followed by Y_l^g. Because $\text{mesh}(X_g)$ is a chain, we have $q^u = p^w$.

4. We have three possibilities:

   - $g = m$ and $k = m_m$. We don't have to prove anything in this situation.
   - $g < m$ and $k = m_g$. Let $h = g + 1$ and $l = 0$, then Y_k^g is followed by Y_0^h. We will have to prove $\lfloor c_0^u; c_1^u, \ldots : c_{n+1}^u + T_q^u \rceil = \lfloor c_0^w; c_1^w, \ldots : c_{n+1}^w + T_p^w \rceil$.

     Because $[X_0, X_1, \ldots, X_m]$ is a chain, we know $\lfloor c_0^g; c_1^g, \ldots : c_n^g + T_q^g \rceil = \lfloor c_0^h; c_1^h, \ldots : c_n^h + T_q^h \rceil$. Combined with the properties of the mesh function, we have $\lfloor c_0^u; c_1^u, \ldots : c_{n+1}^u + T_q^u \rceil = \lfloor c_0^g; c_1^g, \ldots : c_n^g + T_q^g \rceil = \lfloor c_0^h; c_1^h, \ldots : c_n^h + T_q^h \rceil = \lfloor c_0^w; c_1^w, \ldots : c_{n+1}^w + T_p^w \rceil$.
   - $g < m$ and $k < m_g$. Let $h = g$ and $l = k + 1$, then Y_k^g is followed by Y_l^g. Because $\text{mesh}(X_g)$ is a chain, we have $\lfloor c_0^u; c_1^u, \ldots : c_{n+1}^u + T_q^u \rceil = \lfloor c_0^w; c_1^w, \ldots : c_{n+1}^w + T_p^w \rceil$. □

Lemma 97. *For each building block $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$, we have:*

$$\lim_{t \downarrow p} \text{curve}(X)(t) = \lfloor c_0; c_1, \ldots : c_n + T_p \rceil$$

and

$$\lim_{t \uparrow q} \text{curve}(X)(t) = \lfloor c_0; c_1, \ldots : c_n + T_q \rceil.$$

Proof. For each $t \in (p, q)$,

$$\text{curve}(X)(t) = [c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} T_p + \frac{t-p}{q-p} T_q)]$$
$$= \lfloor c_0; c_1, c_2, \ldots, c_{n-1} : c_n + (\frac{q-t}{q-p} T_p + \frac{t-p}{q-p} T_q) \rceil.$$

Because

$$\lim_{t \downarrow p} (\frac{q-t}{q-p} T_p + \frac{t-p}{q-p} T_q) = T_p,$$

we have

$$\lim_{t \downarrow p} \text{curve}(X)(t) = \lfloor c_0; c_1, \ldots : c_n + T_p \rceil.$$

And because

$$\lim_{t \uparrow q} (\frac{q-t}{q-p} T_p + \frac{t-p}{q-p} T_q) = T_q,$$

we have

$$\lim_{t\uparrow q} \text{curve}(X)(t) = \lfloor c_0; c_1, \ldots : c_n + T_q \rceil. \qquad \square$$

Definition 98. *Let $\overline{\text{curve}}$ be the function from a building block $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$ to a function from $[p, q]$ to $\mathbb{C}$, defined by:*

$$\overline{\text{curve}}(X)(t) = \begin{cases} \lfloor c_0; c_1, \ldots : c_n + T_p \rceil & \textit{if } t = p \\ \text{curve}(X)(t) & \textit{if } t \in (p, q) \\ \lfloor c_0; c_1, \ldots : c_n + T_q \rceil & \textit{if } t = q \end{cases}$$

By Lemma 97, for every building block X of our described types, $\overline{\text{curve}}(X)$ is a continuous function on $[p, q]$.

Definition 99. *We extend the function $\overline{\text{curve}}$ to chains. Let $[X_0, X_1, \ldots, X_m]$ be a chain with $X_g = \langle [c_0^g; c_1^g, \ldots, c_n^g], [p^g, q^g], T_p^g, T_q^g \rangle$. Let*

$$\text{curve}([X_0, X_1, \ldots, X_m])(t) = \overline{\text{curve}}(X_g)(t) \textit{ if } t \in [p^g, q^g].$$

We will show $\overline{\text{curve}}([X_0, X_1, \ldots, X_m])$ is a well defined function on the interval $[p^0, q^m]$.
Note that we can create the list $T([X_0, X_1, \ldots, X_m])$ which contains all the endpoints of the building blocks, by $T([X_0, X_1, \ldots, X_m]) = [p^0, p^1, \ldots, p^m, q^m]$, and it is equal to $[p^0, q^0, \ldots, q^{m-1}, q^m]$.
Because $T([X_0, X_1, \ldots, X_m])$ is strictly increasing, we have that for each $t \in [p^0, q^m]$, either $t \in T([X_0, X_1, \ldots, X_m])$, or there exists exactly one m such that $t \in (p^m, q^m)$.

- Let $t \in T([X_0, X_1, \ldots, X_m])$. We have either $t = p^0$ or $t = q^m$ which are defined uniquely, or there exists a g such that $t = p^{g+1} = q^g$. Because $[X_0, X_1, \ldots, X_m]$ is a chain, $\lfloor c_0^g; c_1^g, \ldots : c_n^g + T_q^g \rceil = \lfloor c_0^{g+1}; c_1^{g+1}, \ldots : c_n^{g+1} + T_p^{g+1} \rceil$.
- Let $t \notin T([X_0, X_1, \ldots, X_m])$. There exists a unique g such that $t \in (p^g, q^g)$, and

$$\text{curve}([X_0, X_1, \ldots, X_m])(t) = \overline{\text{curve}}(X_g)(t).$$

Lemma 100. *For every n, for every building block $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$, for every $t \in (p, q)$ we have:*

$$\text{curve}(X)(t) \equiv_{\restriction n-1} [c_0; c_1, \ldots, c_n]$$

Proof. By definition (85)

$$\text{curve}(X)(t) = [c_0; c_1, \ldots, c_{n-1} : c_n + \frac{q-t}{q-p} T_p + \frac{t-p}{q-p} T_q] \equiv_{\restriction n-1} [c_0; c_1, \ldots, c_{n-1}]. \qquad \square$$

Lemma 101. *For every n, for every building block $X = \langle [c_0; c_1, \ldots, c_n], [p, q], T_p, T_q \rangle$,*

$$\overline{\text{curve}}(X) \equiv_{\restriction n-1} \overline{\text{curve}}(\text{mesh}(X))$$

Proof. Let $\text{mesh}(X) = [Y_0, Y_1, \ldots, Y_m]$ with $Y_g = \langle [c_0^g; c_1^g, \ldots, c_{n+1}^g], [p^g, q^g], T_p^g, T_q^g \rangle$.
Let $T([Y_0, Y_1, \ldots, Y_m]) = [p^0, p^1, \ldots, p^m, q^m]$. We will prove, for every $t \in [p, q]$ that

$$\overline{\text{curve}}(X)(t) \equiv_{\restriction n-1} \overline{\text{curve}}(\text{mesh}(X))(t).$$

- When $t \notin T([Y_0, Y_1, \ldots, Y_m])$, there exists exactly one g such that $t \in (p_g, q_g)$.
  Let $t \in (p_g, q_p)$, because $(p_g, q_p) \subseteq (p, q)$, we have using Lemma 100 and Definition 98:

$$\begin{aligned}
&\overline{\text{curve}}(\text{mesh}(X))(t)\\
=&\overline{\text{curve}}(Y_g)(t)\\
=&\,\text{curve}(Y_g)(t)\\
\equiv_{\upharpoonright n}&[c_0^g; c_1^g, \ldots, c_n^g]\\
=&[c_0; c_1, \ldots, c_n]\\
\equiv_{\upharpoonright n-1}&[c_0; c_1, \ldots, c_{n-1}]\\
\equiv_{\upharpoonright n-1}&\,\text{curve}(X)(t)\\
\equiv_{\upharpoonright n-1}&\overline{\text{curve}}(X)(t)
\end{aligned}$$

- When $t = p$, we have

$$\begin{aligned}
&\overline{\text{curve}}(X)(p)\\
=&\lfloor c_0; c_1, \ldots : c_n + T_p\rceil\\
=&\lfloor c_0; c_1, \ldots, c_n : c_{n+1}^0 + T_p^0\rceil\\
=&\overline{\text{curve}}(\text{mesh}(Y_0))(p)\\
=&\overline{\text{curve}}(\text{mesh}(X))(p)
\end{aligned}$$

- When $t = q$, we have

$$\begin{aligned}
&\overline{\text{curve}}(X)(q)\\
=&\lfloor c_0; c_1, \ldots : c_n + T_q\rceil\\
=&\lfloor c_0; c_1, \ldots, c_n : c_{n+1}^m + T_q^m\rceil\\
=&\overline{\text{curve}}(\text{mesh}(Y_m))(q)\\
=&\overline{\text{curve}}(\text{mesh}(X))(q)
\end{aligned}$$

- When $t \in T([Y_0, Y_1, \ldots, Y_m])$, $t \neq p$ and $t \neq q$, there exists a $g < m$ such that $t = q^g (= p^{g+1})$. By definition of the mesh function, there exists an s such that $C_s = \lfloor 0 : c_{n+1}^g + T_q^g\rceil$, thus for every $g < m$ there exists an s:

$$\begin{aligned}
&\overline{\text{curve}}(\text{mesh}(X))(q^g)\\
=&\overline{\text{curve}}(\text{mesh}(Y_g))(q^g)\\
=&\lfloor c_0; c_1, \ldots, c_n : c_{n+1}^g + T_q^g\rceil\\
=&\lfloor c_0; c_1, \ldots : c_n + C_s\rceil\\
=&[c_0; c_1, \ldots : c_n + C_s]\\
\equiv_{\upharpoonright n-1}&[c_0; c_1, \ldots, c_{n-1}]\\
\equiv_{\upharpoonright n-1}&\overline{\text{curve}}(X)(q^g)
\end{aligned}$$

□

Lemma 102. *For each chain $[X_0, X_1, \ldots, X_m]$ of depth n*

$$\overline{\text{curve}}([X_0, X_1, \ldots, X_m]) \equiv_{\upharpoonright n-1} \overline{\text{curve}}(\text{mesh}([X_0, X_1, \ldots, X_m]))$$

Proof. Let $X^g = \langle [c_0^g; c_1^g, \ldots, c_n^g], [p^g, q^g], T_p^g, T_q^g \rangle$, we are going to show that for each $t \in [p^0, q^m]$

$$\overline{\text{curve}}([X_0, X_1, \ldots, X_m])(t) \equiv_{\restriction n-1} \overline{\text{curve}}(\text{mesh}([X_0, X_1, \ldots, X_m]))(t).$$

For every t there exists a g such that $t \in [p_m, q_m]$. With Lemma 101 we have:

$$\begin{aligned} &\overline{\text{curve}}([X_0, X_1, \ldots, X_m])(t) \\ =&\overline{\text{curve}}(X_g)(t) \\ \equiv_{\restriction n-1}&\overline{\text{curve}}(\text{mesh}(X))(t) \\ =&\overline{\text{curve}}(\text{mesh}([X_0, X_1, \ldots, X_m]))(t) \end{aligned}$$

□

5.4 The construction of chains

We are going to describe an infinite sequence of chains of depth g, Chain_g. We start by describing Chain_1 explicit.

Definition 103. *Let* Chain_1 *be*

$$\begin{aligned} &[\langle [0; 2-i], [0, \tfrac{1}{16}], L_R, E_D \rangle, \langle [0; 1-i], [\tfrac{1}{16}, \tfrac{2}{16}], L_D, E_R \rangle, \\ &\langle [0; 1-2i], [\tfrac{2}{16}, \tfrac{3}{16}], L_R, E_D \rangle, \langle [0; -2i], [\tfrac{3}{16}, \tfrac{4}{16}], L_D, E_D \rangle, \\ &\langle [0; -1-2i], [\tfrac{4}{16}, \tfrac{5}{16}], L_D, L_L \rangle, \langle [0; -1-i], [\tfrac{5}{16}, \tfrac{6}{16}], E_L, E_D \rangle, \\ &\langle [0; -2-i], [\tfrac{6}{16}, \tfrac{7}{16}], L_D, L_L \rangle, \langle [0; -2], [\tfrac{7}{16}, \tfrac{8}{16}], E_L, L_L \rangle, \\ &\langle [0; -2+i], [\tfrac{8}{16}, \tfrac{9}{16}], E_L, L_U \rangle, \langle [0; -1+i], [\tfrac{9}{16}, \tfrac{10}{16}], E_U, L_L \rangle, \\ &\langle [0; -1+2i], [\tfrac{10}{16}, \tfrac{11}{16}], E_L, L_U \rangle, \langle [0; 2i], [\tfrac{11}{16}, \tfrac{12}{16}], E_U, L_U \rangle, \\ &\langle [0; 1+2i], [\tfrac{12}{16}, \tfrac{13}{16}], E_U, E_R \rangle, \langle [0; 1+i], [\tfrac{13}{16}, \tfrac{14}{16}], L_R, L_U \rangle, \\ &\langle [0; 2+i], [\tfrac{14}{16}, \tfrac{15}{16}], E_U, E_R \rangle, \langle [0; 2], [\tfrac{15}{16}, 1], L_R, E_R \rangle] \end{aligned}$$

Let us write Chain_1 as $[X_0, \ldots X_{15}]$ with $X_h = \langle [0; c_h], [\frac{h}{16}, \frac{h+1}{16}], T_p^h, T_q^h \rangle$. Then

$$\lfloor 0 : c_h + T_q^h \rceil = C_{h+1} = \lfloor 0 : c_{h+1} + T_p^{h+1} \rceil.$$

The types of the elements of Chain_1 are:

$$\begin{aligned} &[\ll \rho^3(W_{\text{HCF}}), L_R, E_D \gg = \rho^3(\ll W_{\text{HCF}}, E_U, E_R \gg), \\ &\ll \rho^3(J_{\text{HCF}}), L_D, E_R \gg = \rho^3(\ll J_{\text{HCF}}, L_R, L_U \gg), \\ &\ll \rho^3(W_{\text{HCF}}), L_R, E_D \gg = \rho^3(\ll W_{\text{HCF}}, E_U, E_R \gg), \\ &\ll \rho^3(M_{\text{HCF}}), L_D, E_D \gg = \rho^3(\ll M_{\text{HCF}}, L_R, E_R \gg), \\ &\ll \rho^2(W_{\text{HCF}}), L_D, L_L \gg = \rho^3(\ll W_{\text{HCF}}, E_U, E_R \gg), \end{aligned}$$

$$\begin{aligned}
\ll \rho^2(J_{\text{HCF}}), E_L, E_D \gg &= \rho^3(\ll J_{\text{HCF}}, L_R, L_U \gg),\\
\ll \rho^2(W_{\text{HCF}}), L_D, L_L \gg &= \rho^3(\ll W_{\text{HCF}}, E_U, E_R \gg),\\
\ll \rho^2(M_{\text{HCF}}), E_L, L_L \gg &= \rho^3(\ll M_{\text{HCF}}, L_R, E_R \gg),\\
\ll \rho(W_{\text{HCF}}), E_L, L_U \gg &= \rho^3(\ll W_{\text{HCF}}, E_U, E_R \gg),\\
\ll \rho(J_{\text{HCF}}), E_U, L_L \gg &= \rho^3(\ll J_{\text{HCF}}, L_R, L_U \gg),\\
\ll \rho(W_{\text{HCF}}), E_L, L_U \gg &= \rho^3(\ll W_{\text{HCF}}, E_U, E_R \gg),\\
\ll \rho(M_{\text{HCF}}), E_U, L_U \gg &= \rho^3(\ll M_{\text{HCF}}, L_R, E_R \gg),\\
&\ll W_{\text{HCF}}, E_U, E_R \gg,\\
&\ll J_{\text{HCF}}, L_R, L_U \gg,\\
&\ll W_{\text{HCF}}, E_U, E_R \gg,\\
&\ll M_{\text{HCF}}, L_R, E_R \gg]
\end{aligned}$$

Definition 104. *For all $g \geq 1$ we define* Chain_g *inductively by:*

$$\text{Chain}_{g+1} = \text{mesh}(\text{Chain}_g).$$

By Lemma 96, we can prove by induction that for every g, Chain_g is a chain of depth g. With these chains, we can describe closed curves:

Definition 105. *For every $g \geq 1$ we define*

$$\text{curve}_g = \text{curve}(\text{Chain}_g).$$

Theorem 106. [3, Theorem 2] *For the Hurwitz Complex Continued Fraction, we have*

$$\left|\frac{q_{n+2}}{q_n}\right| \geq \frac{3}{2}.$$

Lemma 107. *There exists a simple closed curve in* $\text{HCF}_{\sqrt{5}}$ *which surrounds the square with the corners* $\pm\frac{1}{4} \pm \frac{1}{4}i$.

Proof. This will only be a concept of the proof. As a result of Lemma 97, Definition 86 and the fact that for every g, $\text{curve}_g(0) = \text{curve}_g(1)$, we have that for every g, curve_g is a closed curve. By Theorems 106 and 7 we have that for every $x, y \in \mathbb{C}$ with $x \equiv_{\restriction n} y$, we have:

$$|x - y| \leq \frac{\sqrt{2}}{\frac{3}{2}^{\lfloor \frac{n}{2} \rfloor}(3 - \sqrt{8})}.$$

With the use of induction and Lemma 102, we can see that for all $g, h \in \mathbb{N}$,

$$\text{curve}_g \equiv_{\restriction \min(g,h)-1} \text{curve}_h .$$

So, the sequence $(\text{curve}_g)_{g \in \mathbb{N}^+}$ is a Cauchy sequence, and we can define curve_∞ as

$$\text{curve}_\infty = \lim_{g \to \infty} \text{curve}_g .$$

First, we are going to show that for every $k \in N$,

$$\text{curve}_{k+1} \equiv_{\restriction k} \text{curve}_\infty .$$

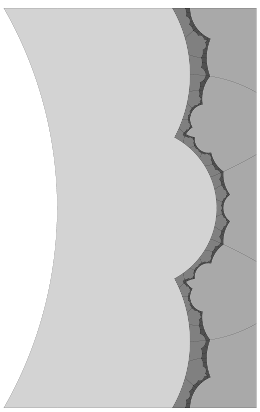
While we know that for all $k' > k$

$$\text{curve}_k \equiv_{\restriction k-1} \text{curve}_{k'},$$

this does not follow immediately. So we have quite some work to do:
Let $X_g = \langle [c_0^g; c_1^g, \ldots, c_k^g], [p^g, q^g], T_p^g, T_q^g \rangle$ be a building block in Chain_k, let Y be the set of all points $y \in \text{HCF}$ such that $y \equiv_{\restriction k-1} [c_0^g; c_1^g, \ldots, c_k^g]$. We will use pictures to show that for every $k' \geq k+1$, for every t in (p^g, q^g), $\text{curve}_{k'}(t)$ is not near the edge of Y. We can formulate this as:

$$\exists_\epsilon \forall_{t \in (p^g, q^g)} \forall_\delta : |\delta| < \min(t - p^g, q^g - t) \cdot \epsilon \implies \text{curve}_{k'}(t) + \delta \equiv_{\restriction k-1} [c_0^g; c_1^g, \ldots, c_k^g].$$

For every type of building block $\lfloor X, T_p, T_q \rceil$, we show the confinements of $\text{curve}(\text{mesh}^n(\langle [0], [0,1], T_p, T_q \rangle))$ while pretending $\text{Shape}_{\sqrt{5}}() = X$. A bigger value of n is represented by a darker gray.

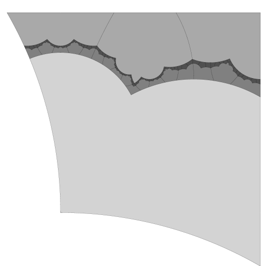


$\ll M_{\text{HCF}}, L_R, E_R \gg$

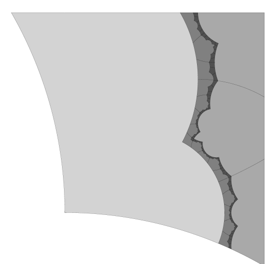
$\ll W_{\text{HCF}}, E_U, E_R \gg$

$\ll J_{\text{HCF}}, L_R, L_U \gg$

$\ll J_{\text{HCF}}, C_5, C_{16} \gg$



$\ll J_{\text{HCF}}, C_5, L_U \gg$



$\ll J_{\text{HCF}}, L_R, C_{16} \gg$

We continue with properties of curve_∞.

For every h, curve_{h+1} is h-bounded as the base of every building block is an element of $\text{HCF}_{\sqrt{5}}$, and we have Lemma 100. Let us assume $\text{curve}_\infty(t) = [c_0; c_1, \ldots] \notin \text{HCF}_{\sqrt{5}}$, so there exists a k such that $|c_k| > \sqrt{5}$. For every $k' > k$, $\text{curve}_{k'}(t)$ is k-bounded and

$$\text{curve}_{k'}(t) \equiv_{\restriction k} \text{curve}_\infty(t).$$

Contradiction, thus for every t

$$\text{curve}_\infty(t) \in \text{HCF}_{\sqrt{5}}.$$

Our mesh splits the interval of building blocks in at least three parts of the same size, except when the type is (a rotation or reverse of) $\ll J_{\text{HCF}}, L_R, L_U \gg$. In that case, the result is a chain of one building block, of type $\rho^3(\ll J_{\text{HCF}}, C_5, C_{16} \gg)^{-1}$. After performing two mesh functions, the individual intervals are at least divided in three pieces. From this follows that for each building block $X^g = \langle [c_0^g; c_1^g, \ldots, c_k^g], [p^g, q^g], T_p^g, T_q^g \rangle$ in Chain_k we have:

$$|q^g - p^g| < \left(\frac{1}{\sqrt{3}}\right)^k.$$

We use this to prove that curve_∞ is injective on $[0, 1)$.
Suppose $x, y \in [0, 1)$ with

$$x \neq y \text{ and } \text{curve}_\infty(x) = \text{curve}_\infty(y).$$

Let g be chosen such that

$$\left(\frac{1}{\sqrt{3}}\right)^g < |x - y|.$$

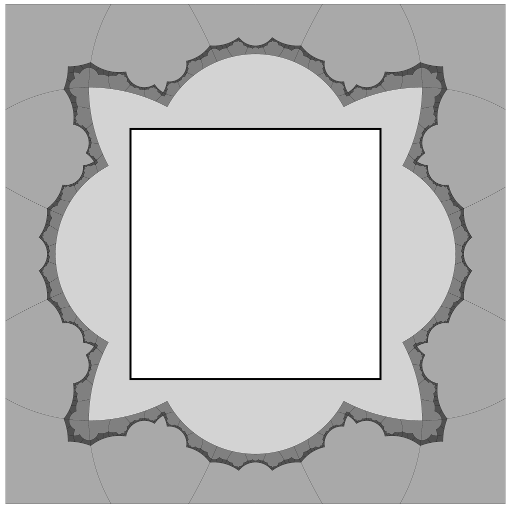
Let $\langle [c_0^k; c_1^k, \ldots, c_g^k], [p^k, q^k], T_p^k, T_q^k \rangle$ and $\langle [c_0^l; c_1^l, \ldots, c_g^l], [p^l, q^l], T_p^l, T_q^l \rangle$ in curve_g such that $x \in [p^k, q^k]$ and $y \in [p^l, q^l]$. Because of our choice of g, we know that $x \notin [p^l, q^l]$ and $y \notin [p^k, q^k]$. By the definition of a Chain, we know there exists a $h \leq g$ such that $c_h^k \neq c_h^l$. Thus for each $g' > g$ we have

$$\text{curve}_{g'}\, x \equiv_{\restriction g} \text{curve}_{g+1}\, x \not\equiv_{\restriction g} \text{curve}_{g+1}\, y \equiv_{\restriction g} \text{curve}_{g'}\, y.$$

This results to

$$\text{curve}_\infty\, x \not\equiv_{\restriction g} \text{curve}_\infty\, y,$$

which leads to a contradiction. We conclude that curve_∞ is injective on $[0, 1)$.
Because the sequence curve_g converged uniformly, curve_∞ is continuous, and $\text{curve}_\infty(0) = \text{curve}_\infty(1)$, from which we can conclude curve_∞ is a simple closed curve.
We use a picture to show that curve_∞ surrounds the square with corners $\pm\frac{1}{4} \pm \frac{1}{4}i$.



∎

Theorem 108. *For every $x = a + bi \in \mathbb{C}$, there exist $c, d \in \mathrm{HCF}_{\sqrt{5}}$ such that $c + d = x$*

Proof. Let $a' = \lfloor a \rceil$ and $b' = \lfloor b \rceil$, then $a' + b'i \in \mathbb{Z}[i]$. Let $x' = x - (a' + b'i)$, then $\frac{x'}{2}$ lies inside the square with corners $\pm\frac{1}{4} \pm \frac{1}{4}i$ and thus inside curve_∞ by Lemma 107.
By Lemma 67, there exist $t_c, t_d \in [0, 1]$ such that

$$\mathrm{curve}_\infty(t_c) + \mathrm{curve}_\infty(t_d) = x'.$$

Because for every $t \in [0, 1], \mathrm{curve}_\infty(t) \in \mathrm{HCF}_{\sqrt{5}}$, we let

$$c = a' + b'i + \mathrm{curve}_\infty(t_c) \in \mathrm{HCF}_{\sqrt{5}}$$

and

$$d = \mathrm{curve}_\infty(t_d) \in \mathrm{HCF}_{\sqrt{5}}\,.$$

Then $c + d = a' + b'i + x' = x$ concludes our proof. ∎

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