# Steinberg's Conjecture

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**Abstract**

*Every planar graph without* 4*-cycles and* 5*-cycles is* 3*-colorable.* This is Steinberg's conjecture, proposed in 1976 by Richard Steinberg. It is listed as an unsolved problem in T.R. Jensen and B. Toft's Graph Coloring Problems (1995). This article provides a collection of references and research journals that analyze this open problem in Graph Theory. Comments on its history, related results and literature are given. This project is part of the course Graph Theory, given by Wieb Bosma at Radboud University Nijmegen.

## 1 Introduction

Graph Coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring. The most famous long-standing mathematical coloring problem is the *Four Color Theorem*, conjectured in 1852 by Francis Guthrie. Its the first problem to be resolved using a computer program.

**Theorem 1.1.** *(Four Color Theorem) The regions of any simple planar map can be colored with only four colors, in such a way that any two adjacent regions have different colors.* [5]

Regarding planar graphs, intuitively the number 4 in Thereom 1.1 seems well-chosen. Whereas the complete graph $K_5$ is nonplanar, $K_4$ is planar but not 3-colorable. This is also true for any wheel whose center point has odd degree. Further, any cycles of odd length are not 2-colorable. Thus the remaining problem for coloring planar graphs is: Which planar graphs can be 3-colored?

Determining which planar graphs are 3-colorable leads to the search for suitable properties. The smallest number of colors needed to color a graph $G$ is called its chromatic number, often denoted by $\mathcal{X}(G)$. Brooks' theorem (1941) states a relationship between the maximum degree $\Delta$ of a graph and its chromatic number.

**Theorem 1.2.** *For any connected (undirected) graph $G$ with maximum degree $\Delta$, the chromatic number of $G$ is at most $\Delta$ unless $G$ is a clique or an odd cycle, in which case the chromatic number is $\Delta + 1$.* [1]

**Remark** A clique in a graph is a subset of its vertices such that every two vertices in the subset are connected by an edge.

A direct result by Theorem 1.2 implies that every connected graph $G$ where every vertex is adjacent to at most 3 vertices is 3-colorable, unless $G$ is $K_4$. Further properties for 3-colorable graphs follow by Grötzsch (1959) and Grünbaum (1963).

**Theorem 1.3.** *Every triangle-free planar graph can be colored with only three colors.* [2]

**Theorem 1.4.** *Any planar graph with fewer than 4 triangles is 3-colorable.* [3]

**Remark** An example for a graph which has four triangles but is not 3-colorable is given by Aksionov and Melnikov (1980). [6]

Another approach to determine 3-colorable graphs implies the question whether planar graphs without 4-cycles can be 3-colored. For example, $K_4$ contains a cycle of length 4 and is not 3-colorable. Unfortunately, not every planar graph without 4-cycles is 3-colorable.

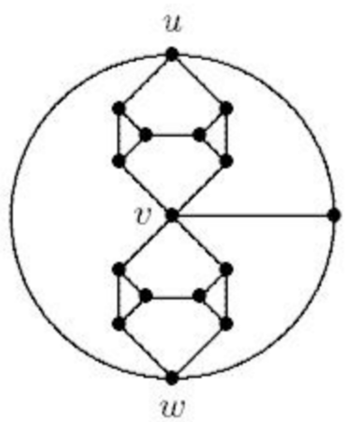


Figure 1: A planar graph without 4-cycles that cannot be 3 -colored.

**Example 1.5.** In Figure 1 the graphs which have been inserted between the verices $u$ and $v$ and between the vertices $v$ and $w$ simulate the edges $uv$ and $vw$. That is, if one 3-colors this subgraph, the vertices marked $u$ and $v$ will receive the same color.

The next question is whether a graph without 4-cycles and 5-cycles is 3-colorable. This is Steinberg's Conjecture, proposed by Richard Steinberg (1976) and listed as an unsolved problem in Jensen and Toft's *Graph Coloring Problems* (1995) [7]. The graph in Figure 1 is used generally to illustrate Steinberg's Conjecture.

# 2 From Steinberg to Borodin

In the past twenty years *Steinberg-type* graphs have been studied in several papers. Borodin and Raspaud's *A sufficient condition for planar graphs to be 3-colorable* (2003) proposes a conjecture which implies Steinberg's Conjecture. The following articles and theorems show the progress towards Borodin and Raspaud.

The famous Hungarian mathematician Paul Erdös (1991) [8] suggests to determine the smallest value of $k$ (if it exists) such that a planar graph without any cycles of length $4, 5, \ldots, k$ can be 3-colored. This extension of Steinberg's Conjecture is constrained by Abbott and Zhou (1991).

**Theorem 2.1.** *Every planar graph without cycles of length* $4, 5, \ldots, 11$ *is* 3*-colorable.* [9]

Further limits on the cycle length $k$ are given by Sanders and Zhao (1995), respectively Salavatipour (2002).

**Theorem 2.2.** *A planar graph without i-circuits,* $4 \leq i \leq 9$*, is* 3*-colorable.* [10]

**Theorem 2.3.** *Any planar graph without cycles of size* $4 \leq k \leq 8$ *is* 3*-colorable.* [11]

**Remark** Proofs of Theorem 2.2 and Theorem 2.3 make use of the term *faces*. For every plane graph $G$, the set $\mathbb{R}^2 \backslash G$ is open; its regions are called the faces of $G$. Thus, standard methods of surface topology are included in these proofs.

The Russian mathematician Borodin contributes several articles on Steinberg's Conjecture. In 1996, he first proved that $k = 10$ is suitable [12] and independently of Sanders and Zhao improves this result to $k = 9$ [13]. His proof is based on the following proposition.

**Proposition 2.4.** *Let $G$ be a planar graph with minimum degree* 3 *such that no two triangles have an edge in common. Then,*

1. *there are two adjacent vertices with degree sum at most* 9*.*

2. *there is a face of size between* 4 *and* 9 *or a* 10*-face incident with ten* 3*-vertices.* [13]

Xu (2006) strengthens Proposition 2.4 from *distance less than 4* to *distance less than 3*. [16]

**Corollary 2.5.** *It follows that every planar graph without cycles between* 4 *and* 9 *is* 3*-colorable.* [13]

Morover Borodin and Raspaud (2003) prove that planar graphs without 3-cycles at distance less than 4 and without 5-cycles are 3-colorable. Their corporation turns out in the *Strong Bordeaux Conjecture* that implies Steinberg's Conjecture.

**Conjecture 2.1. Strong Bordeaux Conjecture**
*Every planar graph without* 5*-cycles and without adjacent triangles is* 3*-colourable.* [14]

The term *Adjacent triangles* denotes two triangles which have at least one vertex in common.

**Remark** In this conjecture, none of the two assumptions can be dropped because there exist planar 4-chromatic graphs without 5-cycles, as well as planar 4- chromatic graphs without intersecting triangles. [14]

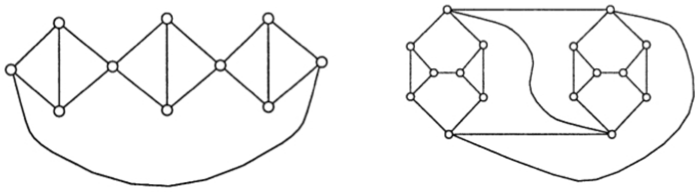

Figure 2: 4-chromatic graphs

In corporation Borodin, Glebov, Raspaud and Salavatipour (2005) strengthen the Bordeaux Conjecture. First, they improve Theorem [11] to $k = 7$. They denote such graphs by $\mathcal{G}_7$.

**Theorem 2.6.** *Any planar graph without cycles of length from* 4 *to* 7 *are* 3*-colorable.* [15]

By use of Theorem 2.6 and the fact that every planar graph without triangles adjacent to cycles of length from 3 to 9 is 3-colourable Borodin et al. propose the following conjecture.

**Conjecture 2.2. Strengthened Bordeaux Conjecture**
*Every planar graph without* 3*-cycles adjacent to cycles of length* 3 *or* 5 *is* 3*-colourable.* [15]

An alternative to Conjectures 2.1 and 2.2 by Chen and Wang (2007) proves that every planar graph without $4, 6, 8$-cycles is 3-colorable [18].
There are many other articles with conditions on 3-colorability of planar graphs: Bu et al. (2009) without $4, 7, 9$-cycles [24] or Borodin et al. (2009) without 5- and 7-cycles without adjacent triangles [25]. Others are [19], [20], [21], [22] or [23].

The most recent approach on the 3-coloring problem is given in *Steinberg's Conjecture and near-colorings* by Chang et al. (2011). Here, Steinberg's conjecture is approached through *near colorings*.

**Definition 2.1.** A graph $G = (V, E)$ is said to be $(i, j, k)$-colorable if its vertex set can be partitioned into three sets $V_1, V_2, V_3$ such that the graphs $G_{V_1}, G_{V_2}, G_{V_3}$ induced by the sets $V_1, V_2, V_3$ have maximum degree at most $i, j, k$ respectively.

Under this terminology, Steinberg's Conjecture says that every Steinberg-type graph is $(0, 0, 0)$-colorable. A result of Xu (2007) implies that every Steinberg-type graph is $(1, 1, 1)$-colorable [17]. Chang et al. prove the following.

**Theorem 2.7.** *Every Steinberg-type graph is* $(2, 1, 0)$*-colorable and* $(4, 0, 0)$*-colorable.* [26]

More illustrations are provided by recent presentation from Raspaud (2013) `http://www.math.uiuc.edu/~lidicky/seminar/2013-raspaud.pdf`.

# 3 Colorability or Choosability

The problem of deciding whether a planar graph is 3-colorable is NP-complete. [4] Some sufficient conditions for planar graphs to be 3-colorable were stated in Section 2, in particular Steinberg's Conjecture. Further, the conjecture has been generalized to list coloring. This provides a different point of view to analyze 3-coloring.

**Definition 3.1.** A list coloring of the graph $G$ is an assignment of colors to $V(G)$ such that each vertex $v$ receives a color from a prescribed list $L(v)$ of colors and adjacent vertices receive distinct colors. $L(G) = (L(v)|v \in V(G))$ is called a color list of $G$. The graph G is called $k$-choosable if $G$ admits a list coloring for all color lists $L$ with $k$ colors in each list.

**Example 3.1.** A graph $G$ is 3-choosable if, for every assignment of sets with 3 elements to the vertices, there is a proper coloring where the vertex $v$ is coloring using a color in its set. Hence 3-choosability implies 3-colorability; just let every set be $\{1, 2, 3\}$.

The corresponding conjecture with list coloring states that every planar graph without 4- and 5-cycles is 3-choosable. Unfortunately Voigt (2007) shows in Figure 3 that there is no direct translation from 3-colorability to 3-choosability. There exist planar graphs without 4- and 5-cycles which are not 3-choosable.

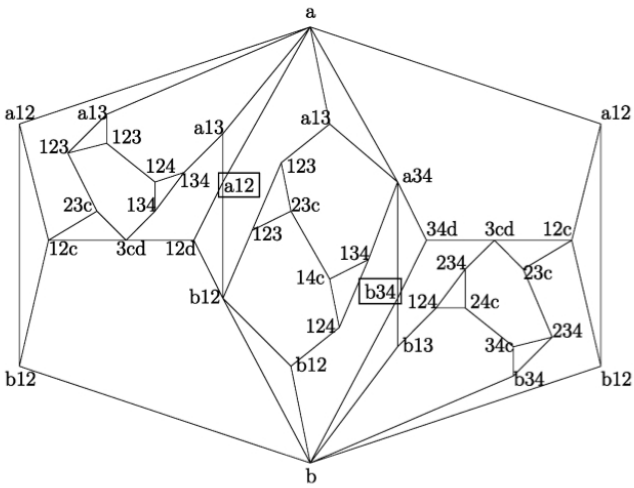


Figure 3: Voigt's counterexample [31]

Earlier studies on list coloring are done by Zhang and Wu (2004, 2005). They find connections from 3-choosability to 3-colorability.

**Theorem 3.2.** *Any planar graph without* 4-*,* 5-,7- *and* 9-*cycles is* 3-*choosable.* [27]

**Theorem 3.3.** *Any planar graph without* 4-*,* 5-,6- *and* 9-*cycles is* 3-*choosable.* [28]

**Remark** Theorem 3.2 and 3.3 emphasize that 3-colorability cannot be just so easily replaced by 3-choosability in Steinberg's Conjecture without further restrictions.

Another connection is given in Lam et al. (2005). This theorem differs from Steinberg's Conjecture and excludes 3-cycles instead of 4-cycles.

**Theorem 3.4.** *Any planar graph without* 3-*,* 5- *and* 6-*cycles is* 3-*choosable.* [29]

The closest attempt to combine colorability and choosability is done by Montassier, Raspaud and Wang (2006). It provides three proven statements.

**Theorem 3.5.** 1. *A graph with no* 4-*, or* 5-*cycles, and no triangles at a distance of less than* 4*, is* 3-*choosable.*

2. *A graph with no* 4-*,* 5-*, or* 6-*cycles, and no triangles at a distance of less than* 3*, is* 3-*choosable.*

3. *There is a graph without* 4- *or* 5-*cycles and without intersecting triangles, which is not* 3-*choosable.*

[30]

# 4 Summary

Steinberg's Conjecture has been studied for the past 35 years. There is neither a rejection of its statement nor a fully accepted proof. The proof by Cahit (2007) using *spiral colorings* is not accepted[32].
The steps from Steinberg's via Bordeaux till the route to 3-choosability shows nicely how thorough a "simple statement" can be analyzed, but still cannot fully be proven.

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