Toric Geometry

An introduction to toric varieties with an outlook towards toric singularity theory

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Introduction

Toric geometry lies in the overlap of algebraic geometry and polyhedral geometry. Algebraic geometry is known as a fascinating but also a technical and 'hard' area of mathematics. It takes some work to develop the technical machinery and the commutative algebra that is used in algebraic geometry. Polyhedral geometry on the other hand is more friendly. The combinatorics and discrete geometry is easier to learn. We have to 'languages' at our disposal:

- In algebraic geometry the building blocks are the *affine varieties*. These can be glued together to give an *algebraic variety*.
- In polyhedral geometry the building blocks are *cones*. A composition of polyhedral cones becomes a *fan*.

In toric geometry we relate the gluing of varieties with the composition of cones. Using the more friendly language of polyhedral geometry we can view toric geometry as an inviting and charming part of algebraic geometry. As noted by Cox [2], "the concreteness of toric varieties provides an excellent context for someone encountering the powerful techniques of modern algebraic geometry for the first time".

The objects of study in toric geometry are **toric varieties**. These are geometric objects defined by combinatorial information. We shall define the *affine toric variety of a cone*. When we have a fan consisting of cones, we can define the *toric variety of a fan* by gluing together the affine toric varieties of the cones in the fan. We study these varieties as we would in algebraic geometry.

Toric varieties are called 'toric' because they are equipped with a 'torus action'. By a torus we mean the linear algebraic group $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$, not the torus from topology. A toric variety contains a torus as an open subset and this defines the *torus action*. We study the torus action to understand what the toric variety looks like. At the heart of this lies the Orbit-Cone correspondence (Section 3.2.2).

What makes toric varieties 'sexy' is that they provide an elementary viewpoint on the basic concepts of algebraic geometry. We can take statements from algebraic geometry and and look for their counterpart in polyhedral geometry, i.e. we look for a combinatorial interpretation. These combinatorial statements are surprisingly simple and this is what makes toric varieties so appealing. The interplay of algebraic and polyhedral geometry is a recurring theme in this thesis.

Literature. Toric varieties were first studied in the 1970's, back then they were referred to as 'toroidal embeddings' [12]. Our main reference is the recent book by Cox et. al. from 2011 [2]. This is a comprehensive introduction to the theory. It assumes only a modest background and "leads to the frontier of this active area of research". There is enough literature on toric geometry, there are three earlier texts which serve as the standard literature in the field.

- The standard book by Fulton [6] introduces toric varieties as an "elementary way to see many examples and phenomena in algebraic geometry".
- The book by Ewald [4] leans on the combinatorial side of the theory, it gives a thorough introduction to polyhedral geometry and builds the theory of toric varieties from there.
- Oda [13] takes a more analytic approach.

We follow the book by Cox et. al. Most of the examples and proofs presented in this thesis are taken from [2]. Throughout the thesis I do not mention this for every example or proof I take from from [2]. I tell a story about toric varieties and work towards the theory of toric resolutions. In the course of the story I use ideas from the literature, I do not claim that these are my original ideas.

Motivation for the project. This thesis is part of the research track of the mathematics' master program. As a student I am interested in algebraic geometry, I do have to admit that I tend to be overwhelmed by the powerful machinery that we encounter in this branch of mathematics, I find the subject both dazzling and fascinating. For my thesis project I wanted to go deeper into the world of algebraic geometry, at the same time I felt the need for a more intuitive approach. Discrete mathematics appeals to me for being a 'more intuitive' branch of mathematics. I informed Ben Moonen if he could point me to an area of algebraic geometry which involves combinatorics. I also asked if he would be available to supervise this project. Much to my satisfaction Ben helped me with this and introduced me to toric geometry.

Toric geometry can be treated as a charming invitation to algebraic geometry. It has given me a deeper understanding of concepts from algebraic geometry. I found that toric varieties are pleasing to work with.

Organization of the thesis. The thesis gives an introduction to the world of toric geometry. We focus on the interplay of algo-geometric concepts and their combinatorial counterparts. We are especially interested in singularity theory. Finding a resolution of singularities is difficult problem in algebraic geometry. We work towards 'toric resolutions', a method for resolving a singularity in a toric variety, such that the method lies within the realm of toric geometry. We give a brief overview of the topics that will be treated. The first chapters develop the basic theory of toric geometry.

(1) We begin by giving a brief overview of the concepts we need from algebraic geometry and polyhedral geometry;

(2) We define affine toric varieties. There are several definitions but we are mainly interested in constructing the affine toric variety of a polyhedral cone;

(3) We define the toric variety of a fan. We describe the relations between fans and varieties. We describe singularities of a toric variety and we give a description of a resolution of singularities within toric geometry;

(4) We describe the divisors of toric varieties. We show how to compute intersection numbers for ‘toric; divisors.

Now that we have set up a theory of toric varieties, we are ready to study toric resolutions.

(5) We give full details on the toric resolution in the case of a *surface*. We describe an algorithm that finds an optimal solution to this problem. There is a surprising relation to *continued fractions*;

(6) Finally, we discuss toric resolutions in higher dimensions.

1. Preliminaries

Before we get started on toric varieties we recall the definitions we need from algebraic geometry. We next describe how a semigroup can define an affine variety. Finally we give an introduction to polyhedral cones.

1.1 Affine varieties

We work in the space $\mathbb{C}^n$ equipped with the Zariski topology. We denote V for an affine variety in $\mathbb{C}^n$ and we denote $\mathbf{I}(V) \subset \mathbb{C}[x_1,\ldots,x_n]$ for its defining ideal. We have the coordinate ring $A(V) = \mathbb{C}[x_1,\ldots,x_n]/\mathbf{I}(V)$. These objects satisfy

$$V \text{ is irreducible } \leftrightarrows \mathbf{I}(V) \text{ is a prime ideal } \leftrightarrows A(V) \text{ is a domain}.$$

There is a one-to-one correspondence between points in a variety and maximal ideals of its coordinate ring.

Let V be an irreducible affine variety. Given a non-zero f in $A(V)$ we denote the localisaton of V at f:

$$D(f) = V_f = \{p \in V \mid f(p) \neq 0\}.$$

This is again an affine variety, its coordinate ring is $A(V)[1/f]$. For example with $V = \mathbb{C}^n$ we have

$$\mathbb{C}[t_1,\ldots,t_n]_{t_1\cdots t_n} = \mathbb{C}[t_1^{\pm 1},\ldots,t_n^{\pm 1}],$$

this is called the ring of *Laurent polynomials*. This construction is a useful way to get open affine subvarieties.

Let $p \in V$ and denote $\mathfrak{m} \subset A(V)$ for the corresponding maximal ideal. We say that p is a *smooth point* if $\dim(\mathfrak{m}/\mathfrak{m}^2) = \dim(V)$. A variety is called smooth if all of its points are smooth.

A ring is called *normal* if it is integrally closed. This means that it is closed under the adding of roots of monic polynomials. An irreducible affine variety V is normal when its coordinate ring $A(V)$ is normal.

For affine toric varieties we will derive combinatorial expressions to capture these properties. Normality has a very nice combinatorial interpretation and we will see in Chapter 3 that this leads to a nice theory of divisors.

Most of the toric varieties we encounter are normal. In particular, a smooth irreducible affine variety is normal.

1.2 Semigroups

A *semigroup* is a set S together with an associative operation + and a neutral element $0 \in S$. This differs from a group in that elements need not have an inverse. A semigroup is *integral* if S can be embedded as a subsemigroup in some lattice $\mathbb{Z}^m$. We say S is *finitely generated* if there exists a finite set $\mathscr{A}$ such that $S = \mathbb{N}\mathscr{A}$. By an *affine semigroup* we mean a finitely generated integral semigroup.

EXAMPLE 1.1. Observe that $(\mathbb{C}, +)$ is a semigroup which is not integral. Examples of affine semigroups are $(\mathbb{N}^n, +) = \mathbb{N}\{e_1, \ldots, e_n\}$ and $S = \{2, 3, \ldots\} = \mathbb{N}\{2, 3\}$.

Given a semigroup we get an associated *semigroup algebra* $\mathbb{C}[S]$. It is generated by elements χ^u indexed by elements $u \in S$. The semigroup operation + induces the multiplication of the χ^u in $\mathbb{C}[S]$, thus $\chi^u \cdot \chi^v = \chi^{u+v}$.

EXAMPLE 1.2. To $\mathbb{N}^n$ we associate $\mathbb{C}[t_1, \ldots, t_n]$, where we denote $t_i = \chi^{e_i}$. To $\mathbb{Z}^n$ we associate $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$, the ring of Laurent polynomials. To $\{2, 3, \ldots\} \subset \mathbb{N}$ we associate $\mathbb{C}[x, y]/(x^3 - y^2)$

Observe that a morphism of semigroups induces a homomorphism of associated $\mathbb{C}$-algebras.

If S is an affine semigroup, then $\mathbb{C}[S]$ is a finitely generated domain, for the embedding $S \hookrightarrow \mathbb{Z}^m$ induces an injective homomorphism $\mathbb{C}[S] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and this is a domain. Hence for an affine semigroup we can take the spectrum $\mathrm{Spec}(\mathbb{C}[S])$ to get an affine variety.

EXAMPLE 1.3. The affine semigroup $(\mathbb{N}^n, +)$ produces the variety $\mathrm{Spec}\,\mathbb{C}[t_1, \ldots, t_n] = \mathbb{C}^n$. Likewise $S = \{2, 3, \ldots\} \subset \mathbb{N}$ defines the affine variety $\mathscr{Z}(x^3 - y^2)$, a zero locus.

Points in affine varieties A point of $\mathrm{Spec}(\mathbb{C}[S])$ is given by a $\mathbb{C}$-algebra homomorphism $f : \mathbb{C}[S] \to \mathbb{C}$. This corresponds to a semigroup morphism $\varphi \colon S \to (\mathbb{C}^*, \cdot)$ that satisfies $u \mapsto f(\chi^u)$.

REMARK 1.4. Let p be a point of $\mathrm{Spec}(\mathbb{C}[S])$ and let $\varphi \colon S \to (\mathbb{C}^*, \cdot)$. Then p corresponds to φ when $\chi^u \in \mathbb{C}[S]$ sends p to $\varphi(u)$.

Given a point $p \in V$, we define the corresponding semigroup morphism $S \to \mathbb{C}$ by sending $m \mapsto \chi^m(p)$.

Given a semigroup morphism $\varphi \colon S \to \mathbb{C}$ we get an induced surjective map $\mathbb{C}[S] \to \mathbb{C}$. The kernel of this map is a maximal ideal and corresponds to a point. This extends the earlier correspondence of points and maximal ideals:

PROPOSITION 1.5. *Let* $V = \mathrm{Spec}(\mathbb{C}[S])$ *be the variety corresponding to the semigroup* S*. Then we have bijective correspondence between*

- *Points* $p \in V$.

- *Maximal ideals* $\mathfrak{m} \subset \mathbb{C}[S]$.
- *Semigroup homomorphisms* $S \to \mathbb{C}$.

□

1.3 Polyhedral cones

A *convex polyhedral cone* is a convex subset of a real vector space. The origin will be the apex of the cone. A finite set of vectors can define a cone.

To see how this works we first take a single vector v. Then we can take the *ray*, or halfline, through v defined to be $\{\lambda v \mid \lambda \in \mathbb{R}, \lambda \geq 0\}$.

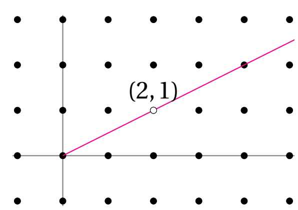

Figure 1: The ray in $\mathbb{R}^2$ given by $(2,1)$.

A finite set of vectors gives a collection of halflines. By taking the convex hull of these lines we obtain a cone. Figure 2 below illustrates how four vectors describe a cone in $\mathbb{R}^3$.

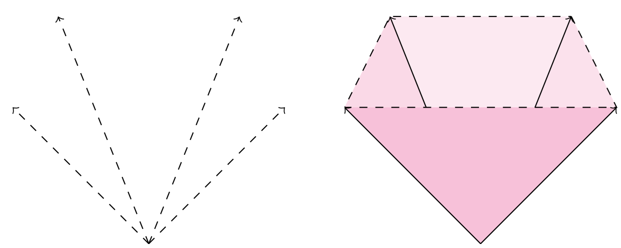

Figure 2: A convex hull spanned by four rays.

Let ρ be a ray in $\mathbb{R}^n$. Then ρ is *rational* if there is a point $p \in \mathbb{Z}^n$ such that ρ is the ray through p. Given a rational ray ρ consider the set of rational points $\rho \cap N$. Then there is a smallest lattice point along the ray other than the origin, this is called the *ray generator* of ρ. In Figure 1 the ray generator is $(2,1)$. We say that a cone in is rational if it can be spanned by rational rays. So a rational cone in $\mathbb{R}^n$ is defined by a set of points in $\mathbb{Z}^n$. The cones we come across are all assumed to be rational. So when we work with cones we may actually work with lattice points.

Any lattice is isomorphic to $\mathbb{Z}^n$ for some n. A lattice N gives the vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, which is isomorphic to some $\mathbb{R}^n$. A finite set of points in a lattice defines a cone in the corresponding vector space. This cone is rational, and this extends the idea of a rational cone to vector spaces which have an underlying lattice. We give this general definition in Definition 1.6.

Throughout this thesis we denote N for a lattice. We denote M for its dual lattice, which we bring into use later on, and we have the usual pairing $\langle,\rangle\colon M \times N \to \mathbb{Z}$. We also have the dual vector spaces $N_{\mathbb{R}} = N \bigotimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \bigotimes_{\mathbb{Z}} \mathbb{R}$.

DEFINITION 1.6 (Convex polyhedral cone). For a finite set S in a lattice N we define the *convex polyhedral cone* $\sigma = \text{Cone}(S) = \left\{\sum_{u \in S} \lambda_u u \,\middle|\, \lambda_u \geq 0\right\} \subset N_{\mathbb{R}}$.

The *dimension* of a cone σ is the dimension of $\text{Span}(\sigma)$, the smallest vector space containing the cone.

EXAMPLE 1.7. Low-dimensional cones make for neat pictures. Figure 3 shows a 2-dimensional cone and Figure 4 shows a 3-dimensional cone.

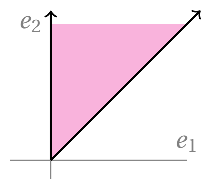

Figure 3: $\text{Cone}(e_1 + e_2, e_2)$

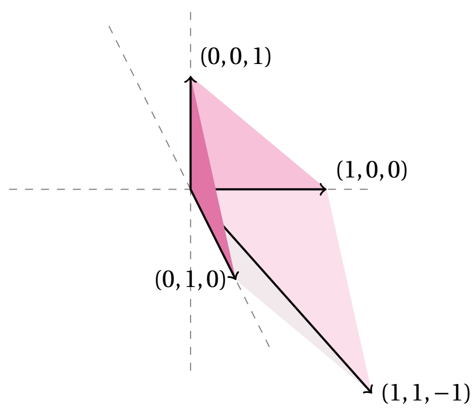

Figure 4: $\text{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3)$

DEFINITION 1.8 (Dual cone). Given a polyhedral cone $\sigma \subset N_{\mathbb{R}}$ we have the *dual cone*

$$\sigma^{\vee} = \left\{m \in M_{\mathbb{R}} \,\middle|\, \langle u, m\rangle \geq 0 \text{ for all } u \in \sigma\right\} \subset M_{\mathbb{R}},$$

so the dual cone sits inside the dual vector space. The dual of a convex polyhedral cone is also a convex polyhedral cone and we have $(\sigma^{\vee})^{\vee} = \sigma$.

EXAMPLE 1.9. For a cone in $\mathbb{R}^2$ its dual is also a cone in $\mathbb{R}^2$, as you can see in Figure 5 below.

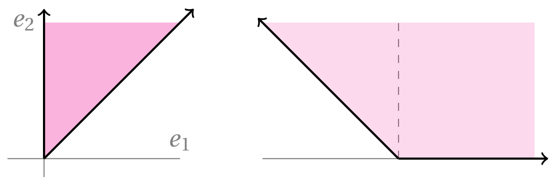

Figure 5: The cone from Figure 3 and its dual $\text{Cone}(e_1, e_2 - e_1)$

The dual of the cone in Figure 4 is also a cone in $\mathbb{R}^3$, namely $\text{Cone}(e_1, e_2, e_3 + e_1, e_3 + e_2)$. It is shown in in Figure 6.

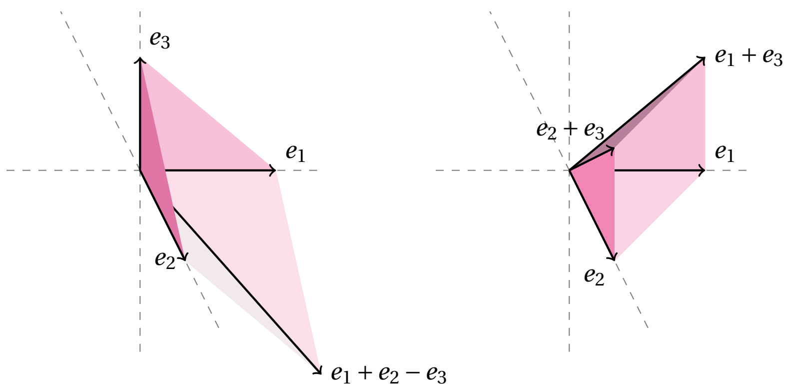

Figure 6: The cone from Figure 4 and its dual $\text{Cone}(e_1, e_2, e_3 + e_1, e_3 + e_2)$.

REMARK 1.10. When σ is a cone of maximum dimension, the dual cone $\sigma^\vee$ is also a cone of maximum dimension.

Faces. We will now discuss what it means for a cone to have *faces*. The cone in Figure 4 clearly has four 2-dimensional faces. Such a 2-dimensional face itself has two faces, namely the boundary rays. These rays are also faces of the 3-dimensional cone. Faces are described using hyperplanes. Let H_m denote the hyperplane given by a dual lattice point $m \in M$:

$$H_m = \{v \in N_{\mathbb{R}} \mid \langle v, m \rangle = 0\}$$

We get the closed half-space:

$$H_m^+ = \{v \in N_{\mathbb{R}} \mid \langle v, m \rangle \geq 0\}$$

A hyperplane $H_m \subset N_{\mathbb{R}}$ *supports* a cone $\sigma \subset N_{\mathbb{R}}$ if H_m^+ fully contains σ, we then call H_m^+ a supporting half-space. Note that H_m supports σ if and only if $m \in \sigma^\vee$. For example, in Figure 4 we see that the hyperplanes defined by the origin, e_1 and $e_1 + e_2 + e_3 \in M$ are all supporting hyperplanes, while H_{e_3} is not as $e_3 \notin \sigma^\vee$. Furthermore, we can describe every polyhedral cone $\sigma \subset N_{\mathbb{R}}$ as an intersection of finitely many closed half-spaces. If $m_1, \dots, m_s$ generate $\sigma^\vee$, then it is easy to check that

$$\sigma = H_{m_1}^+ \cap \ldots \cap H_{m_s}^+.$$

For example, taking the generators of the dual cone in Figure 6 we get

$$\sigma = H_{e_1}^+ \cap H_{e_2}^+ \cap H_{e_3+e_1}^+ \cap H_{e_3+e_2}^+.$$

DEFINITION 1.11 (Face). A face of a cone is the intersection of the cone with a supporting hyperplane. We use the notation $\tau \prec \sigma$ when τ is a face of σ.

A cone is a face of itself, it is the intersection with H_0. An *edge* of a cone σ is a face of dimension 1, so an edge is a ray contained in σ which is also face of σ. A *facet* of a cone is a face of codimension 1. For example, the cone in Figure 4 has 10 faces: the cone itself, four facets, four edges and the origin.

We can also give an algebraic description of when a convex subset of a cone is a face.

PROPOSITION 1.12. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone and let $\tau \subset \sigma$ be a convex subset. Then τ is a face if and only if whenever $u, v \in \sigma$ satisfy $u + v \in \tau$ then both $u \in \tau$ and $v \in \tau$.*

Proof. For a proof we refer to [2, Lemma 1.2.7]. □

For a polyhedral cone σ and its dual cone $\sigma^\vee$ we can relate their faces. Given a face $\tau \preceq \sigma \subset N_{\mathbb{R}}$, we define

$$\begin{aligned}\tau^\perp &= \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\} \\ \tau^* &= \{m \in \sigma^\vee \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\} \\ &= \sigma^\vee \cap \tau^\perp.\end{aligned}$$

We call τ^* the dual face of τ because of the following proposition.

PROPOSITION 1.13. *Let σ be a cone and $\sigma^\vee$ its dual. we can relate the faces. The map $\tau \mapsto \tau^*$ is an order-reversing bijection between the faces of σ and the faces of $\sigma^\vee$.*

Proof. For a proof we refer to [2, Prop. 1.2.10]. □

Strongly convex cone. A convex polyhedral cone σ is called *strongly convex* if $\{0\}$ is a face of σ, or equivalently, if $\sigma \cap (-\sigma) = \{0\}$ so σ does not contain any line through the origin. A strongly convex rational cone is always generated by the ray generators of its edges.

From now on we simply write *cone* instead of strongly convex rational polyhedral cone.

The variety of a cone. Given a cone $\sigma \subset N_{\mathbb{R}}$ we have the dual cone $\sigma^{\vee} \subset M_{\mathbb{R}}$. By *Gordan's lemma* [6, Prop. 1] $S_{\sigma} = \sigma^{\vee} \cap M$ is finitely generated and hence an affine semigroup. This way we may obtain an affine variety from a polyhedral cone, we refer to it as the variety $U_{\sigma} = \mathrm{Spec}(\mathbb{C}[S_{\sigma}])$.

EXAMPLE 1.14. Looking back at the cone from Figure 5 we see that $S_{\sigma} = \sigma^{\vee} \cap M$ is generated by e_1 and $e_2 - e_1$. Writing $x = \chi^{e_1}$ and $y = \chi^{e_2}$ we obtain $\mathbb{C}[S_{\sigma}] = \mathbb{C}[x, x^{-1}y]$. Taking the spectrum we get $U_{\sigma} = \mathbb{C}^2$.

2. Affine toric varieties

In the Introduction we mentioned the origin of the name *toric variety*. A torus is a particular linear algebraic group. A toric variety contains a torus as an open subset and is equipped with a *torus action*. We now make this precise.

DEFINITION 2.1. A linear algebraic group T is a torus if it is isomorphic to some $(\mathbb{C}^*)^n$. The torus T inherits the action of $(\mathbb{C}^*)^n$ on itself. This action on $(\mathbb{C}^*)^n$ is given by pointwise multiplication.

DEFINITION 2.2 (Affine toric variety). An irreducible affine variety $U \subset \mathbb{C}^n$ is *toric* if it contains a torus $T \simeq (\mathbb{C}^*)^n$ as Zariski open subset such that the action of T on itself extends to an algebraic action of T on U.

EXAMPLE 2.3. Clearly $(\mathbb{C}^*)^n$ itself is toric, as well as $\mathbb{C}^n$. Also $V = \mathcal{Z}(x^3 - y^2)$ is toric, with torus embedding $\mathbb{C}^* \hookrightarrow V$ given by $t \mapsto (t^2, t^3)$. The torus action of $\mathbb{C}^*$ on V is given by $t \cdot (u, v) \mapsto (t^2 u, t^3 v)$.

EXAMPLE 2.4. For a slightly bigger example, consider $\widehat{C}_d$, the rational normal cone of degree d. This is the image of the map $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^{d+1}$ given by

$$(s, t) \mapsto (s, st, \ldots, st^d).$$

The torus action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$ carries over to $\widehat{C}_d$.

2.1 Constructing affine toric varieties

We now go through the several ways for an affine toric variety to arise. The construction we will come back to most defines the variety corresponding to a cone.

The toric variety of a cone. We have seen how to get an affine variety U_σ from a (strongly convex rational polyhedral) cone σ. This is in fact a toric variety. Let n denote the rank of the lattice $\mathbb{Z}S_\sigma$. Then there is a torus $T \simeq (\mathbb{C}^*)^n$ acting on U_σ.

DEFINITION 2.5 (The torus corresponding to a lattice). For a lattice $N \simeq \mathbb{Z}^n$ we get a *torus* $T_N \simeq (\mathbb{C}^*)^n$. So the rank of the lattice equals the dimension of the torus.

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \mathrm{Hom}_{\mathbb{Z}}\left(M, C^*\right) \tag{2.1.1}$$

REMARK 2.6. When we view $\{0\}$ as a cone in $N_\mathbb{R}$, the affine semigroup $S_{\{0\}} = M$ produces the variety $\mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}_{\mathbb{Z}}(M, C^*)$. This gives a description of the torus as the variety corresponding to a cone

$$T_N = U_{\{0\}}.$$

REMARK 2.7. For a cone $\sigma \subset N_{\mathbb{R}}$ the variety U_σ is actually a toric variety and it contains the torus T_N as open subset. As we shall see in Proposition 2.13, the fact that $\{0\} \prec \sigma$ implies that $U_{\{0\}} = T_N$ is indeed an open subset. Let us describe the way in which the torus acts on U_σ. For $t \in T_N$ and $\gamma\colon S_\sigma \to \mathbb{C}$ a point in U_σ we get

$$t \cdot \gamma\colon S_\sigma \to \mathbb{C}, \quad m \mapsto \chi^m(t)\gamma(m). \tag{2.1.2}$$

Further constructions. We've seen how affine semigroups, of which the S_σ are special cases, can produce affine toric varieties. We discuss two more ways for an affine toric variety to arise.

In general an affine variety in $\mathbb{C}^n$ can be defined by an ideal in $\mathbb{C}[x_1,\dots,x_n]$. An affine toric variety can be defined by a *toric ideal*.

DEFINITION 2.8 (Toric ideal). A toric ideal in $\mathbb{C}[x_1,\dots,x_n]$ is an ideal generated by binomials, meaning polynomials with precisely two non-zero coefficients.

For instance the ideal $\langle x^3 - y^2\rangle \subset \mathbb{C}[x, y]$, occuring in Example 1.3, is a toric ideal.

The last construction for an affine toric variety that we present is via lattice points. For a finite set of points $\mathscr{A} = \{m_1,\dots,m_s\}$ in the dual lattice M we obtain characters $\chi^{m_i}\colon T_N \to \mathbb{C}^*$. These induce a map $\Phi_{\mathscr{A}}\colon T_N \to \mathbb{C}^s$ given by $t \mapsto \big(\chi^{m_1}(t),\dots,\chi^{m_s}(t)\big)$. We define the affine variety $V = Y_{\mathscr{A}}$ to be the Zariski closure of $\operatorname{im}(\Phi_{\mathscr{A}})$.

We have the following result:

THEOREM 2.9. *Let V be an affine variety. The following are equivalent:*

(1) *V is toric, in the sense of Definition 2.2.*
(2) $V = \operatorname{Spec}(\mathbb{C}[S])$ *for an affine semigroup S.*
(3) $V = V(I)$ *for a toric ideal I.*
(4) $V = Y_{\mathscr{A}}$ *for a finite set of dual lattice points $\mathscr{A}$.*

Proof. For a proof we refer to [2, Theorem 1.1.17]. □

We provide some examples to exhibit the various ways for a toric variety to arise.

EXAMPLE 2.10. Consider $\sigma = \operatorname{Cone}(e_2, 2e_1 - e_2)$ in Figure 7. We see that S_σ is generated by $(1,0)$, $(1,1)$ and $(1,2)$ and we obtain a toric variety with coordinate ring $\mathbb{C}[S_\sigma] = \mathbb{C}[x, xy, xy^2]$.

So we have presented U_σ as in (2) from Theorem 2.9. We now describe U_σ in other ways. (1) We can identify $\mathbb{C}^2$ with U_σ via the map $(s,t) \mapsto (s, st, st^2)$. In this way the torus $(\mathbb{C}^*)^2$ acts on the variety. (3) The variety can be described by the ideal $\langle tv - u^2\rangle$ in $\mathbb{C}[t,u,v]$, which is a toric ideal. (4) The dual lattice points $(1,0)$, $(1,1)$ and $(1,2)$, which generate S_σ, also describe U_σ.

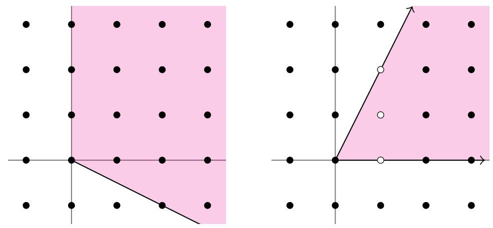

Figure 7: The cone σ and its dual $\text{Cone}(e_1, e_1 + 2e_2)$

EXAMPLE 2.11. We return to the rational normal cone in Example 2.4, which we've seen to be toric in the sense of (1). We defined it as the image of $\Phi(s,t) = (s, st, \dots, st^d)$.

We again describe $\widehat{C}_d$ in other ways. (4) The image of Φ is precisely the affine toric variety defined by the lattice points $(1,0), (1,1), \dots, (1,d)$. (2) These lattice points define a cone which is dual to $\sigma = \text{Cone}(e_2, de_1 - e_2)$. So $\widehat{C}_d = \text{Spec}(\mathbb{C}[S_\sigma])$. (3) The toric ideal $\langle xz - y^d \rangle$ in $\mathbb{C}[x,y,z]$ also defines $\widehat{C}_d$.

Note that the cone from Example 2.10 is the rational normal cone $\widehat{C}_2$.

REMARK 2.12. Given a toric variety there is no unique cone to which the variety relates. The cones in Figure 8 both give the toric variety $\mathbb{C}[x,y,z]/\langle xz - y^2 \rangle$. These cones are seen to be equal to one another through a change of basis. We define a map by setting $(1,0) \mapsto (0,1)$ and $(0,1) \mapsto (-1,1)$, this sends $(2,1) \mapsto (1,1)$ and hence this maps the first cone to the second cone.

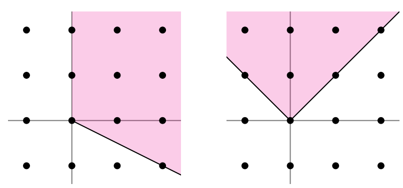

Figure 8: The cones $\text{Cone}(e_2, 2e_1 - e_2)$ and $\text{Cone}(e_2 + e_1, e_2 - e_1)$

Faces and varieties. Now that we have established the relation between cones and varieties, we investigate how the varieties of the faces of a cone relate to each other.

An inclusion of faces $\tau \subset \sigma$ gives an inclusion of semigroups $S_\sigma \subset S_\tau$, hence $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$, which induces a morphism $U_\tau \to U_\sigma$ of affine toric varieties. This morphism satisfies:

PROPOSITION 2.13. *The morphism $U_\tau \to U_\sigma$ is an open embedding if $\tau \preceq \sigma$.*

Proof. Let $\tau \preceq \sigma$ be an inclusion of faces and write $\tau = \sigma \cap H_m$, $m \in M$. Then the algebra of U_τ is the localisation of the algebra of U_σ at χ^m:

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$$

Hence U_τ is a principal open subset of U_σ. □

If $\tau \preceq \sigma$ then U_τ is a principal open subset of U_σ. For this reason we define a toric variety by the dual cone, the ordering of the cones is the same as the ordering on varieties.

EXAMPLE 2.14. Consider the basic cone $\sigma = \text{Cone}(e_1, e_2)$, with variety $U_\sigma = \mathbb{C}^2$. We have the faces $\rho_1 = \text{Cone}(e_1)$, $\rho_2 = \text{Cone}(e_2)$ and the origin. We can describe the faces with hyperplanes generated by e_2, e_1 and $e_1 + e_2$ respectively, obtaining

$$\mathbb{C}[S_{\rho_1}] = \mathbb{C}[x, y]_y = \mathbb{C}[x, y, y^{-1}],$$

as well as $\mathbb{C}[S_{\rho_2}] = \mathbb{C}[x, x^{-1}, y]$ and $\mathbb{C}[S_0] = \mathbb{C}[x, y, x^{-1}y^{-1}]$. These define the varieties $U_\sigma = \mathbb{C}^2$, $U_{\rho_1} = \mathbb{C}^* \times \mathbb{C}$, $U_{\rho_2} = \mathbb{C} \times \mathbb{C}^*$ and $U_0 = (\mathbb{C}^*)^2$. All of them are open subvarieties of $\mathbb{C}^2$.

2.2 Properties of affine toric varieties

In the world of toric geometry we have on one side the polyhedral geometry of the cones and on the other side we have the algebraic geometry of the varieties. In the present section we exhibit some relations between an affine toric variety and its defining cone. We wish to translate properties from the language of algebraic geometry into the language of polyhedral geometry. We begin with normality.

Normality. At the end of section 1.1 we mentioned that most of the toric varieties we encounter are normal. Normality corresponds to semigroups being saturated.

DEFINITION 2.15 (Saturated). An integral semigroup S in a lattice N is *saturated* if for any $s \in N$, whenever a multiple ks, for $(k \in \mathbb{N}_{\geq 0}$, lies in S, then s itself already lies in S.

The semigroup $\{2, 3, \ldots\} \subset \mathbb{N}$ from Example 1.3 is a semigroup which is not saturated. Given a cone $\sigma \subset N_\mathbb{R}$ the affine semigroup S_σ is saturated. In fact we have the following result:

PROPOSITION 2.16. *Let U be an affine toric variety. Then the following are equivalent.*

- *U is normal.*
- *U comes from a saturated affine semigroup*
- *U is the affine variety of a (strongly convex polyhedral) cone.*

Proof. For a proof we refer to [2, Theorem 1.3.5] □

The affine semigroup $\{2, 3, \ldots\}$ is not saturated and the corresponding affine toric variety $\mathcal{Z}(x^2 - y^3)$ is not normal. Hence this variety is not the variety of a cone.

Throughout the thesis we concern ourselves with toric varieties coming from cones, hence the varieties we encounter are all normal.

Smoothness. As we have seen at the end of Section 1.2, for affine varieties there is a correspondence between points, maximal ideals and semigroup homomorphisms.

For a cone σ in $N_{\mathbb{R}}$ we define a point $p_\sigma \in U_\sigma$ by the semigroup morphism $\gamma_\sigma \colon S_\sigma \to \mathbb{C}$, given by

$$\gamma_\sigma(m) = \begin{cases} 1 & \text{if } m \in \sigma^\perp \cap M \\ 0 & \text{else} \end{cases}. \tag{2.2.1}$$

We call p_σ, or γ_σ, the *distinguished point* of σ.

PROPOSITION 2.17. *If $\sigma \subset N_{\mathbb{R}}$ is a cone of maximum dimension, p_σ is the unique fixed point of the torus action and it is given by the maximal ideal*

$$\langle \chi^m \mid m \in S_\sigma \setminus \{0\} \rangle \subset \mathbb{C}[S_\sigma].$$

DEFINITION 2.18 (Hilbert basis). Let $\sigma \subset N_{\mathbb{R}}$ be a cone of maximal dimension. We define the *Hilbert basis*

$$\mathscr{H} = \{ m \in S_\sigma \mid m \text{ is irreducible} \}.$$

REMARK 2.19. The Hilbert basis is a finite set that generates S_σ. It contains the ray generators of the edges of $\sigma^\vee$.

The Hilbert basis will play a crucial role in describing the smooth points of an affine toric variety.

LEMMA 2.20. *Consider a cone of maximal dimension $\sigma \subset N_{\mathbb{R}}$. Then $|\mathscr{H}| = \dim T(p_\sigma)$, where $T(p)$ denotes the tangent space of U_σ at p.*

The lemma enables us to check whether the distinguished point is a smooth point, by only looking at the Hilbert basis rather than computing the actual tangent space.

Proof of the lemma. By definition $T(p_\sigma)$ is dual to $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m} \subset A(U_\sigma)$ is the maximal ideal corresponding to the point p_σ.

$$\mathfrak{m} = \{ f \in \mathbb{C}[S_\sigma] \mid f(p_\sigma) = 0 \}$$

By Remark 1.4 we have the identity $\chi^m(p_\sigma) = \gamma_\sigma(m)$, also, $\gamma_\sigma(m) = 0 \leftrightarrows m \neq 0$. Hence

$$\mathfrak{m} = \{ \chi^m \in \mathbb{C}[S_\sigma] \mid m \in S_\sigma \setminus \{0\} \}.$$

Consider $\mathfrak{m}^2$, which consists of elements of the form χ^{m+n} for $m, n \in S_\sigma \setminus \{0\}$, precisely the reducible elements in S_σ. Hence $\mathfrak{m}/\mathfrak{m}^2$ is generated by χ^m for $m \in \mathscr{H}$, the irreducible elements in S_σ. Thus $\dim(\mathfrak{m}/\mathfrak{m}^2) = |\mathscr{H}|$ and this proves the lemma. □

We are now ready to give the combinatorial interpretation of smoothness.

DEFINITION 2.21 (Smooth cone). We say that a cone σ is smooth if it is generated by (part of) a lattice basis.

THEOREM 2.22. *A cone σ is smooth if and only if the variety U_σ is smooth.*

Proof. We follow the proof of [2, Theorem 1.3.12] ($\Rightarrow$) We assume that σ is smooth, so $\sigma = \mathbb{N}e_1 + \ldots + \mathbb{N}e_r$ where $e_1, \ldots, e_r$ is part of a lattice basis $e_1, \ldots, e_n$. Then $S_\sigma = \mathbb{N}e_1 + \ldots + \mathbb{N}e_r + \mathbb{Z}e_{r+1} + \ldots + \mathbb{Z}e_n$. We get the coordinate ring $\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, \ldots, t_r, t_{r+1}^{\pm 1}, \ldots, t_n^{\pm 1}]$ and this corresponds to the smooth variety $\mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$.

($\Leftarrow$) For the other way around we assume that U_σ is smooth. We first deal with the case when σ has maximal dimension n.

In this case the dual cone $\sigma^\vee$ is strongly convex and $S_\sigma = \sigma^\vee \cap M$ has a Hilbert basis $\mathscr{H}$.

The variety U_σ is smooth in p_σ, so by Lemma 2.20 we have $|\mathscr{H}| = n$. Then $\sigma^\vee$ has at least n edges since $\dim \sigma^\vee = n$ by Remark 1.10. Also, $\sigma^\vee$ has at most $|\mathscr{H}| = n$ edges since by Remark 2.19 each edge of $\sigma^\vee$ is an element of $\mathscr{H}$. Hence $\sigma^\vee$ has exactly n edges thus the same holds true for σ. The ray generators for $\sigma^\vee$ are the elements of $\mathscr{H}$. This means that $\mathbb{Z}S_\sigma$ is generated by the n ray generators of $\sigma^\vee$, and these form a basis for M. So $\sigma^\vee$ is smooth and hence σ is smooth.

When $\dim \sigma = k < n$, then consider

$$N_\sigma = \operatorname{Span}(\sigma) \cap N.$$

We may split $N = N_\sigma \bigoplus N'$ and $\sigma = \sigma' \bigoplus \{0\}$. It suffices to show that σ' is smooth in N'. Decomposing $M = M' \bigoplus M''$ dually, we get a the relation $S_\sigma = S_{\sigma'} \bigoplus M''$, now if we take the varieties corresponding to the semigroups we obtain:

$$U_\sigma \simeq U_{\sigma'} \times T_{N''} \simeq U_{\sigma'} \times (\mathbb{C}^*)^{n-k}.$$

Smoothness of U_σ implies that $U_{\sigma'}$ is smooth. Now σ' is a cone of maximal dimension in N' and we have reduced to the previous case. □

The affine toric varieties given in previous examples are all smooth. Except for $\mathcal{Z}(x^2 - y^3)$.

Simplicial cones. We finish this section by giving the definition of a simplicial cone.

DEFINITION 2.23 (Simplicial). A cone $\sigma \subset N_\mathbb{R}$ is called *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.

REMARK 2.24. It follows immediately from the definitions that a smooth cone is a simplicial cone. Also a cone $\sigma \subset N_\mathbb{R}$ is simplicial if the number of edges equals $\dim \sigma$.

A simplicial cone need not be smooth, see for instance the cone in Figure 7.

DEFINITION 2.25 (Multiplicity of a cone). For a simplicial cone $\sigma \subset N_{\mathbb{R}}$ with minimal generators $u_1, \ldots, u_k$ we obtain the lattice $N_\sigma = (\mathbb{R}u_1 + \ldots \mathbb{R}u_k) \cap N$, we denote L_σ for the sublattice

$$L_\sigma = \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_k \subset N_\sigma.$$

Then we define the *multiplicity* $\mathrm{mult}(\sigma)$ to be the index of the sublattice

$$\mathrm{mult}(\sigma) = [N_\sigma : L_\sigma] = |N_\sigma / L_\sigma|.$$

REMARK 2.26. The variety U_σ is smooth when $\mathrm{mult}(\sigma) = 1$

3. Fans and varieties

By the standard procedure in algebraic geometry, we wish to glue together affine varieties to get more general varieties. Given a collection of cones in the same vector space we obtain a collection of affine toric varieties. We can glue them together depending on how the cones fit together. When two cones σ, σ' meet in a common face $\tau = \sigma \cap \sigma'$, then by Proposition 2.13 the variety U_τ sits inside both U_σ and $U_{\sigma'}$. So we can glue the affine varieties together along the common open part U_τ.

3.1 The toric variety of a fan

A collection of cones which we may turn into a variety is what we call a *fan*.

DEFINITION 3.1. A fan Δ in $N_\mathbb{R}$ is a collection of cones in $N_\mathbb{R}$ such that:

(1) A face of cone in Δ is itself a cone in Δ.

(2) The intersection of two cones in Δ is a face of each.

REMARK 3.2. If we combine (1) and (2) then we have that the intersection of two cones in a fan is again a cone in the fan.

Note that for a cone σ in $N_\mathbb{R}$ the variety U_σ contains the torus T_N. Hence a fan in $N_\mathbb{R}$ produces a collection of affine varieties, all of them containing the same torus T_N. We glue together the affine varieties, as described above, we denote X_Δ for the result. Now T_N is an open part of X_Δ and it is clear that T_N acts on X_Δ. This is how we obtain the *toric variety of a fan*. Note that the gluing together of affine varieties in general gives a pre-variety. In the toric case the result after gluing is separated. Thus the toric variety of a fan is an abstract variety [2, Theorem 3.1.5].

DEFINITION 3.3. Let Δ be an n-dimensional fan. Then for $j \leq n$ the set $\Delta(j) \subset \Delta$ denotes the collection of j-dimensional cones in Δ.

REMARK 3.4. Proposition 2.16 tells us that the affine patches are normal. Gluing together normal varieties gives a normal variety. Hence the toric variety of a fan is always normal.

A cone $\sigma \subset N_\mathbb{R}$ itself can be interpreted as a fan in $N_\mathbb{R}$, namely the cone and all of its faces. Consider the cone σ from Example 2.14. Viewed as a fan σ contains $\rho_1 = \text{Cone}(e_1)$, $\rho_2 = \text{Cone}(e_2)$ and the origin. The varieties involved are $U_\sigma = \mathbb{C}^2$, $U_{\rho_1} = \mathbb{C}^* \times \mathbb{C}$, $U_{\rho_2} = \mathbb{C} \times \mathbb{C}^*$ and $U_0 = (\mathbb{C}^*)^2$. They all sit inside $X_\sigma = \mathbb{C}^2$. In this case there is no gluing.

EXAMPLE 3.5. The fan Δ in Figure 9 consists of two cones σ_1, σ_2, three rays ρ_0, ρ_1, ρ_2 and the origin. Remark 2.13 tells us that U_{ρ_0} is an open subset of U_{σ_1} and of U_{σ_2}. The variety X_Δ has two affine patches

$$U_{\sigma_1} \simeq \operatorname{Spec}\left(\mathbb{C}[x, y, x^{-1}y]\right), \quad U_{\sigma_2} \simeq \operatorname{Spec}\left(\mathbb{C}[x, y, xy^{-1}]\right).$$

The patches are glued along the common open part $U_{\rho_0} = \operatorname{Spec}\left(\mathbb{C}[x, y, x^{-1}y, xy^{-1}]\right)$. Note that σ_1, σ_2 and ρ are generated by (part of) a lattice basis so they are smooth cones. The patches are copies of $\mathbb{C}^2$ and they are glued along $\mathbb{C} \times \mathbb{C}^*$. In the next section we describe how X_Δ is the blowing-up of $\mathbb{C}^2$ at the origin.

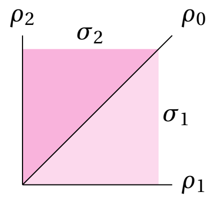

Figure 9: The fan Δ.

EXAMPLE 3.6. The Hirzebruch surface $\mathscr{H}_r$, for $r \in \mathbb{N}$, is the toric variety given by the the fan in Figure 10. The fan has minimal generators

$$u_1 = e_2, \quad u_2 = e_1, \quad u_3 = -e_2, \quad u_4 = -e_1 + re_2.$$

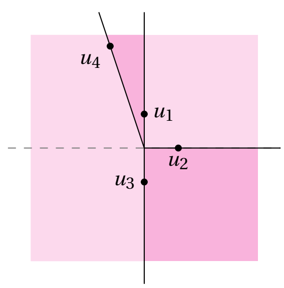

Figure 10: The fan of the Hirzebruch surface $\mathscr{H}_r$.

The four cones in the fan are all smooth and we have the affine patches

$$\begin{aligned}
U_{\sigma_1} &\simeq \operatorname{Spec}\left(\mathbb{C}[x, y]\right) \\
U_{\sigma_2} &\simeq \operatorname{Spec}\left(\mathbb{C}[x, y^{-1}]\right) \\
U_{\sigma_3} &\simeq \operatorname{Spec}\left(\mathbb{C}[x^{-1}, x^{-r}y]\right) \\
U_{\sigma_4} &\simeq \operatorname{Spec}\left(\mathbb{C}[x, y, x^{-1}y]\right),
\end{aligned}$$

neighbouring patches $U_{\sigma_i}, U_{\sigma_{i+1}}$ are glued together via $\text{Cone}(u_{i+1}) \simeq \mathbb{C} \times \mathbb{C}^*$.

EXAMPLE 3.7. Consider the fan in Figure 11, this contains the origin and two rays spanned by e_1 and $-e_1$. This defines the toric variety $\mathbb{P}^1$. The rays define two copies of $\mathbb{C}$, which are the two charts of $\mathbb{P}^1$, they are glued along $U_0 = \mathbb{C}^*$. The gluing data is given by $x \mapsto x^{-1}$.

Figure 11: The fan corresponding to $\mathbb{P}^1$.

EXAMPLE 3.8. Consider the fan in Figure 12. The fan has minimal generators $\pm e_1, \pm e_2$. There are four cones which span the quarters of the plane. Each of these represents a copy of $\mathbb{C}^2$. Two neighbouring copies are glued together via $\mathbb{C} \times \mathbb{C}^*$, just like in the previous example. This way we obtain $\mathbb{P}^1 \times \mathbb{P}^1$ as a toric variety.

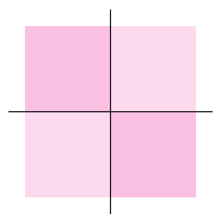

Figure 12: The fan of $\mathbb{P}^1 \times \mathbb{P}^1$.

The variety $\mathbb{P}^1 \times \mathbb{P}^1$ is an example of a Hirzebruch surface. When $r = 0$ we get the fan from Figure 12, so $\mathscr{H}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. We return to Hirzebruch surfaces when we classify smooth toric surfaces in Chapter 5.

3.2 Relations between the fan and the variety

A fan Δ is called smooth if all cones in Δ are smooth. Theorem 2.22 holds for fans as well:

PROPOSITION 3.9. *The variety X_Δ is nonsingular if and only if the Δ is a smooth fan.*

3.2.1 Toric morphisms

We have described what it means for a variety to be toric. We now describe what it means for a morphism between toric varieties to be a *toric morphism*. This can be expressed on the level of the varieties as well as on the level of the fans defining the varieties.

DEFINITION 3.10. Let N_1, N_2 be lattices with fans Δ_i in $(N_i)_\mathbb{R}$. A map $f\colon N_1 \to N_2$ is *compatible* with the fans if for every cone $\sigma_1 \in \Delta_1$ there is a cone $\sigma_2 \in \Delta_2$ such that $f(\sigma_1) \subseteq \sigma_2$.

DEFINITION 3.11 (Toric morphism). Let $X_{\Delta_1}, X_{\Delta_2}$ be toric varieties with fans Δ_i in $(N_i)_\mathbb{R}$. A morphism $\varphi\colon X_{\Delta_1} \to X_{\Delta_2}$ is said to be a *toric morphism* if φ maps the torus $T_{N_1} \subset X_{\Delta_1}$ into $T_{N_2} \subset X_{\Delta_2}$ and the induced map $T_{N_1} \to T_{N_2}$ is a group homomorphism.

We can relate compatible maps to toric morphisms in the following way.

THEOREM 3.12. *Let N_1, N_2 be lattices with fans Δ_i in $(N_i)_\mathbb{R}$.*

Let $f\colon N_1 \to N_2$ be a linear map of lattices that is compatible with Δ_1 and Δ_2. Then there is a toric morphism $\varphi\colon X_{\Delta_1} \to X_{\Delta_2}$ such that $\varphi|_{T_{N_1}}$ is the map

$$f \otimes 1\colon N_1 \otimes_\mathbb{Z} \mathbb{C}^* \to N_2 \otimes_\mathbb{Z} \mathbb{C}^*.$$

Conversely, let $\varphi\colon X_{\Delta_1} \to X_{\Delta_2}$ be a toric morphism, then φ induces a linear map $f\colon N_1 \to N_2$ that is compatible with Δ_1 and Δ_2.

Proof. For a proof we refer to [2, Theorem 3.3.4]. □

DEFINITION 3.13 (Complete). A variety V is called complete if for every variety X the projection morphism $V \times X \to X$ is a closed map.

It turns out that you can easily tell when a toric variety is complete. This relies on the properness criterion for toric morphisms.

THEOREM 3.14 (Properness criterion). *Let $\varphi\colon X_\Delta \to X_{\Delta'}$ be a toric morphism corresponding to a linear map $f\colon N \to N'$ that is compatible with fans $\Delta \subset N_\mathbb{R}$ and $\Delta' \subset N'_\mathbb{R}$. Then*

$$\varphi \text{ is a proper morphism } \leftrightarrows\ f^{-1}(|\Delta'|) = |\Delta|,$$

where $|\Delta|$ denotes the support of the fan.

Proof. For a proof we refer to [2, Thm. 3.4.11] □

As a consequence we get:

COROLLARY 3.15. *A toric variety X_Δ is complete if and only if the support of the fan Δ is the whole vector space, i.e., $\bigcup_{\sigma\in\Delta} \sigma = N_\mathbb{R}$.*

For example, the fans of $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$, which are complete varieties, cover the whole of $\mathbb{R}^1$ and $\mathbb{R}^2$, respectively.

3.2.2 Orbit-cone correspondence

In the previous chapter we mentioned how the torus acts on a toric variety. As always when working with a group action we may view the variety as a union of its orbits. For the toric variety of a fan it turns out that each cone σ in the fan determines an orbit $O(\sigma)$. We set out to show how we can obtain the orbit structure of a toric variety from its cone(s), this is called the *orbit-cone correspondence*.

EXAMPLE 3.16. Consider the basic cone $\sigma = \mathrm{Cone}(e_1, e_2)$, with variety $U_\sigma = \mathbb{C}^2$. It contains $(\mathbb{C}^*)^2$ as torus. The torus action is simply given by $(t_1, t_2) \cdot (x_1, x_2) = (t_1 x_1, t_2 x_2)$. There are four orbits, generated by the elements $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. We may relate these 4 orbits to the 4 cones. Observe that $(0,0)$ is a fixed point and that it is the distinguished point of σ. Likewise we relate the orbits of $(0,1)$ and $(1,0)$ to the rays ρ_1 and ρ_2. Finally we relate $(\mathbb{C}^*)^2$, the orbit of $(1,1)$, to the origin.

Each cone $\sigma \in \Delta$ has a distinguished point $\gamma_\sigma \in U_\sigma \subset X_\Delta$, as defined in (2.2.1). Recall how the torus acts on the semigroup morphisms, described in (2.1.2). This gives an orbit

$$T_N \cdot \gamma_\sigma = \{t \cdot \gamma_\sigma \mid t \in T_N\}.$$

We describe this orbit in the following lemma.

LEMMA 3.17. *Let Δ be a fan in $N_\mathbb{R}$. Given a cone σ in Δ, we can define a T_N-orbit in X_Δ by*

$$O(\sigma) = \left\{\gamma\colon S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \leftrightarrows m \in \sigma^\perp \cap M\right\}.$$

This is the orbit of the distinguished point γ_σ under T_N.

REMARK 3.18. Before we prove Lemma 3.17, we note that

$$\left\{\gamma\colon S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \leftrightarrows m \in \sigma^\perp \cap M\right\} \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*).$$

We have that $M(\sigma) = \sigma^\perp \cap M$ is a sublattice of M, however this is not in general true for $\sigma \cap N$. We can look at the subgroup $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and we get a quotient lattice $N(\sigma) = N/N_\sigma$. The lattices $N(\sigma)$ and $M(\sigma)$ are dual to each other hence by (2.1.1) we see:

$$O(\sigma) \simeq \mathrm{Hom}_\mathbb{Z}(M(\sigma), \mathbb{C}^*) \simeq T_{N(\sigma)}.$$

The orbit $O(\sigma)$ is the torus corresponding to the quotient lattice $N(\sigma)$.

Proof of Lemma 3.17. By definition of the distinguished point we see $\gamma_\sigma \in O(\sigma)$ and it is clear that $O(\sigma)$ is invariant under T_N. We show that T_N acts transitively on $O(\sigma)$, this means that $O(\sigma)$ is the orbit of γ_σ. By Remark 3.18 we have that $O(\sigma)$ is the torus $T_{N(\sigma)}$ of the quotient lattice $N(\sigma)$. The torus $T_{N(\sigma)}$ is a quotient of T_N itself via the surjection

$$T_N \simeq N \otimes_\mathbb{Z} \mathbb{C}^* \to N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^* \simeq T_{N(\sigma)},$$

so T_N acts transitively on $O(\sigma) \simeq T_{N(\sigma)}$. □

We have seen how a cone σ in Δ defines an orbit $O(\sigma)$ in X_Δ. Now we are ready to formulate the *orbit-cone correspondence.*

PROPOSITION 3.19. *Let Δ be a fan in $N_\mathbb{R}$ defining the toric variety X_Δ. Then:*

(1) *There is a bijective correspondence between cones in Δ and T_N-orbits in X_Δ, given by $\sigma \leftrightarrow O(\sigma)$.*

(2) *For each cone σ, $\dim O(\sigma) = n - \dim\sigma$, where $n = \dim N_\mathbb{R}$.*

(3) *The affine variety U_σ is the union of orbits*

$$U_\sigma = \bigcup_{\tau\preceq\sigma} O(\tau).$$

Proof. (1) The injectivity of $\sigma \mapsto O(\sigma)$ is immediate from the definition of $O(\sigma)$. It is a bijection by the following claim: Given a torus orbit $\mathcal{O}$ there exists a smallest cone σ such that $\mathcal{O} \subset U_\sigma$. This cone satisfies $O(\sigma) = \mathcal{O}$.
Proof of the claim. The variety X_Δ is covered by open subsets U_σ. For two cones σ_1, σ_2 we have $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1\cap\sigma_2}$, hence there is a unique smallest affine patch U_σ containing $\mathcal{O}$. Take $\gamma \in \mathcal{O}$. Our aim is to show that $\gamma \in O(\sigma)$, then we may conclude that $\mathcal{O}$ and $O(\sigma)$ are the same orbit. Consider the set $\gamma^{-1}(\mathbb{C}^*) \subset S_\sigma$. For $u, v \in S_\sigma$ we have $\gamma(u+v) = \gamma(u)\cdot\gamma(v)$. So $u+v \in \gamma^{-1}(\mathbb{C}^*)$ implies $u, v \in \gamma^{-1}(\mathbb{C}^*)$, hence by Proposition 1.12 $\gamma^{-1}(\mathbb{C}^*)$ is a face of $\sigma^\vee$. By Proposition 1.13 we can find a face $\tau \prec \sigma$ such that

$$\gamma^{-1}(\mathbb{C}^*) = \sigma^\vee \cap \tau^\perp \cap M.$$

We see that $\gamma(m) = 0$ for $m \notin \tau^\perp$, hence γ can be identified with a semigroup homomorphism $\overline{\gamma}\colon \tau^\vee \cap M \to \mathbb{C}$. Now $\gamma \in U_\tau$ implies $\mathcal{O} \subset U_\tau$ and hence $\tau = \sigma$ by minimality of σ. We have $\gamma(m) \neq 0 \leftrightarrows m \in \sigma^\perp$ and we conclude $\gamma \in O(\sigma)$.

(2) The lattice $N(\sigma) = N/(\mathrm{Span}(\sigma)\cap N)$ has rank $n - \dim\sigma$. As seen in Lemma 3.17 this lattice has $O(\sigma)$ as a torus. Hence $\dim O(\sigma) = n - \dim\sigma$.

(3) As a variety U_σ is a union of orbits. For a face $\tau \preceq \sigma$ we get $O(\tau) \subset U_\tau \subseteq U_\sigma$. If we now take an orbit $\mathcal{O}$ in U_σ then by part (1) this is $O(\tau)$ for the minimal τ such that $\mathcal{O} \subset U_\tau$. Then $\tau \preceq \sigma$ since σ is a cone such that $\mathcal{O} \subset U_\sigma$. □

REMARK 3.20. Note that every fan contains the origin as a cone. This corresponds to the orbit $O(0) = T_N$. Since the origin does not have any faces, we get $U_{\{0\}} = O(\{0\}) = T_N$.

EXAMPLE 3.21. Reconsider the cone σ from Example 3. Viewed as a fan we have σ, two rays ρ_1, ρ_2 and the origin. Then σ corresponds to p_σ, the fixed point of the torus action. The rays give 1-dimensional subvarieties of the form $\mathbb{C}\times\mathbb{C}^*$. The origin corresponds to $T_N \simeq (\mathbb{C}^*)^2$.

EXAMPLE 3.22. The variety $\mathbb{P}^1\times\mathbb{P}^1$ corresponds to the fan given in Figure 12. The fan has nine cones, four maximal cones, four rays and the origin.

The torus $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^1 \times \mathbb{P}^1$ by $(s,t)\cdot\big((X:Y),(Z:W)\big) = \big((sX:Y),(tZ:W)\big)$ There are nine torus orbits:

- four fixed points of the form $\big((0:1),(1:0)\big)$,
- four 1-dimensional orbits generated by elements of the form $\big((1:0),(1:1)\big)$,
- one 2-dimensional orbit generated by $\big((1:1),(1:1)\big)$.

This agrees with the orbit-cone correspondence.

Orbit Closures. We now discuss orbit closures. Consider a cone τ in $N_{\mathbb{R}}$. We want to view the orbit closure $\overline{O(\tau)}$ as a toric variety, such that $O(\tau)$ acts on it. Recall from Remark 3.18 that $O(\tau)$ is the torus corresponding to the quotient lattice

$$N(\tau) = N/N_\tau, \quad N_\tau = N \cap \operatorname{Span}(\tau).$$

We construct a fan in $N(\tau)_{\mathbb{R}}$ that corresponds to the orbit closure $\overline{O(\tau)}$.

Let Δ be a fan in $N_{\mathbb{R}}$. Each cone $\sigma \in \Delta$ with $\tau \preceq \sigma$ determines a cone in the quotient lattice.

$$\overline{\sigma} = \big(\sigma + \operatorname{Span}(\tau)\big)/\operatorname{Span}(\tau) \subset N(\tau)_{\mathbb{R}}$$

LEMMA 3.23. *Let $\tau \in \Delta$ for a fan Δ in $N_{\mathbb{R}}$. Then the collection*

$$\operatorname{Star}(\tau) = \left\{\overline{\sigma} \;\middle|\; \tau \preceq \sigma\right\}$$

is a fan in $N(\tau)_{\mathbb{R}}$.

Proof. Note that if $\tau \prec \sigma' \prec \sigma$ then $\overline{\sigma'} \prec \overline{\sigma}$. This goes both ways, a face of $\overline{\sigma}$ is of the form $\overline{\sigma'}$ for a face $\sigma' \prec \sigma$ containing τ.

Note that $\overline{\tau} = \{0\} \in \operatorname{Span}(\tau)$. Hence for any cone $\tau \preceq \sigma$ we see that $\{0\}$ is a face of $\overline{\sigma}$ and so σ is strongly convex.

So $\operatorname{Star}(\tau)$ is a collection of strongly convex cones in $N(\tau)_{\mathbb{R}}$. We proceed to check that it is indeed a fan. (1) For a cone $\overline{\sigma}$ a face is of the form $\overline{\rho}$ for $\tau \preceq \rho \preceq \sigma$, so any face of $\overline{\sigma}$ is itself a cone in $N(\tau)_{\mathbb{R}}$. (2) Given two cones $\overline{\sigma}, \overline{\sigma'} \subset N(\tau)_{\mathbb{R}}$, their intersection is the cone $\overline{\sigma \cap \sigma'}$ which is a face of each. □

EXAMPLE 3.24. Reconsider the cone $\sigma = \operatorname{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$ from Figure 6. The ray τ through $e_1+e_2+e_3$ is contained in σ. In $N(\tau)$ the points e_3 and $-e_1-e_2$ become equivalent. Hence $\operatorname{Star}(\tau) = \overline{\sigma} = \operatorname{Cone}(e_1, e_2, -e_1, -e_2) = \mathbb{Z}^2$.

DEFINITION 3.25. Let $\tau \in \Delta$ for a fan Δ in $N_{\mathbb{R}}$, we define the toric variety

$$V(\tau) = X_{\operatorname{Star}(\tau)}.$$

The variety $V(\tau)$ contains the torus $T_{N(\tau)}$. As we have shown in the proof of Lemma 3.17, the torus $T_{N(\tau)}$ coincides with $O(\tau)$. In fact $V(\tau)$ is the closure of the orbit $O(\tau)$, as we will show in Proposition 3.26.

Consider an affine patch $U_{\overline{\sigma}}$ of $V(\tau)$. The dual cone $\overline{\sigma}^\vee$ is the image of $\sigma^\vee$ in $M(\tau) = \tau^\perp \cap M$, which is the dual lattice of $N(\tau)$, so we identify $S_{\overline{\sigma}}$ with $\sigma^\vee \cap \tau^\perp \cap M$. This describes a projection of S_σ onto $S_{\overline{\sigma}}$, by sending $m \mapsto 0$ for $m \notin \tau^\perp$. This induces a projection of $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\overline{\sigma}}]$ and we get a closed embedding

$$U_{\overline{\sigma}} \hookrightarrow U_\sigma.$$

These embeddings combine to give an embedding $X_{\mathrm{Star}(\tau)} \hookrightarrow X_\Delta$. We have the following result:

PROPOSITION 3.26. *For $\tau \in \Delta$ there is a closed embedding $V(\tau) \hookrightarrow X_\Delta$ such that $V(\tau)$ is the closure of $O(\tau)$.*

LEMMA 3.27. *Let τ, σ be cones in a fan in $N_\mathbb{R}$. If $\tau \npreceq \sigma$ then we have*

$$V(\tau) \cap U_\sigma = \emptyset.$$

Proof. As a subset of X_Δ, the variety $V(\tau)$ is covered by the open sets U_π for $\tau \preceq \pi$. It suffices to show that for cones $\pi \succeq \tau$ we have

$$\big(V(\tau) \cap U_\pi\big) \cap U_\sigma = \emptyset.$$

The intersection equals $V(\tau) \cap U_{\pi\cap\sigma}$. Write $\pi' = \pi \cap \sigma$. Then π' is a face of σ and π' does not contain τ since $\tau \npreceq \sigma$. We may find an element $m \in \sigma^\vee \setminus \tau^\perp$ such that $\pi' = \sigma \cap H_m$. This means that $U_{\pi'} \subset U_\sigma$ is the principal open subset where χ^m is non-zero, as shown in Proposition 2.13.

On the other hand, since $m \notin \tau^\perp$ we have $\chi^m(x) = 0$ for $x \in V(\tau)$.

Hence $\big(V(\tau) \cap U_\pi\big) \cap U_\sigma = V(\tau) \cap U_{\pi'} = \emptyset$, as desired. □

Proof of Proposition 3.26. We have to show that $V(\tau) \hookrightarrow X_\Delta$ is a closed embedding, we prove that $V(\tau) \cap U_\sigma$ is closed in U_σ for all $\sigma \in \Delta$.

When $\tau \preceq \sigma$ we have the closed embedding $U_{\overline{\sigma}} \hookrightarrow U_\sigma$.

When $\tau \npreceq \sigma$ then by Lemma 3.27 we have $V(\tau) \cap U_\sigma = \emptyset$, which is closed.

Now it easily follows that $V(\tau)$ is the orbit closure, since $O(\tau)$ lies dense in $V(\tau)$. □

We now have a nice description of the closure of a torus orbit. We can extend the orbit-cone correspondence to orbit closures.

PROPOSITION 3.28. *Let $\tau \subset N_\mathbb{R}$ be a cone in a fan. Them we can view $\overline{O(\tau)}$ as the union of orbits*

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma).$$

Proof. The variety $\overline{O(\tau)} = V(\tau)$ is a union of orbits so it suffices to prove

$$\tau \preceq \sigma \leftrightarrows O(\sigma) \subseteq V(\tau).$$

When $\tau \prec \sigma$ we get the inclusions $O(\sigma) \subset V(\sigma) \subset V(\tau)$. For the converse direction we reason by contradiction. Suppose $\tau \not\prec \sigma$, then by Lemma 3.27 we have $V(\tau) \cap U_\sigma = \emptyset$. Hence $V(\tau) \cap O(\sigma) = \emptyset$ and $O(\sigma) \not\subset V(\tau)$. □

REMARK 3.29. For any fan Δ, the origin is a face of every cone, hence the closure of $T_N = O(0)$ contains all the orbits, $\overline{O(\{0\})} = X_\Delta$.

3.2.3 Refinements and blow-up

DEFINITION 3.30 (Refinement). We say that a fan Δ' *refines* the fan Δ if:

(1) every cone in Δ is a union of cones in Δ' ;

(2) the fans have the same *support*, meaning $\bigcup\{\sigma \mid \sigma \in \Delta'\} = \bigcup\{\sigma \mid \sigma \in \Delta\}$.

An illustration of a refinement of fans is given in Figure 13.

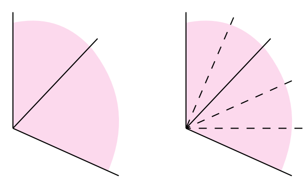

Figure 13: A refinement of fans.

Blow-up. Recall that the *blowing up of* $\mathbb{C}^n$ *at the origin* is the closed subvariety of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ given by $\{(x_1,\dots,x_n; Y_1 : \dots : Y_n) \mid x_i Y_j = x_j Y_i \text{ for all } i,j\}$. It is covered by principal open subsets $W_i = \{(x_1,\dots,x_n; Y_1 : \dots : Y_n) \in \mathrm{Bl}_0(\mathbb{C}^n) \mid Y_i \neq 0\}$, so $W_i = D(Y_i)$.

EXAMPLE 3.31. Reconsider the fan Δ of Example 3.5. We show that the blowing $\mathbb{C}^2$ at the origin is a toric variety, with defining fan Δ. we set out to show that $X_\Delta = \mathrm{Bl}_0(\mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{P}^1$. Both of them are covered by two open sets:

$$\begin{aligned} U_{\sigma_1} &= \mathrm{Spec}(\mathbb{C}[x, x^{-1}y]) \\ U_{\sigma_2} &= \mathrm{Spec}(\mathbb{C}[x, xy^{-1}]) \\ W_1 &= \{(x_1, x_2; Y_1 : Y_2) \in \mathrm{Bl}_0(\mathbb{C}^2) \mid Y_1 \neq 0\} \\ W_2 &= \{(x_1, x_2; Y_1 : Y_2) \in \mathrm{Bl}_0(\mathbb{C}^2) \mid Y_2 \neq 0\} \end{aligned}$$

We identify U_{σ_1} and U_{σ_2} with the principal open subsets W_1 and W_2 of $\mathrm{Bl}_0(\mathbb{C}^2)$. We view both of U_{σ_1} and U_{σ_2} as $\mathbb{C}^2$ with coordinates $(x_1, Y_2/Y_1)$ respectively $(x_2, Y_1/Y_2)$.

Going the other way around, we identify $(u,v) \in U_{\sigma_1}$ with $(u, uv; 1 : v) \in W_1$. This gives bijective correspondences, so the varieties have the same open cover.

Figure 14 demonstrates that the fan Δ is also a refinement of the upper right quadrant. This corresponds to $\mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$. We replace the 2-dimensional cone for

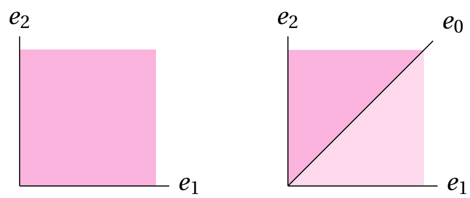

Figure 14: The basic cone $\sigma = \mathrm{Cone}(e_1, e_2)$ and its refinement Δ.

two 2-dimensional cones and a ray. On the level of varieties we replace the origin for a copy of $\mathbb{P}^1$. This shows that introducing a new ray within a cone changes the variety at its distinguished point.

We proceed with the blowing-up of $\mathbb{C}^n$ at the origin. The toric variety $\mathbb{C}^n$ corresponds to the fan $\Delta \subset \mathbb{R}^n$ consisting of $\mathrm{Cone}(e_1, \dots, e_n)$ and its faces. Now define $e_0 = e_1 + \dots + e_n$ and define the fan

$$\Delta^* = \left\{ \mathrm{Cone}(S) \,\middle|\, S \subset \{e_0, e_1, \dots, e_n\} \,\middle|\, \{e_1, \dots, e_n\} \not\subset S \right\}.$$

Then we establish the blowing up as the toric variety $\mathrm{Bl}_0(\mathbb{C}^n) = X_{\Delta^*}$. The reader may verify that the refinement in Figure 14 is Δ^* for $n = 2$.

PROPOSITION 3.32. *The blowing-up of $\mathbb{C}^n$ at the origin is the toric variety*

$$\mathrm{Bl}_0(\mathbb{C}^n) = X_{\Delta^*}.$$

Proof. Denote $\sigma_i = \mathrm{Cone}(e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n)$. Then Δ^* is the fan consisting of $\sigma_1, \dots, \sigma_n$ and all their faces. We get affine semigroups

$$S_{\sigma_i} = \langle e_i, -e_i + e_1, \dots, -e_i + e_n \rangle.$$

The sets $W_i = D(Y_i)$ form an open cover of $\mathrm{Bl}_0(\mathbb{C}^n) = X_{\Delta^*}$. Just as in Example 3.31 we can provide bijections between U_{σ_i} and W_i. The variety $U_{\sigma_i} \simeq \mathbb{C}^n$ has coordinates $u_1, \dots, u_n$, and W_i has coordinates $(x_1, \dots, x_n; Y_1 : \dots : Y_n)$. We get the map $U_{\sigma_i} \to W_i$

$$(u_1, \dots, u_n) \mapsto \left(u_1 u_i, \dots, u_{i-1} u_i, u_i, u_{i+1} u_i, \dots, u_n u_i; u_0 : \dots : u_{i-1} : 1 : u_{i+1} : \dots : u_n\right),$$

and the map $W_i \to U_{\sigma_i}$

$$(x_1, \dots, x_n; Y_1 : \dots : Y_n) \mapsto \left(\frac{Y_1}{Y_i}, \dots, \frac{Y_{i-1}}{Y_i}, x_i, \frac{Y_{i+1}}{Y_i}, \dots, \frac{Y_n}{Y_i}\right)$$

It follows that the varieties $\mathrm{Bl}_0(\mathbb{C}^n)$ and X_{Δ^*} have the same open cover. □

Stellar refinement. The refinement Δ^* from Proposition 3.32 is a an example of a so-called stellar refinement.

DEFINITION 3.33 (Stellar refinement). Let $\sigma \subset N_{\mathbb{R}}$ be a cone and let ρ be any ray in $N_{\mathbb{R}}$. Then we define

$$\sigma^*(\rho) = \begin{cases} \sigma & \text{if } \rho \not\subseteq \sigma \\ \{\rho + \tau \mid \tau \prec \sigma \mid \rho \not\subseteq \tau\} & \text{if } \rho \subseteq \sigma \end{cases}$$

By $\rho + \tau$ we mean their sum as subsets of $N_{\mathbb{R}}$. If ρ is generated by u_ρ and $\tau = \text{Cone}(u_1, \ldots, u_k)$ then $\rho + \tau = \text{Cone}(u_\rho, u_1, \ldots, u_k)$.

If Δ is a fan in $N_{\mathbb{R}}$ and ρ a ray, then $\Delta^*(\rho)$ is the union of the $\sigma^*(\rho)$ for $\sigma \in \Delta$. We call $\Delta^*(\rho)$ the *stellar refinement of* Δ *with center* ρ.

For a fan Δ and a ray generator u_ρ consider the stellar refinement $\Delta^*(\rho)$. Then $\Delta^*(\rho)$ admits those cones in Δ not containing u_ρ. A cone in Δ that contains u_ρ is replaced for a set of cones which have ρ as an edge, as shown in Figure 15.

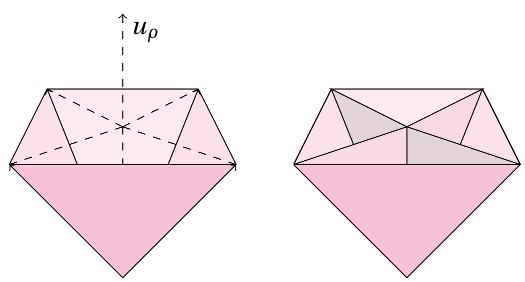

Figure 15: Stellar refinement.

Indeed $\Delta^*(\rho)$ is a fan and it refines Δ. This is proven in [2, Lemma 11.1.3].

EXAMPLE 3.34. The refinement in Figure 14 is the stellar refinement of $\text{Cone}(e_1, e_2)$ with center $\rho = \text{Cone}(e_1 + e_2)$. More generally the fan Δ^* from Proposition 3.32 is the stellar refinement of $\text{Cone}(e_1, \ldots, e_n)$ with the ray through e_0 as its center.

REMARK 3.35. For a smooth cone $\sigma = \text{Cone}(u_1, \ldots, u_k) \subset N_{\mathbb{R}}$, the blowing up at the distinguished point of the toric variety U_σ is also an instance of stellar refinement. Define ρ_0 to be the ray through $u_0 = u_1 + \ldots + u_k$. Then it turns out that $\text{Bl}_{\gamma_\sigma}(U_\sigma) = X_{\sigma^*(\rho_0)}$, this is proved in [2, Theorem 3.3.15]. This generalizes Example 3.34.

3.3 Resolving singularities

We now have the know-how to develop concept of 'toric resolutions' of singularities.

The resolution of singularities is a classic problem in algebraic geometry. A variety X can be separated into a smooth locus and a singular locus. We want to get red of the singularities, so we wish to change X at the singular locus to obtain a smooth variety $\widetilde{X}$. The idea of a *resolution* is that we do not change X at the smooth locus. Let X_{Sing} denote the singular locus.

DEFINITION 3.36. A *resolution of singularities* of a (nonsmooth) variety X is a smooth variety $\widetilde{X}$, together with a proper birational morphism $\varphi\colon \widetilde{X} \to X$, such that φ induces an isomorphism outside of the singular locus

$$\widetilde{X} \setminus \varphi^{-1}(X_{\text{Sing}}) \simeq X \setminus X_{\text{Sing}}.$$

For general varieties it is a difficult problem to find a resolution of singularities. A well-known result due to Hironaka [9] states that, in characteristic 0, every variety admits a resolution of singularities. In the toric case things turn out to be much simpler and we can give resolutions in a 'combinatorial' way, we can describe a resolution in terms of the fan.

Recall that a toric variety is smooth, or nonsingular, if and only if its defining fan (or cone) is smooth. If a toric variety X_Δ has a singular point p, then there is a nonsmooth cone σ such that $p \in U_\sigma$. In order to resolve the singularity at p, we would like to replace the cone σ by a collection of smooth cones, which comes down to a refinement.

REMARK 3.37. A refinement of fans gives a proper birational moprhism on the level of varieties. Let Δ' be a fan that refines the fan Δ in $N_\mathbb{R}$, then Theorem 3.12 gives us a morphism $X_{\Delta'} \to U_\Delta$. Since it corresponds to a refinement of fans, Theorem 3.14 tells us that this morphism is proper.

If we now find a smooth fan $\widetilde{\Delta}$ that refines a given fan Δ, then by Proposition 3.9 the variety $X_{\widetilde{\Delta}}$ is smooth. Now we have the first two ingredients of a resolution of singularities $\varphi\colon \widetilde{X} \to X$. We still need that φ induces an isomorphism outside of the singular locus.

We consider the singular locus. Since toric varieties are normal, the singular locus is of codimension at least 2 [8]. So for a toric surface X_Δ the singular locus consists of points, the distinguished points of the nonsmooth cones in the fan. In general we can describe the singular locus in terms of orbit closures.

PROPOSITION 3.38. *For a toric variety X_Δ we have the singular locus*

$$(X_\Delta)_{\text{Sing}} = \bigcup_{\sigma \text{ nonsmooth}} V(\sigma),$$

where $V(\sigma)$ denotes the orbit closure. The smooth locus is

$$X_\Delta \setminus (X_\Delta)_{\text{Sing}} = \bigcup_{\sigma \text{ smooth}} U_\sigma.$$

We need the following lemma, a proof of which can be found in [2, Proposition 11.1.2]

LEMMA 3.39. *Let σ be a nonsmooth cone. Then the distinguished point p_σ is singular and the same holds true for all points in its orbit $O(\sigma)$.* □

Proof of Proposition 3.38. Whenever a cone is nonsmooth then so is every cone τ that contains σ. By Proposition 3.28 we see that

$$\bigcup_{\sigma \text{ nonsmooth}} V(\sigma) = \bigcup_{\sigma \text{ nonsmooth}} O(\sigma). \tag{3.3.1}$$

The complement is union of orbits $O(\sigma)$ for the smooth cones. Note that whenever a cone σ is smooth then so are all of its faces $\tau \preceq \sigma$. Hence the orbit-cone correspondence tells us that the complement of (3.3.1) is

$$\bigcup_{\sigma \text{ smooth}} O(\sigma) = \bigcup_{\sigma \text{ smooth}} U_\sigma. \tag{3.3.2}$$

By Lemma 3.39 all points in (3.3.1) are singular. By Theorem 2.22 all points in (3.3.2) are smooth. Since the sets are complementary they are the singular and the smooth locus. □

It follows that when we wish to find a resolution of X_Δ we should leave the smooth cones in Δ unchanged. This brings us to our main result:

THEOREM 3.40. *Let $\widetilde{\Delta}$ be a smooth fan that refines a given fan Δ, we write $\varphi\colon X_{\widetilde{\Delta}} \to X_\Delta$ for the associated map on varieties. If $\widetilde{\Delta}$ does not refine any smooth cone in Δ then $\varphi\colon X_{\widetilde{\Delta}} \to X_\Delta$ is a resolution of singularities.*

Proof. The variety $X_{\widetilde{\Delta}}$ is smooth and the map φ is a proper birational map. Since the smooth cones of Δ are still cones in $\widetilde{\Delta}$, the map φ induces an isomorphism outside of the singular locus

$$X_{\widetilde{\Delta}} \setminus \varphi^{-1}\left((X_\Delta)_{\text{Sing}}\right) \simeq X_\Delta \setminus (X_\Delta)_{\text{Sing}}.$$

□

The theorem tells us that a smooth refinement is enough to resolve a singularity. It does not provide a way to find such a refinement. In Chapter 5 we describe an algorithm for finding a smooth refinement in the 2-dimensional case. We deal with the general case in Chapter 6, where we shall use stellar refinements.

Before we get to any resolutions we provide the necessary background of the theory of toric divisors.

4. Divisors on toric varieties

In the present chapter we describe the divisors on a toric variety. We show how to compute intersection products on a toric variety by working on the combinatorial side of things, i.e., with lattice objects. This chapter can be seen as an intermezzo before we get to the results on resolutions of singularities. In the next chapter we describe a method for resolving a singularity on a toric surface. In order to show that this method provides a resolution we need to compute some intersection numbers. We work towards a result on self-intersection numbers of divisors on a toric surface. When the reader simply assumes the results in this chapter, the reader may skip ahead to Chapter 5 on resolutions of toric surface singularities.

We state the basic definitions of divisors before we discuss them in the toric context. The theory of toric divisors is worked out in full detail in [2, Chapter 4].

4.1 Divisors

DEFINITION 4.1 (Divisor). A prime divisor D in a variety X is an irreducible subvariety of codimension one, $\dim D = \dim X - 1$. To a prime divisor we assign the following subring of the field of rational functions $\mathbb{C}(X)$

$$\mathscr{O}_{X,D} = \left\{\varphi \in \mathbb{C}(X) \,\middle|\, \varphi \text{ is defined on an open } U \subseteq X \text{ with } U \cap D \neq \emptyset\right\}.$$

When X is normal we have a discrete valuation $\nu_D\colon \mathbb{C}(X)^* \to \mathbb{Z}$, we call $\nu_D(f)$ the order of vanishing of f along D. Then $\mathscr{O}_{X,D}$ consists of 0 and those rational functions $f \in \mathbb{C}(X)^*$ with order of vanishing $\nu_D(f) \geq 0$. So when X is normal we call $\mathscr{O}_{X,D}$ a discrete valuation ring (DVR).

REMARK 4.2. Let $f \in \mathbb{C}(X)^*$ a rational function on a normal variety X. Then $\nu_D(f) \neq 0$ for only a finite number of prime divisors D on X. This statement is proven in [2, Theorem 4.0.9].

We define $\mathrm{Div}(X)$ the free abelian group generated by the prime divisors on X. Elements of $\mathrm{Div}(X)$ are called *Weil divisors*. Such a divisor can be written as a formal sum

$$D = \sum_i a_i D_i \in \mathrm{Div}(X).$$

A divisor is *effective* if all the a_i are non-negative. The *support* of a divisor is

$$\mathrm{Supp}(D) = \bigcup_{a_i \neq 0} D_i.$$

We now define a special class of divisors.

DEFINITION 4.3 (Principal divisor). Let $f \in \mathbb{C}(X)^*$ be a rational function, to f we associate the *principal divisor*

$$\operatorname{div}(f) = \sum_D v_D(f) D \quad ,$$

this is well defined by Remark 4.2. We denote $\operatorname{Div}_0(X)$ for the set of principal divisors.

The principal divisors form a subgroup $\operatorname{Div}_0(X) \subset \operatorname{Div}(X)$.

We are interested in local properties of divisors. We may restrict a divisor on X to an open subset U. If $D = \sum_i a_i D_i$ is a Weil divisor on X and $U \subseteq X$ is a nonempty open subset, then

$$D|_U = \sum_{U \cap D_i \neq 0} a_i (U \cap D_i)$$

is a Weil divisor on U called the *restriction* of D to U.

Now that we have the notion of local properties of divisors we are ready to define an important class of (Weil) divisors, the *Cartier divisors.*

DEFINITION 4.4 (Cartier divisor). We say that a divisor D on X is *Cartier* if it is locally principal. So there exists an open cover $X = \bigcup_{i=1}^k U_i$ such that $D|_{U_i}$ is principal. If $D|_{U_i} = \operatorname{div}(f_i)|_{U_i}$ for $1 \leq i \leq k$, then we say that D has local data $\{(U_i, f_i)\}_{(1 \leq i \leq k)}$.

The Cartier divisors, denoted $\operatorname{CDiv}(X)$, form a subgroup of the Weil divisors. Since a principal divisor is obviously Cartier, we obtain

$$\operatorname{Div}_0(X) \subset \operatorname{CDiv}(X) \subset \operatorname{Div}(X).$$

The divisor class group. The principal divisors induce an equivalence relation on the Weil divisors.

DEFINITION 4.5 (Equivalent divisor). Two Weil divisors D, E are said to be equivalent, denoted $D \sim E$, if $D - E \in \operatorname{Div}_0(X)$.

Indeed this is an equivalence relation, the quotient group

$$\operatorname{Cl}(X) = \operatorname{Div}(X) / \operatorname{Div}_0(X).$$

is called the divisor class group.

REMARK 4.6. The equivalence relation $\sim$ on $\operatorname{Div}(X)$ descends to $\operatorname{CDiv}(X)$. Let D, E be equivalent Weil divisors on a normal variety, then $D - E$ is Cartier since it is principal. As the Cartier divisors form a subgroup it follows that

$$D \text{ is Cartier } \leftrightarrows E \text{ is Cartier} .$$

We define the *Picard group*

$$\operatorname{Pic}(X) = \operatorname{CDiv}(X) / \operatorname{Div}_0(X).$$

4.2 Toric divisors

Let Δ be a fan in $N_{\mathbb{R}}$. By the orbit-cone correspondence we know that a ray $\rho \in \Delta$ corresponds to an orbit $O(\rho)$ and the orbit closure $V(\rho)$ both of codimension one. Hence $V(\rho)$ is a divisor on X_Δ, the *ray divisor*. We use the notation D_ρ to emphasize that it is a divisor.

A ray divisor is a prime divisor on the normal variety X_Δ. Then $\mathscr{O}_{X_\Delta, D_\rho}$ is a discrete valuation ring hence we get a valuation $v_\rho \colon \mathbb{C}(X_\Delta)^* \to \mathbb{Z}$. For $m \in M$ the corresponding character $\chi^m \colon T_N \to \mathbb{C}^*$ is a rational function in $\mathbb{C}(X_\Delta)^*$. We can compute the valuation of this character in terms of lattice objects:

PROPOSITION 4.7. *Let Δ be a fan in $N_{\mathbb{R}}$ and let $\rho \in \Delta$ be a ray with ray generator u_ρ. Denote χ^m for the character corresponding to $m \in M$. Then*

$$v_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Proof. For a proof we refer to [2, Proposition 4.1.1]. □

We can use this result to compute the divisor of a character.

COROLLARY 4.8. *For $m \in M$ the principal divisor of χ^m is given by*

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Delta(1)} \langle m, u_\rho \rangle D_\rho.$$

Torus invariant divisors. We describe how the T_N-action works on divisors on the toric variety X_Δ. On the level of points we have $t \cdot p \in X_\Delta$ for $t \in T_N$ and $p \in X_\Delta$. If D is a prime divisor, the T_N-action gives the prime divisor $t \cdot D$. Note that for a Weil divisor $D = \sum_i a_i D_i$ we get $t \cdot D = \sum_i a_i (t \cdot D_i)$. We say that a divisor D is T_N*-invariant* if $t \cdot D = D$.

DEFINITION 4.9 (Toric divisor). Let X_Δ be a toric variety for a fan Δ in $N_{\mathbb{R}}$. We say that a divisor on X_Δ is a *toric divisor* if it is T_N-invariant. We also refer to toric divisors as *T-Weil* and *T-Cartier* divisors.

REMARK 4.10. A ray divisor is a toric divisor. By the orbit-cone correspondence a ray divisor $D_\rho = V(\rho)$ is a union of torus orbits, hence T_N-invariant.

In fact any toric divisor can be decomposed as a sum of ray divisors:

PROPOSITION 4.11. *Given a fan Δ in $N_{\mathbb{R}}$, the group of T-Weil divisors on X_Δ is given*

$$\operatorname{Div}_{T_N}(X_\Delta) = \bigoplus_{\rho \in \Delta(1)} \mathbb{Z} D_\rho \subseteq \operatorname{Div}(X_\Delta).$$

Proof. Since ray divisors are T_N-invariant, it follows that a divisor $\sum_\rho a_\rho D_\rho$ is also T_N-invariant. Hence we have the inclusion $\bigoplus \mathbb{Z} D_\rho \subseteq \operatorname{Div}_{T_N}(X_\Delta)$.

For the converse, let $D = a_i D_i$ be a T-Weil divisor. We show that each D_i is a ray divisor. Each D_i is T_N-invariant and thus a union of orbits. Since D_i has codimension 1, the torus T_N, viewed as an orbit which has maximum dimension, cannot be contained in D_i. So we have $D_i \subset X_\Delta \setminus T_N$. Hence the support of D satisfies $\bigcup_i D_i \subset X_\Delta \setminus T_N$. By the orbit-cone correspondence we have $X_\Delta \setminus T_N = \bigcup_{\rho\in\Delta(1)} V(\rho)$. Hence each D_i is a ray divisor. □

Computing the class group. The following result makes it easy to compute the divisor class group of a toric variety.

PROPOSITION 4.12. *Given a fan Δ in $N_{\mathbb{R}}$ we have the exact sequence*

$$M \longrightarrow \mathrm{Div}_{T_N}(X_\Delta) \longrightarrow \mathrm{Cl}(X_\Delta) \longrightarrow 0, \tag{4.2.1}$$

where the first map sends $m \mapsto \mathrm{div}(\chi^m)$ and the second map sends a divisor to its class.

Proof. For a proof we refer to [2, Theorem 4.1.3]. □

By the exact sequence every divisor class has a representative in Div_{T_N}. So any divisor, not necessarily toric, is equivalent to a toric divisor. Using Proposition 4.11 we see that $\mathrm{Cl}(X_\Delta)$ is generated by the classes $[D_\rho]$ of the ray divisors. We also know that for $m \in M$ divisors of the form $\mathrm{div}(\chi^m)$ are equivalent to 0. We now use these facts to compute the divisor class group for some toric varieties.

EXAMPLE 4.13. Let σ be the cone in $\mathbb{R}^2$ with ray generators $u_1 = de_1 - e_2$ and $u_2 = e_2$. This gives the rational normal cone of degree d from Example 2.4. We compute the class group. It is generated by the classes of the ray divisors D_1, D_2. Using the above proposition, we see that the divisors are subject to the relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = \langle e_1, u_1\rangle D_1 + \langle e_1, u_2\rangle D_2 = dD_1$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = \langle e_2, u_1\rangle D_1 + \langle e_2, u_2\rangle D_2 = -D_1 + D_2.$$

This means that the class group is generated by $[D_1]$ with $d \cdot [D_1] = 0$, in other words $\mathrm{Cl}(\widehat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}$.

EXAMPLE 4.14. We return to the fan of Example 3.5 which corresponds to $\mathrm{Bl}_0(\mathbb{C}^2)$ as seen in Example 3.31. The ray generators are $u_1 = e_1$, $u_2 = e_2$ and $u_0 = e_1 + e_2$. The corresponding ray divisors satisfy the relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = \textstyle\sum_{i=0}^{2} \langle e_1, u_i\rangle D_i = D_0 + D_1$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = \textstyle\sum_{i=0}^{2} \langle e_2, u_i\rangle D_i = D_0 + D_2.$$

We see that the class group is generated by $[D_1] = [D_2] = -[D_0]$ and $\mathrm{Cl}(\mathrm{Bl}_0(\mathbb{C}^2)) \simeq \mathbb{Z}$.

EXAMPLE 4.15. We return to the Hirzebruch surface $\mathscr{H}_r$ from Example 3.6. We have four ray divisors corresponding to the ray generators $u_1 = e_2$, $u_2 = e_1$, $u_3 = -e_2$ $u_4 = -e_1 + re_2$. This gives relations

$$0 \sim \operatorname{div}(\chi^{e_1}) = \sum_{i=1}^4 \langle e_1, u_i\rangle D_i = -D_2 + D_4$$
$$0 \sim \operatorname{div}(\chi^{e_2}) = \sum_{i=1}^4 \langle e_2, u_i\rangle D_i = D_1 + D_3 + rD_4.$$

We get generators $[D_4]$ and $[D_1]$, the class group is $\mathrm{Cl}(\mathscr{H}_r) \simeq \mathbb{Z}^2$.

4.2.1 Toric Cartier divisors

Let D be a Cartier divisor on a toric variety. Since it is a divisor, by Proposition 4.12 we may find integers a_ρ such that $D \sim \sum_\rho a_\rho D_\rho$. Remark 4.6 tells us that $\sum_\rho a_\rho D_\rho$ is Cartier as well, it even is T-Cartier. Let $\mathrm{CDiv}_{T_N}(X_\Delta)$ denote the group of T-Cartier divisors. For $m \in M$ the divisor $\operatorname{div}(\chi^m)$ is T-Cartier. Now (4.2.1) reduces to the exact sequence

$$M \longrightarrow \mathrm{CDiv}_{T_N}(X_\Delta) \longrightarrow \mathrm{Pic}(X_\Delta) \longrightarrow 0. \tag{4.2.2}$$

We want to describe when a T_N-invariant divisor is Cartier. We first describe the affine case.

PROPOSITION 4.16. *Let $\sigma \subset N_\mathbb{R}$ be a cone. Then every T-Cartier divisor on U_σ is of the form* $\operatorname{div}(\chi^m)$ *for a unique $m \in S_\sigma$.*

Proof. For a proof we refer to [2, Proposition 4.2.2]. □

COROLLARY 4.17. *So in this case every Cartier divisor is principal. Hence for a cone σ in $N_\mathbb{R}$ the Picard group is*

$$\mathrm{Pic}(U_\sigma) = 0$$

and (4.2.2) *reduces to*

$$M \longrightarrow \mathrm{Div}_0(U_\sigma) \longrightarrow 0. \tag{4.2.3}$$

EXAMPLE 4.18. We return to σ from Example 4.13. We've seen that $\mathrm{Cl}(U_\sigma) \simeq \mathbb{Z}/d\mathbb{Z}$ and that it is generated by $[D_1] = [D_2]$. If D_1 or D_2 is Cartier then we would have $\mathrm{Cl}(U_\sigma) = \mathrm{Pic}(U_\sigma)$. Since $\mathrm{Pic}(U_\sigma) = 0$ we see that D_1, D_2 are not Cartier if $d > 1$.

EXAMPLE 4.19. Let σ be the cone in $\mathbb{R}^3$ with ray generators

$$u_1 = e_1, \quad u_2 = e_2, \quad u_3 = e_1 + e_3, \quad u_4 = e_2 + e_3$$

given in Figure 6. There are four ray divisors D_i. Let $D = \sum_{i=1}^4 a_i D_i$ be a divisor. Then by Proposition 4.16 the divisor D on U_σ is Cartier if there exists $m = m_1 e_1 + m_2 e_2 + m_3 e_3$ in M such that D is the divisor of χ^m, which is

$$\operatorname{div}(\chi^m) = m_1 D_1 + m_2 D_2 + (m_1 + m_3) D_3 + (m_2 + m_3) D_4.$$

We see $m_1 = a_1$, $m_2 = a_2$, $m_3 = a_3 - a_1$ and $m_3 = a_4 - a_2$, hence

$$D \text{ is Cartier } \leftrightarrows\ a_1 + a_4 = a_2 + a_3.$$

As in earlier examples, a computation gives $0 \sim D_1 + D_3 \sim D_2 + D_4 \sim D_3 + D_4$, hence $\mathrm{Cl}(U_\sigma) \simeq \mathbb{Z}$. Since $\mathrm{Pic}(U_\sigma) = 0$ none of the ray divisors D_i is Cartier.

Cartier data. In general, for a toric variety we can describe the T-Cartier divisors in terms of *Cartier data*.

PROPOSITION 4.20. *Let X_Δ be the toric variety of a fan Δ in $N_\mathbb{R}$ and let $D = \sum_\rho a_\rho D_\rho$. Then the following are equivalent:*

(1) *D is Cartier.*
(2) *D is principal on the affine open subsets U_σ for $\sigma \in \Delta$.*
(3) *For each $\sigma \in \Delta$ there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho\rangle = a_\rho$ for all rays $\rho \prec \sigma$.*
(4) *For each maximal cone $\sigma \in \Delta$ there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho\rangle = a_\rho$ for all rays $\rho \prec \sigma$.*

Proof. The divisor $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$ is toric. For an affine patch we have $D|_{U_\sigma} = \sum_{\rho\in\sigma(1)} a_\rho D_\rho$. If D is Cartier then so is $D|_{U_\sigma}$ and this is then principal by Proposition 4.16. When (2) holds true then by definition D is Cartier. This shows (1) $\leftrightarrows$ (2). The divisor D is principal on the affine patch U_σ if and only if we can find m_σ such that $D|_{U_\sigma} = \mathrm{div}(\chi^{m_\sigma})$. By Corollary 4.8 we have $\mathrm{div}(\chi^{m_\sigma}) = \sum_{\rho\in\sigma(1)}\langle m_\sigma, u_\rho\rangle D_\rho$. This shows (2) $\leftrightarrows$ (3). It is immediate that (3) implies (4). For the converse, any cone τ is the face of a maximal cone $\sigma \in \Delta$. If m_σ works for all the rays $\rho \prec \sigma$, then m_σ also works for the rays $\rho \prec \tau \prec \sigma$. This shows (3) $\leftrightarrows$ (4). □

So for a T-Cartier divisor D we have $D|_{U_\sigma} = \mathrm{div}(\chi^{m_\sigma})$, in other words D has local data $\left\{(U_\sigma, \chi^{m_\sigma})\right\}_{(\sigma\in\Delta)}$.

DEFINITION 4.21. When D is a Cartier divisor then the data $\{m_\sigma\}_{\sigma\in\Delta}$ as in Proposition 4.20 is called the *Cartier data.*

PROPOSITION 4.22. *The Cartier data satisfies:*

(1) *m_σ is unique modulo $M(\sigma) = M \cap \sigma^\perp$.*
(2) *If $\tau \prec \sigma$ then $m_\sigma \equiv m_\tau \mod M(\tau)$.*

Proof. Suppose m_σ and m'_σ both work for all rays $\rho \prec \sigma$. Then for all rays $\rho \prec \sigma$ we have

$$\langle m_\sigma - m'_\sigma, u_\rho\rangle = 0.$$

Then for all $u \in \sigma$ we have

$$\langle m_\sigma - m'_\sigma, u\rangle = 0,$$

which is equivalent to saying

$$m_\sigma - m'_\sigma \in \sigma^\perp \cap M = M(\sigma).$$

So m_σ is unique $\mod M(\sigma)$. For (2) note that m_σ also works for the rays $\rho \prec \tau \prec \sigma$. So both m_σ and m_τ work for all the rays $\rho \prec \tau$, we conclude $m_\sigma - m_\tau \in M(\tau)$. □

REMARK 4.23. By part (4) of Proposition 4.20, a Cartier divisor on an affine toric variety U_σ corresponds to an element of M, which is determined $\mod M(\sigma)$. This shows that $\mathrm{CDiv}_T(U_\sigma)$, the group of T-Cartier divisors on U_σ, is isomorphic to $M/M(\sigma)$. This agrees with (4.2.3), as $\sigma^\perp$ is precisely the kernel of $m \mapsto \mathrm{div}(\chi^m)$.

We use the Cartier data to compute intersection numbers in the next section. Let us give an example of how to find the Cartier data for a toric divisor.

EXAMPLE 4.24. Consider the toric surface whose fan Δ in $\mathbb{R}^2$ has ray generators

$$u_1 = e_1, \quad u_2 = e_2, \quad u_0 = 2e_1 + 3e_2$$

and maximal cones

$$\sigma = \mathrm{Cone}(u_1, u_0), \quad \sigma' = \mathrm{Cone}(u_2, u_0).$$

We have ray divisors D_0, D_1, D_2. Consider a divisor $D = aD_0 + bD_1 + cD_2$, we want to describe conditions on the integers for D to be Cartier.

The divisor is Cartier if we have Cartier data $m_\sigma, m_{\sigma'}$. Let us write

$$m_\sigma = \xi e_1 + \zeta e_2, \quad m_{\sigma'} = \eta e_1 + \theta e_2.$$

Then

$$\begin{aligned} a &= \langle m_\sigma, u_0\rangle = 2\xi + 3\zeta & \qquad b &= \langle m_\sigma, u_1\rangle = \xi \\ a &= \langle m_{\sigma'}, u_0\rangle = 2\eta + 3\theta & \qquad c &= \langle m_{\sigma'}, u_2\rangle = \theta \end{aligned}$$

A small computation gives $6(\xi + \zeta - \eta - \theta) = -a + 2b + 3c$, hence

$$D \text{ is Cartier} \leftrightarrows a \equiv 2b + 3c \mod 6,$$

and when this is the case we have the Cartier data

$$m_\sigma = be_1 + \tfrac{a-2b}{3} e_2, \quad m_{\sigma'} = \tfrac{a-3c}{2} e_1 + ce_2.$$

Fans and divisors. On a toric variety we can relate Weil and Cartier divisors. We finish this section by stating two propositions, a proof can be found in [2, section 4.2]. When the defining fan is smooth the Weil and Cartier divisors coincide:

PROPOSITION 4.25. *Let X_Δ be the toric variety of the fan Δ. Then the following are equivalent:*

- Δ *is smooth.*
- *Every Weil divisor is Cartier.*
- $\mathrm{Pic}(X_\Delta) = \mathrm{Cl}(X_\Delta)$.

This result has a simplicial analog:

PROPOSITION 4.26. *Let X_Δ be the toric variety of the fan Δ. Then the following are equivalent:*

- Δ *is simplicial.*
- *Every Weil divisor is $\mathbb{Q}$-Cartier, i.e., has a positive multiple that is Cartier.*
- $\mathrm{Pic}(X_\Delta)$ *has finite index in* $\mathrm{Cl}(X_\Delta)$.

4.3 Intersection products

On a normal variety we can consider the intersection of a (Cartier) divisor, which has codimension one, and a (irreducible complete smooth) curve, which has dimension one. When this intersection is finite and the divisor and the curve meet transversally, then we may count the number of points where they intersect. This cannot be done in general, but we do have the concept of an *intersection product*. This product is a pairing of Cartier divisors and complete curves satisfying the following rules.

DEFINITION 4.27. Given a smooth irreducible complete curve C on a normal variety X. Then for Cartier divisors D, D' on X we want to define an *intersection product* $D \cdot C$ such that:

- $(D + D') \cdot C = D \cdot C + D' \cdot C$,
- $D \cdot C = D' \cdot C$ whenever $D \sim D'$,
- When $D \cap C$ is finite and C, D meet transversally, we get $D \cdot C = \#(D \cap C)$.

For the usual definition of the intersection product we refer to the standard literature[5]. There you can also find the precise meaning of a divisor and a curve *meeting transversally*.

REMARK 4.28. We can extend the definition of the intersection product for Cartier divisors to $\mathbb{Q}$-Cartier divisors. Let D be a divisor such that kD is Cartier for a positive integer k. Given a curve C we may compute the intersection product

$$D \cdot C = \frac{1}{k}(kD) \cdot C \in \mathbb{Q}.$$

4.3.1 Intersection products on a toric variety

We review the intersection product on a toric variety X_Δ for a fan in $N_\mathbb{R}$. We show how to compute the intersection product for a T-Cartier divisor D and a T_N-invariant irreducible complete smooth curve C. We are only interested in such curves so from now on we drop the adjectives.

Walls. A curve on a toric variety X_Δ corresponds to a wall $\tau \in \Delta$.

DEFINITION 4.29 (Wall). Let Δ be a fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. Then $\tau \in \Delta(n-1)$ is a *wall* if it is the intersection of two maximal cones $\sigma, \sigma' \in \Delta(n)$. Then τ is a face of both and we say that the wall τ separates σ and σ', as illustrated in Figure 16.

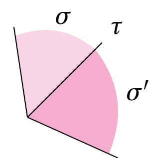

Figure 16: A wall seperating two cones in a 2-dimensional fan.

REMARK 4.30. In a complete fan Δ, every $\tau \in \Delta(n-1)$ is a wall.

PROPOSITION 4.31. *Let C be a (torus-invariant irreducible smooth complete) curve in a toric variety X_Δ for a fan Δ in $N_\mathbb{R} \simeq \mathbb{R}^n$. Then $C = V(\tau)$ for a wall $\tau \in \Delta(n-1)$ separating two cones $\sigma, \sigma' \in \Delta(n)$.*

Proof. Since $\dim C = 1$ it corresponds to a cone τ in $\Delta(n-1)$ and $C = V(\tau)$. Recall that the orbit closure is $V(\tau) = X_{\mathrm{Star}(\tau)}$ for the fan $\mathrm{Star}(\tau)$ in $N(\tau)_\mathbb{R}$. Since $\dim \tau = n-1$ the quotient lattice $N(\tau) = N/N_\tau$ is 1-dimensional and we have $N(\tau)_\mathbb{R} \simeq \mathbb{R}^1$. The fan corresponds to the complete variety C so by Corollary 3.15 the support of the fan $\mathrm{Star}(\tau)$ is the whole of $\mathbb{R}^1$. Hence we can find $\sigma, \sigma' \in \Delta$ such that τ is a face of both cones and $1 \in \overline{\sigma}$ and $-1 \in \overline{\sigma'}$. Then it follows that $\sigma, \sigma' \in \Delta(n)$ and τ is the wall separating these cones. □

Toric intersection product. Let D be a Cartier divisor on a toric variety X_Δ. Let $\tau \in \Delta(n-1)$ be a wall separating two cones $\sigma, \sigma' \in \Delta(n)$. These cones correspond to Cartier data m_σ, $m_{\sigma'}$. Also, pick a $t_0 \in \sigma' \cap N$ that maps to a minimal generator $\overline{t_0}$ of $\overline{\sigma'}$ in $N(\tau)_\mathbb{R}$. These are the lattice objects we need to compute the intersection product of D with the curve $C = V(\tau)$:

PROPOSITION 4.32. *Let D be a Cartier divisor on X_Δ and let $\tau = \sigma \cap \sigma'$ be a wall in Δ. Then $C = V(\tau)$ is a curve and in this case the intersection product satisfies*

$$D \cdot C = \langle m_\sigma - m_{\sigma'}, t_0 \rangle.$$

Proof. For a proof we refer to [2, Proposition 6.3.8]. □

EXAMPLE 4.33. We return to Example 4.24. We have the divisor $D = aD_0 + bD_1 + cD_2$ such that $a \equiv 2b + 3c \mod 6$, so D is Cartier with Cartier data

$$m_\sigma = be_1 + \tfrac{a-2b}{3} e_2, \quad m_{\sigma'} = \tfrac{a-3c}{2} e_1 + ce_2.$$

Let τ be the ray through u_0, which defines a curve $C = V(\tau)$. The wall τ separates the maximal cones $\sigma = \mathrm{Cone}(u_1, u_0), \sigma' = \mathrm{Cone}(u_2, u_0)$. Recall that $N_\tau = \mathrm{Span}(u_0) \cap \mathbb{Z}^2$ and note that $e_1 + 2e_2$ and u_0 form a basis for $\mathbb{Z}^2$. Hence $t_0 = e_1 + 2e_2$ maps to a minimal generator $\overline{t_0}$ of $\overline{\sigma'}$ in $N(\tau) = N/N_\tau$. We are now ready to compute the intersection product:

$$D \cdot C = \langle m_\sigma - m_{\sigma'}, t_0 \rangle = \tfrac{a-2b-3c}{6}.$$

The fan Δ is simplicial so every divisor is $\mathbb{Q}$-Cartier, hence the above formula works for arbitrary integers a, b, c. This gives the intersections of the ray divisors with the curve $V(\tau)$:

$$D_0 \cdot V(\tau) = -\tfrac{1}{6}, \quad D_1 \cdot V(\tau) = \tfrac{1}{3}, \quad D_2 \cdot V(\tau) = \tfrac{1}{2}.$$

It will turn out that for simplicial cones we can compute the intersection numbers for ray divisors in terms of the multiplicities of cones in the fan. This brings us to wall relations in the next subsection.

REMARK 4.34. Remark that $D_0 = V(\tau)$, so in Example 4.33 we have computed the intersection number of D_0 with itself. This is something particular for surfaces.

A ray divisor $D_\rho = V(\rho)$ on a toric surface is also a curve. When this curve is complete we define the self-intersection number $D_\rho^2 = D_\rho \cdot D_\rho$.

4.3.2 Intersection products on toric surfaces

We give details for computing intersection numbers on toric surfaces. Let Δ be a fan in $N_\mathbb{R} \simeq \mathbb{R}^2$. Then we can give the ray generators $u_1, \dots, u_r \in N$ in clockwise order, as illustrated in Figure 17. The rays $\rho_i = \mathrm{Cone}(u_i)$ correspond to ray divisors D_i. The goal is to compute the intersection numbers among the ray divisors.

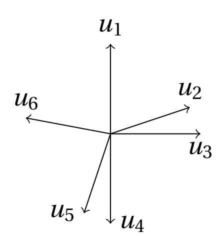

Figure 17: The rays of a 2-dimensional fan in clockwise order.

REMARK 4.35. A 2-dimensional cone which is given by two generators is simplicial. So the fans we are dealing with are all simplicial. By Proposition 4.26 it follows that on a toric surface all divisors are $\mathbb{Q}$-Cartier.

Relations among the ray generators. Let $m \in M$, then we have the principal divisor

$$\sum_{i=1}^{k} \langle m, u_i \rangle D_i = \mathrm{div}(\chi^m) \sim 0.$$

Now let C be a curve in X_Δ, intersecting the divisor with C we find

$$\sum_{i=1}^{k} \langle m, u_i \rangle (D_i \cdot C) = 0,$$

which we rewrite into $\left\langle m, \sum_{i=1}^{k} (D_i \cdot C) u_i \right\rangle = 0$. Since this holds for all $m \in M$, this gives the relation:

$$\sum_{i=1}^{k} (D_i \cdot C) u_i = 0. \tag{4.3.1}$$

We next describe the *wall relation.* Let τ be a wall in the toric surface X_Δ. So $\tau = \mathrm{Cone}(u)$ separates the cones $\sigma_1 = \mathrm{Cone}(u, v)$ and $\sigma_2 = \mathrm{Cone}(u, w)$ for ray generators u, v, w. Then the vectors $u,\ v,\ w \in \mathbb{Z}^2$ are linear dependent, this gives rise to the wall relation:

$$\kappa u + \lambda v + \mu w = 0, \quad \kappa, \lambda, \mu \in \mathbb{Z}. \tag{4.3.2}$$

Since v, w lie on opposite sides of the wall, we may assume $\lambda, \mu > 0$. Note that (4.3.2) is the wall relation for surfaces. The general version is given in [2, (6.4.4)].

Intersection formulas. We now combine the above relations to compute the intersection product of a ray divisor with a curve on a toric surface.

LEMMA 4.36. *Let τ be a wall separating two cones σ and σ' in a 2-dimensional fan. So τ is an edge of σ, we use ρ to refer to the other edge of σ. The ray ρ corresponds to a ray divisor D_ρ and we may compute*

$$D_\rho \cdot V(\tau) = \frac{1}{\mathrm{mult}(\sigma)},$$

where $\mathrm{mult}(\sigma)$ *is the multiplicity of the cone from Definition 2.25.*

Proof. Let u and v denote the minimal generators of τ and ρ. We can find $m \in M_{\mathbb{Q}}$ such that $\langle m, u \rangle = 0$ and $\langle m, v \rangle = -1$. By Remark 4.35 the divisor D_ρ is $\mathbb{Q}$-Cartier, we take $k \in \mathbb{N}$ such that kD_ρ is Cartier and we obtain the Cartier data for D_ρ:

$$m_\sigma = km, \quad m_{\sigma'} = 0.$$

We may now compute

$$D_\rho \cdot V(\tau) = \frac{1}{k}(kD_\rho) \cdot V(\tau) = \frac{1}{k}\langle km, t_0 \rangle = \langle m, t_0 \rangle,$$

where $t_0 \in \sigma' \cap M$ maps to a minimal generator $\overline{t_0}$ of $\overline{\sigma'}$ in $N(\tau)$. Remark that t_0 and u generate the lattice N, as $N_\tau = \mathbb{Z}u$ and $N(\tau) = N/N_\tau$. Hence we may find positive integers α, β such that $v = -\alpha t_0 + \beta u$. The minus sign appears because v and t_0 lie on opposite sides of the wall. We have the inclusion

$$\mathbb{Z}v + \mathbb{Z}u \subset \mathbb{Z}t_0 + \mathbb{Z}u = N$$

and we remark that $N_\sigma = (\mathbb{R}u + \mathbb{R}v) \cap N = N$. We conclude that $\text{mult}(\sigma) = \alpha$ is the index of $\mathbb{Z}v + \mathbb{Z}u$ in N_σ. We may write $t_0 = -\frac{1}{\alpha}(v-u) = -\frac{1}{\text{mult}(\sigma)}(w-v)$ and we get

$$D_\rho \cdot V(\tau) = \langle m, t_0 \rangle = -\frac{1}{\text{mult}(\sigma)}\big(\langle m, u\rangle - \langle m, v\rangle\big) = \frac{1}{\text{mult}(\sigma)}.$$

□

PROPOSITION 4.37. *Let* $\tau = \text{Cone}(u)$ *be a wall separating* $\sigma = \text{Cone}(u, v)$ *and* $\sigma' = \text{Cone}(u, w)$ *in a 2-dimensional fan* Δ*. Let* D_u*,* D_v *and* D_w *denote the corresponding ray divisors on* X_Δ*. Then the wall relation* (4.3.2) *equals the relation* (4.3.1) *up to a constant. Furthermore*

(1) $D_\rho \cdot V(\tau) = 0$ *for all rays* $\rho \notin \{\rho_u, \rho_v, \rho_w\}$.

(2) $D_v \cdot V(\tau) = \dfrac{1}{\text{mult}(\sigma)}$ *and* $D_w \cdot V(\tau) = \dfrac{1}{\text{mult}(\sigma')}$.

(3) $D_u \cdot V(\tau) = \dfrac{\kappa}{\lambda\,\text{mult}(\sigma)} = \dfrac{\kappa}{\mu\,\text{mult}(\sigma')}$.

Proof. (2) This follows form Lemma 4.36. (1) Observe that if $\rho \notin \{\rho_u, \rho_v, \rho_w\}$, then ρ and τ never lie in the same cone of Δ, so that $D_\rho \cap V(\tau) = \emptyset$ by the Orbit-Cone Correspondence. It follows that $D_\rho \cdot V(\tau) = 0$.

Taking the curve $C = V(\tau)$, the relation (4.3.1) reduces to the equation

$$(D_u \cdot C)u + (D_v \cdot C)v + (D_u \cdot C)w = 0. \tag{4.3.3}$$

By part (2) the numbers $(D_u \cdot C)$ and $(D_u \cdot C)$ are positive, this means that (4.3.1) equals the wall relation up to multiplication by a constant. This implies

$$\lambda(D_u \cdot V(\tau)) = \kappa(D_v \cdot V(\tau)), \quad \lambda(D_w \cdot V(\tau)) = \mu(D_v \cdot V(\tau)),$$

and the assertion in (3) follows. □

Complete surfaces. In a complete toric surface every ray is a wall, so we can compute the intersection numbers among all of the ray divisors.

COROLLARY 4.38. *Let* Δ *be a complete fan in* $N_\mathbb{R} \simeq \mathbb{R}^2$ *with ray generators* $u_1, \ldots, u_r \in N$ *in clockwise order. We have ray divisors* D_i *on* X_Δ*. We may compute*

$$D_i \cdot D_j = \begin{cases} 0 & \textit{if} \quad |i-j| > 1 \\ \dfrac{1}{\text{mult}(\sigma)} & \textit{if} \quad |i-j| = 1 \;\; \textit{and} \;\; \sigma = \text{Cone}(u_i, u_j) \\ \dfrac{\kappa}{\lambda\,\text{mult}(\sigma)} = \dfrac{\kappa}{\mu\,\text{mult}(\sigma')} & \textit{if} \quad i = j \end{cases}$$

In the last case the coefficients come from the wall relation $\kappa u_i + \lambda u_{i-1} + \mu u_{i+1} = 0$ *and the cones are* $\sigma = \text{Cone}(u_{i-1}, u_i)$, $\sigma' = \text{Cone}(u_i, u_{i+1})$.

EXAMPLE 4.39. We re-do the computations in Example 4.33, this time using the wall relations. The wall τ separates the maximal cones $\sigma_1 = \text{Cone}(u_1, u_0)$, $\sigma_2 = \text{Cone}(u_2, u_0)$. We compute the multiplicities

$$\text{mult}(\sigma_1) = 3, \quad \text{mult}(\sigma_2) = 2.$$

The curve is $V(\tau) = D_0$ and Corollary 4.38 implies

$$D_1 \cdot D_0 = \tfrac{1}{3}, \quad D_2 \cdot D_0 = \tfrac{1}{2},$$

and the relation

$$(-1) \cdot u_0 + 2 \cdot u_1 + 3 \cdot u_2 = 0$$

implies

$$D_0 \cdot D_0 = \tfrac{-1}{2 \cdot 3} = \tfrac{-1}{3 \cdot 2} = -\tfrac{1}{6}.$$

For a Cartier divisor $D = aD_0 + bD_1 + cD_2$ we get the same intersection product as computed in Example 4.33.

4.3.3 Smooth surfaces.

We investigate the theory of toric divisors in the special case of divisors on a *smooth* toric surface. When the variety X_Δ is smooth, all multiplicities of the cones are 1. Hence the relation (4.3.3) reduces to

$$\big(D_{\rho_u} \cdot V(\tau)\big) u + v + w = 0.$$

So we can assign an integer to each ray which is also a wall:

LEMMA 4.40. *Let* Δ *be a smooth complete fan in* $N_{\mathbb{R}} \simeq \mathbb{R}^2$ *with ray generators* $u_1, \ldots, u_r$. *Then we can find integers* $b_1, \ldots, b_{r-1}$ *such that*

$$u_{i-1} + u_{i+1} = b_i u_i.$$

□

So for smooth complete toric surfaces Corollary 4.38 reduces to:

PROPOSITION 4.41. *Let Δ be a smooth complete fan in $N_{\mathbb{R}} \simeq \mathbb{R}^2$ with ray generators $u_1, \ldots, u_r \in N$ in clockwise order. We have ray divisors D_i on X_Δ. We may compute*

$$D_i \cdot D_j = \begin{cases} 0 & \text{if} \quad |i-j| > 1 \\ 1 & \text{if} \quad |i-j| = 1 \\ -b_i & \text{if} \quad i = j \end{cases},$$

where b_i is the integer from Lemma 4.40. □

COROLLARY 4.42. *For a divisor $D = \sum_i a_i D_i$ we get get the intersection product*

$$D \cdot D_j = a_{j-1} - b_i a_i + a_{j+1}.$$

EXAMPLE 4.43. The Hirzebruch surface $\mathscr{H}_r$ is a smooth complete toric surface. There are four ray divisors coming from the ray generators

$$u_1 = e_2, \quad u_2 = e_1, \quad u_3 = -e_2, \quad u_4 = -e_1 + re_2.$$

The ray ρ_2 gives the relation

$$u_1 + u_3 = r \cdot u_2$$

and we find $b_2 = -r$. Likewise we can find $b_1 = 0$, $b_3 = 0$ and $b_4 = r$. This gives selfintersection numbers

$$D_1^2 = 0, \quad D_2^2 = -r, \quad D_3^2 = 0, \quad D_4^2 = r,$$

and the the intersection numbers

$$D_1 \cdot D_2 = D_2 \cdot D_3 = D_3 \cdot D_4 = D_4 \cdot D_1 = 1.$$

REMARK 4.44. For a smooth toric surface which is not complete we can still use the intersection formulas. If a ray generator u_j corresponds to a wall then D_j is a curve, and in this case the formulas from Proposition 4.41 make sense.

5. Resolution of toric surface singularities

In this chapter we review the singularities in a toric surface X_Δ. As we discussed at the end of Chapter 3, a refinement of the fan can lead to a resolution of singularities. We repeat our main result:

THEOREM 3.40. *Let $\widetilde{\Delta}$ be a smooth fan that refines a given fan Δ, we write $\varphi\colon X_{\widetilde{\Delta}} \to X_\Delta$ for the associated map on varieties. If $\widetilde{\Delta}$ does not refine any smooth cone in Δ then $\varphi\colon X_{\widetilde{\Delta}} \to X_\Delta$ is a resolution of singularities.*

In general we have to be careful *not* to refine any smooth cone. For instance, a nonsmooth 3-dimensional cone σ may have a smooth 2-dimensional face τ. So when we provide a smooth provide refinement of σ we should not refine τ.

However, in dimension 2 this is not a problem. We fix a lattice $N \simeq \mathbb{Z}^2$ for the remainder of this chapter.

Let σ be a nonsmooth cone in $N_\mathbb{R}$. Its edges are smooth cones. A smooth refinement of σ still contains the edges of σ. This means that we only change U_σ at the singular point p_σ.

Now consider a 2-dimensional fan Δ in $N_\mathbb{R}$. Suppose we find a smooth refinement for each nonsmooth cone in Δ. Let $\widetilde{\Delta}$ denote the refinement of Δ where the nonsmooth cones are replaced by their smooth refinements. Then the smooth cones of Δ are also cones in $\widetilde{\Delta}$ and Theorem 3.40 leads to a resolution of singularities of the surface X_Δ.

5.1 Problem statement

We see that the problem of a resolution of singularities for toric surfaces reduces to refining nonsmooth cones. Refining a 2-dimensional cone σ means introducing rays $\operatorname{Cone}(u_i)$ for u_i in σ. To resolve the singularities in U_σ we want to insert rays in such a way that they subdivide σ into smooth cones. The problem we set out to solve is:

PROBLEM 1. Let σ be a cone in $N_\mathbb{R}$ with ray generators u, v. We need to find a sequence $u = u_1, \ldots, u_n = v$ of primitive vectors in σ such that subsequent pairs form a basis for the lattice. This gives then a smooth refinement of σ, this is illustrated in Figure 18.

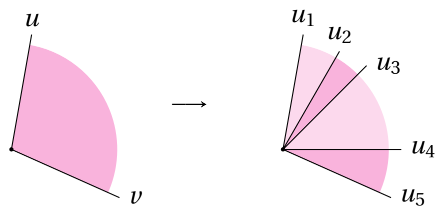

Figure 18: A smooth refinement of the cone.

5.2 The algorithm for solving the problem

In this section we develop a recipe to solve Problem 1. We describe an algorithm that finds the optimal solution to the problem.

5.2.1 A toy example and the normal form.

EXAMPLE 5.1. Fix a lattice basis (e, f). Consider the cone generated by f and $2e - f$ in N. This corresponds to the affine variety $\mathcal{Z}(xz - y^2) \subset \mathbb{C}^3$ which is singular at 0. We refine σ by inserting the ray through e. This subdivides σ into the smooth cones $\sigma_1 = \text{Cone}(f, e)$ and $\sigma_2 = \text{Cone}(e, 2e - f)$.

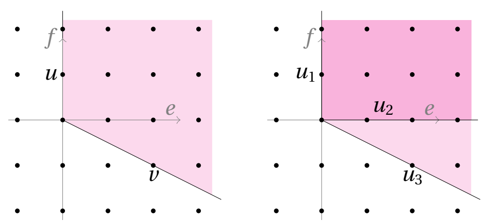

Figure 19: Resolution of $\text{Cone}(f, 2e - f)$.

In this example it was slightly obvious to insert the ray through e. Consider the cone $\sigma' = \text{Cone}(5e + 3f, e + f)$. In this case it is not immediately clear which ray to insert, consider the dashed arrows in the left side of Figure 20.

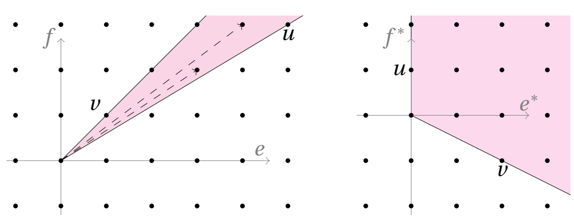

Figure 20: The cone σ' presented in two different ways.

Note however that this is the cone generated by f^* and $2e^* - f^*$, for $e^* = (3,2)$; $f^* = (5,3)$. Hence via change of basis we see $U_{\sigma'} = U_\sigma$. It follows naturally to insert the ray through $e^* = 3e + 2f$.

It turns out that through a change of basis we may arrive at a cone for which it is obvious to insert the ray through e. This change of basis will be the main ingredient for resolving a surface singularity. Let us first describe the form of a cone for which it is obvious to introduce the ray e.

DEFINITION 5.2 (Normal Form). Fix a lattice basis (e, f). Given a cone $\sigma = \text{Cone}(u, v)$ for primitive vectors u, v, we say that σ is in normal form, with respect to the basis, if:

- $u = f$ and
- $v = \lambda e - \mu f$, for natural numbers $0 \le \mu < \lambda$.

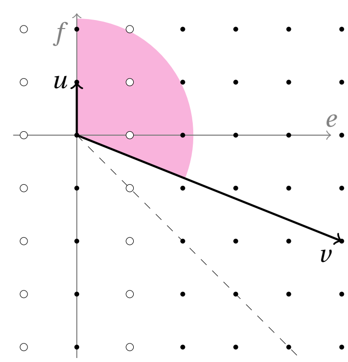

Figure 21: A cone in normal form.

The normality means that v lies within the checkered area in Figure 21. The circled coordinates are the ones which extend f to a basis. When refining a cone as in Problem 1 these are the candidates for u_2. Indeed it is obvious to choose $u_2 = e$.

REMARK 5.3. If $\mu = 0$ then $v = e$ since it is a primitive vector, and σ is a smooth cone.

REMARK 5.4. A nice property of the normal form is that the dual cone is contained in the upper-right quadrant. This means $A_\sigma \subset \mathbb{C}[x, y] \subset \mathbb{C}[x, x^{-1}, y, y^{-1}]$.

We claim that using a change of basis we can always write a cone in normal form. In fact:

PROPOSITION 5.5. *Given a cone σ there exists a unique basis with respect to which σ is in normal form.*

We prove this using modified euclidean division, note the minus sign:

LEMMA 5.6. *Given integers a, b with $a > 0$ we can find integers k, r with $0 \le r < a$ such that $b = ka - r$.* □

Proof of Proposition 5.5. Let σ be a cone with ray generators u, v . We first prove existence. Since u is primitive we can extend it to a lattice basis (e^*, u), for some (primitive) vector e^*. Write $v = ae^* + bu$ and note that $a \neq 0$, otherwise we would have $v = \pm u$. We may assume $a > 0$, replacing e^* for $-e^*$ if necessary. By Lemma 5.6 we get integers k, r with $0 \leq r < a$ such that $b = ka - r$. We obtain

$$v = ae^* + (ka - r)u.$$

Now we define $e = e^* + ku$ and we may write $v = ae - ru$. Setting $f = u, \mu = r$ and $\lambda = a$ we see that σ is in normal form with respect to the basis (e, f):

$$v = \lambda e - \mu f, \quad 0 \leq \mu < \lambda.$$

We now prove uniqueness. Suppose (e, f) and (e', f') are both bases with respect to which σ is in normal form. Then $f' = u = f$. We may write $e' = \alpha e + \beta f$ for some natural numbers α, β. Combining this with $v = \lambda e - \mu f$ and $v = \lambda' e' - \mu' f'$ we obtain

$$\begin{aligned} 0 &= (\lambda e - \mu f) - (\lambda' e' - \mu' f') \\ &= \lambda e - \lambda' e' + (\mu' - \mu) f \\ &= (\lambda - \alpha\lambda')e + (\mu' - \mu - \beta\lambda) f. \end{aligned}$$

This implies $\lambda = \alpha\lambda'$ and $\mu = \mu' - \beta\lambda$.

Recall that we have $0 \leq \mu < \lambda$ and $0 \leq \mu' < \lambda'$. Rewriting $\mu \geq 0$ gives $\mu' - \alpha\beta\lambda' \geq 0$. Combining this with $\mu' < \lambda'$ we must have $\alpha\beta = 0$. Since λ, λ' are both positive we have $\alpha \neq 0$, hence $\beta = 0$. Now $e' = \alpha e$ and since e' is a primitive vector we have $\alpha = 1$. We conclude $(e, f) = (e', f')$. □

5.2.2 The algorithm for resolving an affine toric surface singularity.

We arrive at an algorithm to resolve a singularity in an affine toric variety U_σ for some cone σ in $N_\mathbb{R}$.

To solve Problem 1 we need to introduce primitive vectors. To this end we write σ in normal form and we introduce the ray generated by the basis vector e. This subdivides σ into the upper right quadrant and a cone which lies inside the checkered area of Figure 21. So we have reduced the problem to finding a smooth refinement of this smaller cone.

Algorithm 1 repeats this process of writing a cone in normal form and proceeding with the bottom cone. We prove that the algorithm terminates and produces a resolution. It will turn out that this coincides with the calculation of a continued fraction.

ALGORITHM 1: Resolution of affine toric surfaces

input : A cone $\sigma = \text{Cone}(u, v)$ as in Problem 1.
output : A sequence of primitive vectors in σ.
Set a counter $i = 1$ and set $\sigma_1 = \sigma$, $e^{(1)} = 0$;
while $v \neq e^{(i)}$ **do**
  Apply Proposition 5.5 to σ_i, finding a basis (e, f) and natural numbers $0 \leq \mu < \lambda$;
  Store $f^{(i)} = f, e^{(i)} = e, \mu^{(i)} = \mu, \lambda^{(i)} = \lambda$;
  Set $u_i = f$;
  Update the counter ;
  Set $\sigma_i = \text{Cone}(e, v)$;
end

REMARK 5.7. Iterating the process produces a sequence of bases $\left(e^{(i)}, f^{(i)}\right)$ such that:

- $f^{(i)} = u_i$
- $f^{(i+1)} = e^{(i)}$
- $v = \lambda^{(i)} e^{(i)} - \mu^{(i)} f^{(i)}$

THEOREM 5.8. *The produced sequence u_i is a solution to Problem 1, i.e. (1) the output is sensible and (2) the algorithm terminates.*

Proof. For (1), Remark 5.7 tells us that $\text{Cone}(u_i, u_{i+1}) = \text{Cone}(f^{(i)}, e^{(i)})$ is smooth. For (2) we investigate how the sequence develops. The algorithm terminates when we arrive at an i such that $v = e^{(i)}$, or, by Remark 5.3, when $\mu^{(i)} = 0$.

Given $f^{(i+1)} = e^{(i)}$ and assuming $\mu^{(i)} \neq 0$ we follow the construction as in Proposition 5.5 to get $e^{(i+1)}$. Setting $e^* = -f^{(i)}$ gives a basis $(e^*, f^{(i+1)})$. Rewriting $v = \lambda^{(i)} e^{(i)} - \mu^{(i)} f^{(i)}$ we obtain

$$v = \mu^{(i)} e^* + \lambda^{(i)} f^{(i+1)}.$$

Since $\mu^{(i)} > 0$ we may find integers k, r with $0 \leq r < \mu^{(i)}$ such that $\lambda^{(i)} = k\mu^{(i)} - r$. We now define:

- $e^{(i+1)} = e^* + k f^{(i+1)}$,
- $\lambda^{(i+1)} = \mu^{(i)}$,
- $\mu^{(i+1)} = r$.

Observe that we indeed maintain

$$\begin{aligned} \lambda^{(i+1)} e^{(i+1)} - \mu^{(i+1)} f^{(i+1)} &= \mu^{(i)} e^* + (k\mu^{(i)} - r) f^{(i+1)} \\ &= -\mu^{(i)} f^{(i)} + \lambda^{(i)} e^{(i)} \\ &= v. \end{aligned}$$

We see that the $\mu^{(i)}$ form a strictly decreasing sequence. Hence there is an i for which $\mu^{(i)} = 0$ and the algorithm terminates. □

REMARK 5.9. Rewrite $\lambda^{(i)} = k\mu^{(i)} - r$ to get $\mu^{(i+1)} = k\mu^{(i)} - \mu^{(i-1)}$. Note that k can be defined as $\lceil \mu^{(i-1)}/\mu^{(i)} \rceil$, providing a recursive definition for the $\mu^{(i)}$.

This recursive relation extends to the vectors u_i. Rewrite $e^{(i+1)} = e^* + kf^{(i+1)}$ to get

$$u_{i+2} = -u_i + ku_{i+1}, \quad \text{for } k = \lceil \mu^{(i-1)}/\mu^{(i)} \rceil.$$

This recursion brings us to the relation with continued fractions.

5.2.3 Continued fractions

The Hirzebruch-Jung continued fraction [10, 11] of a rational number q is the expansion

$$q = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \frac{1}{a_r}}}$$

for natural numbers $a_i \geq 2$, these are the *partial quotients* of the continued fraction. Note the minus signs in the continued fraction, we abbreviate the expression by

$$[[a_1, \ldots, a_r]].$$

Given a (finite) expansion we define numbers m_i, n_i recursively:

$$m_i = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ a_{i-2}m_{i-1} - m_{i-2} & \text{if } i \geq 2 \end{cases} \quad n_i = \begin{cases} -1 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ a_{i-2}n_{i-1} - n_{i-2} & \text{if } i \geq 2 \end{cases}$$

The rational numbers $\frac{m_i}{n_i}$ are called the *convergents* of the HJ-continued fraction. These numbers are the result of 'cutting off' the continued fraction, i.e.

$$\frac{m_i}{n_i} = [[a_1, \ldots, a_{i-2}]], \quad i \geq 3.$$

The convergents are rational approximations of q satisfying

$$\frac{m_3}{n_3} > \ldots > \frac{m_{r-1}}{n_{r-1}} > \frac{m_r}{n_r} = q,$$

These properties are proven in [2, Proposition 10.2.2]. We represent these numbers in Table 1 for good measure.

Given a cone σ as in Problem 1, denote (e, f) for the basis with respect to which σ is in normal form, we get associated natural numbers $0 \leq \mu < \lambda$.

PROPOSITION 5.10. *The algorithm computes the Hirzebruch-Jung continued fraction of λ/μ. Furthermore, the columns of Table 1 represent the introduced rays:*

$$u_i = m_i e - n_i f.$$

i	1	2	3	4	$\dots$
a_i	a_1	a_2	$\dots$		
m_i	0	1	m_3	m_4	$\dots$
n_i	-1	0	n_3	n_4	$\dots$

Table 1: The table of convergents.

Proof. Run the algorithm on σ, producing the decreasing sequence

$$\mu^{(1)} > \mu^{(2)} > \ldots > \mu^{(n)} = 0.$$

Set $\mu^{(0)} = \lambda$ and define $a_i = \lceil \mu^{(i-1)} / \mu^{(i)} \rceil$, for $1 \le i \le n-1$. We now claim:

$$\lambda / \mu = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{n-2}}}}. \tag{5.2.1}$$

Remark 5.9 gives the recursive relation $\mu^{(i-1)} = a_i \mu^{(i)} - \mu^{(i+1)}$, for $i \le n-2$ we have $\mu^{(i)} > \mu^{(i+1)} > 0$ and we may rewrite the relations as

$$\mu^{(i-1)} / \mu^{(i)} = a_i - \frac{1}{\mu^{(i)} / \mu^{(i+1)}}. \tag{5.2.2}$$

We splice these relations together. We begin with $\mu^{(0)} / \mu^{(1)}$ and we can use (5.2.2) to construct a 'chain' of fractions. Splicing the relations for $i = 1, \ldots, n-2$ we get the continued fraction

$$\mu^{(0)} / \mu^{(1)} = [[a_1, a_2, \ldots, \mu^{(n-2)} / \mu^{(n-1)}]].$$

As $\mu^{(n)} = 0$ the recursive relation $\mu^{(n-2)} = a_{n-1} \mu^{(n-1)} - \mu^{(n)}$ reduces to

$$\mu^{(n-2)} / \mu^{(n-1)} = a_{n-2}$$

and we arrive at the continued fraction (5.2.1).

We now show that the convergents m_i / n_i correspond to the introduced rays $m_i e - n_i f$. We define $u_1 = f, u_2 = e$ in accordance with

$$\begin{aligned} m_1 &= 0, \qquad & n_1 &= -1, \\ m_2 &= 1, \qquad & n_2 &= 0. \end{aligned}$$

We prove by induction on i. Suppose $u_i = m_i e - n_i f$ and $u_{i+1} = m_{i+1} e - n_{i+1} f$. By Remark 5.9 we have $u_{i+2} = a_i u_{i+1} - u_i$. So $u_{i+2} = (a_i m_{i+1} - m_i) e - (a_i n_{i+1} - n_i) f = m_{i+2} e + n_{i+2} f$, which finishes the induction. □

EXAMPLE 5.11. Consider Cone$(f, 11e - 7f)$, which is in normal form. It is a nonsmooth cone. We run the algorithm to find a smooth refinement.

In each step we are given $f^{(i)}$, $e^{(i)}$, $\lambda^{(i)}$, $\mu^{(i)}$, $k = \lceil \lambda^{(i)}/\mu^{(i)} \rceil$; we compute

$$f^{(i+1)} = e^{(i)}, \quad e^{(i+1)} = -f^{(i)} + ke^{(i)}, \quad \lambda^{(i+1)} = \mu^{(i)}, \quad \mu^{(i+1)} = k\mu^{(i)} - \lambda^{(i)}.$$

The result is given in Table 2. The rays u_i are represented in Figure 22.

i	$f^{(i)}$	$e^{(i)}$	$\lambda^{(i)}$	$\mu^{(i)}$	$\lceil \lambda^{(i)}/\mu^{(i)} \rceil$	u_i
1	f	e	11	7	2	f
2	e	$-f+2e$	7	3	3	e
3	$-f+2e$	$-3f+5e$	3	2	2	$-f+2e$
4	$-3f+5e$	$-5f+8e$	2	1	2	$-3f+5e$
5	$-5f+8e$	$-7f+11e$	1	0		$-5f+8e$
6						$-7f+11e$

Table 2: The results of running the algorithm on Cone$(f, 11e - 7f)$.

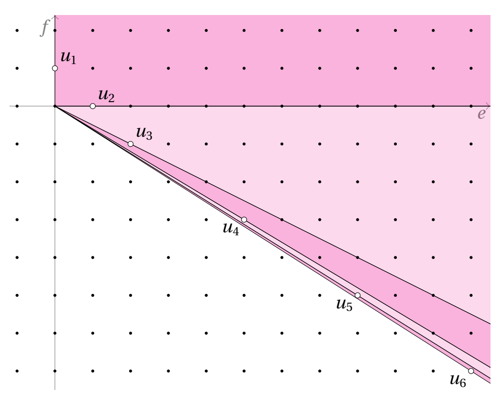

Figure 22: Resolution of Cone$(f, 11e - 7f)$.

The rays in Figure 22 can also be given by means of a continued fraction, as seen in Proposition 5.10. We calculate the HJ continued fraction expansion of λ/μ. We consider the sequence of the $\mu^{(i)}$. We have $\mu^{(0)} = \lambda^{(1)} = 11$ and $\mu^{(1)} = 7$. By Remark 5.9 we have the recursive relations for the $\mu^{(i)}$:

$$\begin{aligned}
\mu^{(2)} &= \boxed{2}\cdot 7 - 11 &= 3 \\
\mu^{(3)} &= \boxed{3}\cdot 3 - 7 &= 2 \\
\mu^{(4)} &= \boxed{2}\cdot 2 - 3 &= 1 \\
\mu^{(5)} &= \boxed{2}\cdot 1 - 2 &= 0
\end{aligned}$$

The HJ continued fraction expansion is

$$\lambda/\mu = 2 - \cfrac{1}{3 - \cfrac{1}{2-\frac{1}{2}}}.$$

The convergents are shown in Table 3, in which the columns also represent the rays in Figure 22. The convergents give the approximations $\frac{2}{1} > \frac{5}{3} > \frac{8}{5} > \frac{11}{7}$.

i	1	2	3	4	5	6
a_i	2	3	2	2		
m_i	0	1	2	5	8	11
n_i	−1	0	1	3	5	7

Table 3: The convergents of λ/μ.

5.2.4 Optimality of the algorithm

The algorithm produces a resolution of singularities in affine toric surfaces by refining a given cone σ. For a toric surface X_Δ in general we may refine all cones $\sigma \in \Delta$ to obtain a smooth refinement $\widetilde{\Delta}$ of Δ and hence a resolution of singularities $X_{\widetilde{\Delta}} \to X_\Delta$. We set out to show that the resolution found by the algorithm is the optimal solution, in the following sense.

DEFINITION 5.12 (Minimal resolution). We say that a resolution of singularities $\varphi\colon \widetilde{X} \to X$ is *minimal* if for every other resolution $\psi\colon Y \to X$ there is a morphism $\rho\colon Y \to \widetilde{X}$ such that $\rho \circ \varphi = \psi$.

The theorem we set out to prove is:

THEOREM 5.13. *The resolution of singularities given by the algorithm is minimal.*

REMARK 5.14. For surfaces we can state the concept of a minimal smooth refinement of fans. A smooth refinement $\widetilde{\Delta}$ of the fan Δ is *minimal* if for another smooth refinement Γ of Δ, the fan Γ refines $\widetilde{\Delta}$.

Theorem 5.13 implies that the smooth refinement $\widetilde{\Delta}$ of Δ provided by the algorithm is the minimal refinement. In this sense the algorithm produces the optimal solution to Problem 1.

The remainder of this section is devoted to proving Theorem 5.13. It suffices to prove this in the affine case. Let σ be a singular cone. We run the algorithm, introducing a sequence of rays $\rho_1, \dots, \rho_r$.

A resolution is minimal if there were no 'unnecessary' blow-ups. The following theorem tells us when this is the case.

THEOREM 5.15 (Castelnuovo). *Given a divisor D on a smooth surface we have $D^2 = -1$ if and only if D is the exceptional curve of some blow-up (it can be 'blown down').*

Proof. For a proof we refer to [8, Theorem 5.7]. □

So we set out to find possible divisors on $X_{\widetilde{\Delta}}$ with self-intersection -1.

LEMMA 5.16. *Let σ be a 2-dimensional cone. If σ is nonsmooth then U_σ has a unique singular point p_σ.*

REMARK 5.17. As a consequence, the resolution produced by the algorithm is a finite sequence of blow-ups in p_σ. We may conclude that the only candidates for divisors with self-intersection -1 are those corresponding to the introduced rays.

The following lemma investigates these ray divisors.

LEMMA 5.18. *For a divisor D_ρ corresponding to an introduced ray ρ in the algorithm. If $\rho = \text{Cone}(u_i)$ then the self-intersection number $D_\rho^2 = -a_{i-1}$ where a_j is the j^{th} coefficient in the continued fraction.*

Proof. By Remark 5.9 the vectors $u_0, \ldots, u_r$ satisfy

$$u_{i-1} + u_{i+1} = a_{i-1} u_i.$$

The lemma follows from Proposition 4.41. □

REMARK 5.19. In Table 1 and Table 3 the columns represent the introduced rays and the self-intersection numbers of the corresponding divisors are given at the top.

Proof of Theorem 5.13. The coefficients in a HJ-continued fraction expansion are all ≥ 2, hence there is no divisor on $X_{\widetilde{\Delta}}$ with self-intersection number -1. The affine case of Theorem 5.13 now follows directly from Theorem 5.15. The general case follows. □

5.3 Further remarks on surfaces and resolutions

Resolution graph. We can visualize the concept of resolutions of a surface with a *resolution graph.* This is a graph displaying the curves of a resolution of singularities and shows the self-intersection numbers.

We first explain how we can provide a graph of the T-invariant curves on a smooth complete surface X. The complete curves on X come from walls in the fan. We can show how these curves intersect. When two curves meet, their intersection number is 1 since we work on a smooth toric surface. In the picture we also show the self-intersection numbers for the curves. In Example 5.20 we produce a graph for the curves on a Hirzebruch surface from Example 3.6.

EXAMPLE 5.20. The Hirzebruch surface is smooth and complete. There are four curves coming from the walls. Neighboring wall divisors intersect each other. We picture this as four lines forming a square. In Example 4.43 we computed the self-intersection numbers of the corresponding divisors.

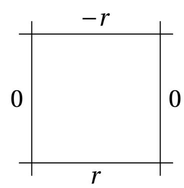

Figure 23: The complete curves on the Hirzebruch surface $\mathscr{H}_r$.

For any toric surface X_Δ we may run the algorithm to find a smooth variety $X_{\tilde{\Delta}}$. The resolution graph of X_Δ is the graph showing the curves in $X_{\tilde{\Delta}}$ corresponding to new rays introduced by the algorithm.

EXAMPLE 5.21. Reconsider the resolution of $\text{Cone}(f, 2e - f)$ from Figure 19. When we write $\sigma = \text{Cone}(v, w)$ and introduce the ray $\rho = \text{Cone}(e)$ we find

$$v + w = 2e.$$

Hence D_ρ has self-intersection number -2.

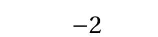

Figure 24: The resolution graph for $\text{Cone}(f, 2e - f)$.

EXAMPLE 5.22. In the resolution of $\text{Cone}(f, 11e-7f)$ we find four curves. Neighboring wall divisors intersect, hence the four divisors form a chain. The self-intersection numbers are the coefficients in the HJ-continued fraction of 11/7.

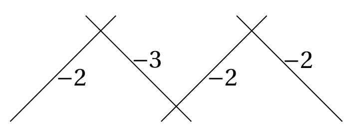

Figure 25: The resolution graph for $\text{Cone}(f, 11e - 7f)$.

The Hilbert basis. We can view the vectors given by the algorithm in another way. They form the Hilbert basis from Definition 2.18.

PROPOSITION 5.23. *Let $\sigma = \text{Cone}(e_2, \lambda e - \mu f)$ be in normal form and let $u_1, \dots, u_n$ be the ray generators found by the algorithm. Define $S = \{e_2, u_1, \dots, u_n, \lambda e - \mu f\}$, then S is the Hilbert basis of the semigroup $\sigma \cap N$.*

Proof. For a proof we refer to [2, Proposition 10.2.8]. □

We return to $\text{Cone}(f, 11e - 7f)$ from Example 5.11. The ray generators produced by the algorithm form the Hilbert base as shown in Figure 26.

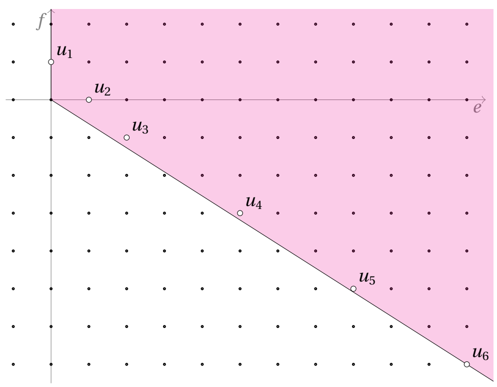

Figure 26: The Hilbert base inside $\text{Cone}(f, 11e - 7f)$.

Classification of toric surfaces. We called a variety *minimal* if it cannot be 'blown down'. A classic problem in algebraic geometry is to find a minimal model of a given variety. By a model we mean a representative of the birational equivalence class of a variety. For surfaces every birational equivalence class has a relatively minimal model[8, Remark V 5.8.4].

Toric varieties are rational varieties, which means that they are birationally equivalent to a projective space. To see this, let X_Δ be a toric variety of dimension n, then X_Δ contains an n-dimensional torus $T \simeq (\mathbb{C}^*)^n$ as open subset. The varieties X_Δ and $\mathbb{P}^n$ are birationally equivalent since $(\mathbb{C}^*)$ is an open subset of $\mathbb{P}^n$.

In the case of *rational surfaces* the minimal models are precisely $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathscr{H}_r$ for $r = 0$ or $r \geq 2$ [8, Remark V 5.8.2].

This means that the toric surfaces which are minimal, i.e. they cannot be 'blown down', are $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathscr{H}_r$ for $r = 0$ or $r \geq 2$. We note that $\mathscr{H}_1$ is a blow-up of $\mathbb{P}^2$. Hence we have the result:

THEOREM 5.24. *Every smooth complete toric surface is obtained from either $\mathbb{P}^2$ or a Hirzebruch surface after a finite sequence of blow-ups.* □

6. More on toric resolutions

We have investigated resolution of singularities in a toric surface. We now turn our attention to toric varieties in dimension ≥ 3. In higher dimensions we lose some of the nice properties from the surface case. The key ingredient of the algorithm was to bring a cone into *normal form*. This concept of a normal form does not translate to dimensions ≥ 3. Furthermore, for surfaces the alogirthm provided the *optimal* solution, in dimension ≥ 3 there is no unique optimal solution, as we shall show in Example 6.1.

What does not change is that we can resolve a singularity by finding a smooth refinement as in Theorem 3.40. So we set out to solve:

PROBLEM 2. Let Δ be a fan in $N_{\mathbb{R}}$ where $\dim(N_{\mathbb{R}}) = n$. We want to find a smooth fan $\widetilde{\Delta}$ refining Δ such that the smooth cones in Δ are in the refinement $\widetilde{\Delta}$.

We mentioned at the end of Chapter 3 that we can use stellar refinements to find such a smooth refinement.

6.1 Stellar refinements

EXAMPLE 6.1. Consider the 3-dimensional cone σ in Figure 27. This cone has four edges so the ray generators are certainly not part of a (lattice) basis, hence σ is not smooth. We now want to find a smooth refinement of σ.

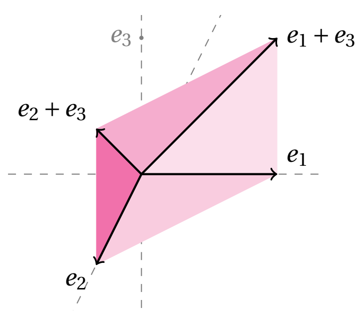

Figure 27: The cone $\sigma = \mathrm{Cone}(e_1, e_2, e_3 + e_1, e_3 + e_2)$.

We reconsider stellar refinements from Definition 3.33. Let ρ_1 and ρ_2 be the rays generated by e_1 respectively e_2. We take the stellar refinement of σ with ρ_1 as center. By definition this is the fan

$$\sigma^*(\rho_1) = \{\rho_1 + \tau \mid \tau \prec \sigma \mid \rho_1 \not\subseteq \tau\}.$$

So we take the faces of σ which do not contain e_1, and we gather them in a fan by adding e_1 to these cones. The resulting fan consists of

$$\mathrm{Cone}(e_1, e_1 + e_3, e_2 + e_3), \quad \mathrm{Cone}(e_1, e_2, e_2 + e_3)$$

and all their faces, see Figure 28. This figure also shows the fan $\sigma^*(\rho_2)$.

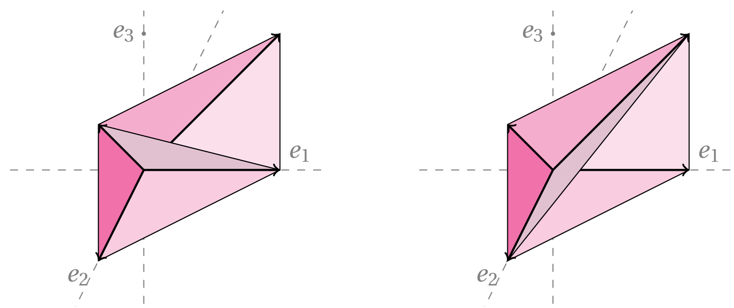

Figure 28: The stellar refinements $\sigma^*(\rho_1)$ and $\sigma^*(\rho_2)$

Both these fans are smooth refinements of σ. Both of these are minimal resolutions. However they are not refinements of each other. So Remark 5.14 no longer holds true in dim ≥ 3.

Example 6.1 illustrates how stellar refinement can lead to a smooth refinement. Whilst this is in general not true, we can find a simplicial refinement in this manner.

Consider a cone in normal form $\sigma = \mathrm{Cone}(e_2, \lambda e_1 - \mu e_2), \quad 0 \leq \mu < \lambda$. Let ρ be the ray through e_1, then $\sigma^*(\rho)$ consists of two cones $\mathrm{Cone}(e_1, e_2)$ and $\mathrm{Cone}(e_1, \lambda e_1 - \mu e_2)$ and their faces. So the introduction of the ray ρ is an instance of stellar refinement. Hence we have:

REMARK 6.2. The introducing of rays in the algorithm is an instance of iterated stellar refinement.

6.2 Resolutions

We follow the idea by Fulton [6] and break down the Problem 2 in two steps. Let Δ be a fan as in the problem.

(1) We first find a simplicial fan Δ' refining Δ;

(2) Next we find a smooth fan $\widetilde{\Delta}$ refining Δ'.

(1) Simplicializing. We first find a *simplicial refinement* of Δ. We can refine in such a way that the simplicial cones are unchanged:

LEMMA 6.3. *Every fan Δ has a simplicial refinement Δ' such that the simplicial cones of Δ are also cones in Δ'. This can be achieved by iterated stellar refinement.*

We follow the proof from Barthel-Fieseler-Kaup[7, Lemma 3.2]. We define:

DEFINITION 6.4. A ray ρ is a *splitting edge* of a cone σ if there is a complementary facet τ of σ, i.e. a facet such that $\sigma = \tau + \rho$. A cone is *stout* if it has no splitting edges.

The cone in Figure 27 is a stout cone, none of the four edges has a complementary facet.

Recall from Remark 2.24 that a d-dimensional cone is simplicial if and only if it has exactly d edges and the corresponding ray generators are linearly independent over $\mathbb{R}$. This means that all the edges are splitting edges. Furthermore, each face of a simplicial cone has splitting edges. Therefore:

REMARK 6.5. A cone is simplicial if (and only if) it does not include any stout face. Hence a fan that does not contain any stout cones is a simplicial fan.

Proof of Lemma 6.3. Let Δ be a fan and let ρ be an edge included in a cone σ in the fan. Then the stellar refinement $\sigma^*(\rho)$ does not contain any stout cones, since by construction a cone $\tau + \rho$ in $\sigma^*(\rho)$ has ρ as splitting edge. The idea is now to use stellar refinements to lower the number of stout cones. Taking edges of stout cones as centers, we see that we can use a sequence of stellar refinements to get rid of all the stout cones. The result is a fan Δ', which is simplicial by Remark 6.5.

Finally, we claim that no simplicial cone get in Δ get subdivided, i.e. the simplicial cones of Δ are still in the refinement Δ'. Note that the fans have the same edges

$$\Delta(1) = \Delta'(1). \tag{6.2.1}$$

Let σ be a simplicial cone in Δ. We show that σ is also a cone in Δ'. If $\sigma' \in \Delta'$ lies in σ, then (6.2.1) implies $\sigma'(1) \subset \sigma(1)$, hence σ' is a face of σ as the latter is simplicial. So the collection $\{\sigma' \in \Delta' \mid \sigma' \subseteq \sigma\}$ supports σ and all of the σ' are faces of σ. It follows that some σ' in the collection equals σ which shows $\sigma \in \Delta'$ as desired. □

(2) Smoothening. Let Δ be a simplicial fan. Then we can refine in such a way that the smooth cones are unchanged:

LEMMA 6.6. *Every simplicial fan Δ has a smooth refinement $\widetilde{\Delta}$, such that the smooth cones of Δ are also cones in $\widetilde{\Delta}$. This can be achieved by iterated stellar refinement.*

As stated in Remark 2.26 a (simplicial) cone is smooth whenever its multiplicity is 1. So a fan Δ is smooth when $\text{mult}(\sigma) = 1$ for all cones $\sigma \in \Delta$. To find a smooth refinement of a simplicial fan we would like to lower the multiplicity of cones in the fan. The idea is that we subdivide a cone of mult > 1 into cones of lower multiplicity. We first give some useful properties of the multiplicity of a cone.

PROPOSITION 6.7. *Let $\sigma \subset N_{\mathbb{R}}$ be a simplicial cone with minimal generators $u_1, \ldots, u_d$ and let $e_1, \ldots, e_d$ be a basis for $N_\sigma = \mathrm{Span}(\sigma) \cap N$. When we write $u_i = \sum_j a_{ij} e_j$, then we have*

$$\mathrm{mult}(\sigma) = \left|\det(a_{ij})\right|.$$

Proof. It is standard linear algebra that the determinant is the index of the sublattice $\mathbb{Z}u_1 + \ldots + \mathbb{Z}u_d$ inside N_σ, see for instance [14, corollary 9.63]. □

PROPOSITION 6.8. *Let $\sigma \subset N_{\mathbb{R}}$ be a simplicial cone with minimal generators $u_1, \ldots, u_d$. Then the generators span the fundamental parallelotope*

$$P_\sigma = \left\{ \sum_{i=1}^{d} \lambda_i u_i \;\middle|\; \lambda_i \in \mathbb{R} \text{ and } 0 \le \lambda_i < 1 \right\},$$

and the multiplicity of the cone is the number of lattice points inside the parallelotope

$$\mathrm{mult}(\sigma) = |P_\sigma \cap N|.$$

Proof. Observe that the composition of maps

$$P_\sigma \cap N \hookrightarrow N_\sigma \to N_\sigma / (\mathbb{Z}u_1 + \ldots + \mathbb{Z}u_d)$$

is a bijection and by definition $\mathrm{mult}(\sigma) = [N_\sigma : \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_d]$. □

Note that if σ is a face of τ then $P_\sigma \subset P_\tau$. So counting points inside the parallelotope gives the following result:

COROLLARY 6.9. *For cones $\sigma \preceq \tau$ we have* $\mathrm{mult}(\sigma) \le \mathrm{mult}(\tau)$.

These properties make it easier to compute multiplicities. The following lemma shows that a stellar refinement may lower the multiplicity of a cone.

LEMMA 6.10. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone which has multiplicity > 1. Denote $u_1, \ldots, u_d$ for the minimal generators of σ and assume we have a lattice point in the parallelotope:*

$$u_\rho = \textstyle\sum_{i=1}^{d} \lambda_i u_i \in P_\sigma \cap N, \quad \text{for } 0 \le \lambda_i < 1.$$

Let $\sigma^*(\rho)$ be the stellar refinement with the ray through u_ρ as center. Then we have

$$\mathrm{mult}(\tau + \rho) < \mathrm{mult}(\sigma), \quad \text{for cones in } \sigma^*(\rho). \tag{6.2.2}$$

Proof. Let $\tau + \rho$ be a cone in $\Delta^*(\rho)$, i.e. τ is a face of σ that does not contain the ray ρ as subset. We denote $u_1, \ldots, u_d$ for the minimal generators of σ. Then there is a minimal generator u_i which is not contained in τ. Let δ_j denote the facet of σ not containing u_j, so δ_j is generated by the other minimal generators. Then $(\tau + \rho) \prec (\delta_j + \rho)$ and Corollary 6.9 gives

$$\mathrm{mult}(\tau + \rho) \le \mathrm{mult}(\delta_j + \rho)$$

Recall that ρ is the ray generated by $u_\rho = \sum_{i=1}^{d} \lambda_i u_i$, for $0 \le \lambda_i < 1$. Compare the cones $\delta_j + \rho$ and σ, these cones have mostly the same generators except for u_ρ and u_j. We use Proposition 6.7 to compute

$$\operatorname{mult}(\delta_j + \rho) = \lambda_j \operatorname{mult}(\sigma)$$

and since $\lambda_j < 1$ we conclude $\operatorname{mult}(\tau + \rho) < \operatorname{mult}(\sigma)$ □

So Lemma 6.10 describes a suitable refinement in the sense that we subdivide a cone σ into cones of lower multiplicity. This is the key ingredient to finding a smooth refinement of a simplicial fan.

Proof op Proposition 6.6. Let Δ be a simplicial fan. Define the multiplicity of the fan to be

$$\operatorname{mult}(\Delta) = \max_{\sigma \in \Delta}\{\operatorname{mult}(\sigma)\}.$$

We assume Δ is nonsmooth so $\operatorname{mult}(\Delta) > 1$ and we pick a cone σ of maximal multiplicity, i.e. $\operatorname{mult}(\Delta) = \operatorname{mult}(\sigma)$. Since mult > 1 we have by Proposition 6.8 that $P_\sigma \cap N$ is non-empty. So we may pick a lattice point in the parallelotope, let ρ denote the ray though this point. Now consider the stellar refinement of Δ with center ρ. Now $\Delta^*(\rho)$ does not contain σ and each newly introduced cone satisfies (6.2.2). We conclude that $\Delta^*(\rho)$ is a refinement of Δ such that

- either $\operatorname{mult}\big(\Delta^*(\rho)\big) < m$,
- or $\operatorname{mult}\big(\Delta^*(\rho)\big) = m$ but $\Delta^*(\rho)$ has less cones of maximal multiplicity.

Note that $\Delta^*(\rho)$ does not change the simplicial cones in Δ. We may repeat this process until we have lowered all the multiplicities in the cone. This gives the desired refinement. □

Note that a smooth cone is a simplicial cone. So when we combine Lemmas 6.3 and 6.6 we see that any fan has a smooth refinement leaving the smooth cones unchanged. This solves Problem 2 with a sequence of stellar refinements. The result is a resolution of singularities by Theorem 3.40.

Exceptional locus. We have arrived at a resolution of singularities $\varphi \colon \widetilde{X}_\Delta \to X_\Delta$, so we have an isomorphism outside of the singular locus:

$$X_{\widetilde{\Delta}} \setminus \varphi^{-1}\big((X_\Delta)_{\text{Sing}}\big) \simeq X_\Delta \setminus (X_\Delta)_{\text{Sing}}.$$

The pre-image $\varphi^{-1}\big((X_\Delta)_{\text{Sing}}\big)$ is what we call the *exceptional locus* In Proposition 3.38 we described the singular locus of a toric variety in terms of orbit closures

$$X_\Delta \setminus (X_\Delta)_{\text{Sing}} = \bigcup_{\sigma \text{ smooth}} O(\sigma).$$

We can do the same for the exceptional locus. The locus corresponds to the new cones $\widetilde{\Delta}$, these cones are proper subsets of the nonsmooth cones in Δ.

$$E = \bigcup_{\sigma \in \widetilde{\Delta} \setminus \Delta} O(\sigma).$$

Note that if a cone σ' is nonsmooth, then so is every cone that contains σ' as a face. Hence by the Orbit-Cone correspondence we have $E = \bigcup_{\sigma \in \widetilde{\Delta} \setminus \Delta} V(\sigma)$.

Techniques to solve resolutions. We have proved the existence of toric resolutions. Every toric variety has a resolution of singularities which is also a toric variety. A toric resolution is given by a smooth refinement. We 'smoothen' the fan by choosing lattice points, for a nonsmooth cone σ in the fan we pick a lattice point in the parallelotope P_σ. These points describe a sequence of stellar refinements and hence a sequence of blow-ups. However, given a toric variety it is not clear which are 'good choices' for lattice points.

In dimension 2 we had a unique minimal refinement of a fan, as explained this no longer exists in dim ≥ 3. There are some tactics which lead to choose lattice points. For more information of toric singularities in higher dimensions we refer to the literature.

- [2] discusses the Barycentric refinement at the end of section 11.1. Next, section 11.2 discusses other types of resolutions.
- [3] provides an overview of toric singularities in dimension 3, "it focuses, in particular, on a toric version of Reid's desingularization strategy". In [3, p. 176] the reader may also find an overview of the classification of singularities in terms of cones.

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