Radboud University Nijmegen

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Products of Graphs and Hedetniemi's Conjecture

Bachelor Thesis in Mathematics

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1 Introduction

When one thinks about products in the mathematical sense, graph theory might not be the first subject that comes to mind. One might instead imagine products in the algebraic sense, or Cartesian products in set theory. Products of graphs are not too dissimilar to the latter of those two. In this thesis, we will not just discuss these products of graphs, but a conjecture to do with this concept as well.

In 1966, Stephen T. Hedetniemi conceived a conjecture as a part of his PhD thesis [8]. He pursued his PhD studies at the University of Michigan and in the spring of 1961 he followed a course on graph theory, taught by Prof. Frank Harary. Harary was a pioneer when it came to modern graph theory. He incited much curiosity in the students he taught by introducing them to conjectures and open problems within the field.

Hedetniemi was one of the many students who was fascinated by these. Later on in his studies, Hedetniemi showed Prof. Harary a simplified version of a lengthy proof of a theorem on homomorphisms of graphs. Harary was impressed by his work and subsequently invited him to work on a PhD thesis with him on the subject of graph theory.

It was during the making of this thesis that he realized that very little was known about homomorphisms of graphs at the time, which made him ponder on the topic of homomorphisms of products of graphs, also known as categorical products or tensor products of graphs. Following a few observations by Hedetniemi on this topic, he made the following conjecture:

Conjecture 1.0.1 (Hedetniemi's conjecture [7]). *For graphs G and H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Where $G \times H$ denotes the product of G and H. Hedetniemi did not think much of it at the time, but it turned out that this conjecture was quite difficult to either prove or disprove. Though steady progress was made on the conjecture over the years, it ended up taking over half a century before a general refutation was written.

However, the article showing the refutation is short but nonetheless very complex. The proof only consists of three main steps to the conclusion, but it leaves a lot of the reasoning and smaller thinking steps to the reader, making it a difficult paper to grasp for those not properly familiarized with the topic of products of graphs. That is why the main goal of my thesis is to add to this article by providing the needed definitions and going over these in-depth, supplying additional lemmas, and giving some examples when needed.

In this thesis, we will first go over some standard definitions in Section 2.1 and explain products of graphs in Section 2.2, so that we have a better understanding of what Hedetniemi's conjecture entails when we discuss it again in Section 2.3. Then in Section 3, some specific types and variants of graphs for which Hedetniemi's conjecture has already been proven or disproven will be discussed, i.e. the conjecture in the case of Kneser graphs, fractional coloring, and directed graphs, and we will go over the proof detailing that the conjecture is either true or false depending on the case. In Section 4 we discuss the refutation of Hedetniemi's conjecture in greater detail, starting by explaining the concept of exponential graphs in Section 4.1, and then we walk through the article disproving Hedetniemi's conjecture in Section 4.2, with many additions and examples to make it easier to follow. Finally, in Section 5, we will go over the progress that has been made on the conjecture aside from variants of graphs and the general refutation. This includes showing that the conjecture is true if the minimal chromatic number of G and H is 4. We will also show that the conjecture is asymptotically true, and discuss how the minimal chromatic number for two graphs G and H that produce a counterexample has decreased over the years, all the way down to proving that the chromatic number of the product of graphs G and H with a chromatic number of 5 or higher, can indeed be 4. There will be some results we do not fully prove in this thesis due to their length or complexity, one in our section on fractional coloring and one in proving that the conjecture is true in the case of the minimum chromatic number between G and H being 4 or less. Lastly, the section on showing that the conjecture is generally not true if the minimum chromatic number is 5 will not be discussed in much detail but merely summarized due to the sheer complexity of the proof.

2 Preliminaries

In this section, a brief overview of some standard definitions within graph theory will be given, along with an introduction to products of graphs, how we denote them, and a few examples. We end this section by discussing Hedetniemi's conjecture and why it seems plausible at first glance.

2.1 Definitions

We start off with some likely familiar yet still crucial definitions about graph theory which we will need for the rest of this paper. Most definitions are cited or adapted from [16]. This book has a lot of additional examples for those definitions, so a read-through is recommended if any concepts explained in this section are still unclear.

Definition 2.1.1. A *graph* is an ordered pair $G = (V, E)$, mainly denoted as simply G. It consists of a set of vertices V and a set of edges E. Vertices are single elements which we typically write using the letters v or u, and edges are pairs of vertices. These are shown in figures as a line between the vertices. For example, if there is an edge connecting v_0 and v_1, we say that they are adjacent, and that $\{v_0, v_1\} \in E(G)$.

Definition 2.1.2. A graph H is a *subgraph* of a graph G if each of its vertices belongs to $V(G)$ and each of its edges belongs to $E(G)$. We say it is an *induced subgraph* if the subgraph contains *all* edges in $V(G)$ with both endpoints in $V(H)$. We denote H a subgraph of G with $H \subset G$ and H an induced subgraph of G with $H \subseteq G$

This definition, specifically for induced subgraphs, is important for one of our proofs in the section about Kneser graphs.

Example 2.1.3. The following is an example showing the distinction between these definitions.

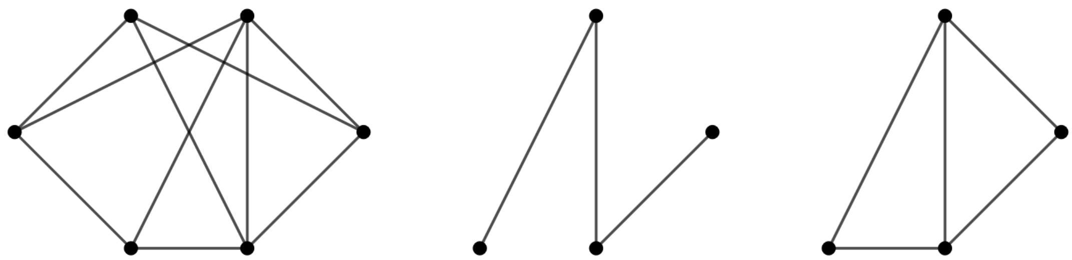

Figure 1: From left to right, a graph G, a subgraph of G, an induced subgraph of G.

We will also go over the definition of complete graphs and cliques.

Definition 2.1.4. The *complete graph* on n vertices, denoted by K_n, is a graph with an edge between every possible pair of different vertices (so excluding self loops).

Definition 2.1.5. We say a graph G has a *k-clique* if it has a subset on k vertices such that there is an edge between every possible pair of those k vertices. In other words, if K_k is a subgraph of G. We use the term *clique number* of G as the largest k such that G has a k-clique.

As the conjecture we discuss in this paper has to do with graph coloring, we will also go over the definitions of a k-colorable graph and the chromatic number of a graph.

Definition 2.1.6. We define a *k-coloring* of a graph G as the assignment of a color from the set $\{1, \ldots, k\}$ to every vertex in $V(G)$. We say a coloring is *proper* if the graph G is colored such that no two adjacent vertices $\{x, y\} \in E(G)$ share the same color. Conversely, we say a coloring is *not proper* if this is not the case.

Specifying whether a coloring of a graph G is proper or not will not be important until Section 4, so every coloring up until that point is proper unless stated otherwise.

Definition 2.1.7. We say that a graph G is *k-colorable* if we can properly color the graph with k colors. Sometimes we will use a function $\Psi : V(G) \to \{1, \ldots, k\}$ to describe the assigning of colors to every vertex in a graph G.

Definition 2.1.8. The *chromatic number* of a graph G is equal to k if k is the least integer such that our graph is k-colorable. We denote this by $\chi(G) = k$.

This means that if a graph is k-colorable, this gives us an upper bound for the chromatic number of said graph.

Definition 2.1.9. A *cycle* of length n, denoted by C_n is a graph with n vertices where we can number each vertex in a certain way $(v_1, v_2, \ldots, v_n)$ such that v_i is only adjacent to v_{i+1} and v_{i-1}, and $\{v_1, v_n\} \in E(G)$.

Example 2.1.10. Below is an example of a proper graph coloring.

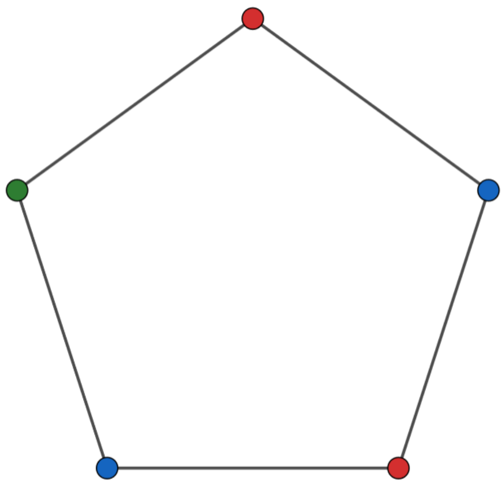

Figure 2: Coloring of the graph C_5, the cycle graph with 5 vertices.

Proposition 2.1.11. *If H is a subgraph of G, and $\chi(H) = k$, then $\chi(G) \geq k$*

Proof. Assume that $\chi(G) < k$. Then, since H is a subgraph of G, this means that we must be able to color H with fewer than k colors. But since $\chi(H) = k$, H is not $k-1$-colorable. Thus we have $\chi(G) \geq k$ by contradiction. □

Proposition 2.1.12. *Any graph with $\chi(G) = 3$ contains a cycle of odd length.*

Proof. Assume a graph G to contain no odd cycles, so it contains either no cycles or only cycles of even length. Then we can color the graph as follows: take a random vertex $v \in V(G)$, and assign it the color red. Then, assign all the vertices that are directly adjacent to it the color blue. Then assign the adjacent vertices of all the now blue vertices the color red. Repeat this process until the graph is fully colored. This gives us a proper 2-coloring of G, since we know that, for any vertex $v \in V(G)$, all its neighbors (adjacent vertices) are not adjacent amongst each other. If there was such a vertex with two adjacent neighbors, this would give us a cycle of odd length. □

2.2 Products of graphs

The other crucial part of the conjecture is the concept of products of graphs and how they are constructed. In this section, we will explain what they are and give a few examples in order to clearly illustrate them.

Definition 2.2.1. The *product* of two graphs G and H, which we will write as $G \times H$, has the set of vertices $\{(v, u) \mid v \in V(G), u \in V(H)\}$, where

$$\{(v_0, u_0), (v_1, u_1)\} \in E(G \times H) \iff \{v_0, v_1\} \in E(G) \text{ and } \{u_0, u_1\} \in E(H).$$

In other words, the vertices of the product $G \times H$ are all pairs, where the first element of the pair is a vertex selected from G, and the second is one selected from H. Two vertices in $G \times H$ are connected via an edge if and only if the first elements in both pairs, so the vertices belonging to G, also have an edge between them in G, and the vertices taken from H also have an edge in H. This definition is best illustrated at the hand of a few examples.

Example 2.2.2. Our first example is the product of $G = K_2$ and $H = K_2$, shown on the right. Note that the vertices of the product $K_2 \times K_2$ are labeled as pairs, where the first vertex in the pair is taken from G, and the second is taken from H.

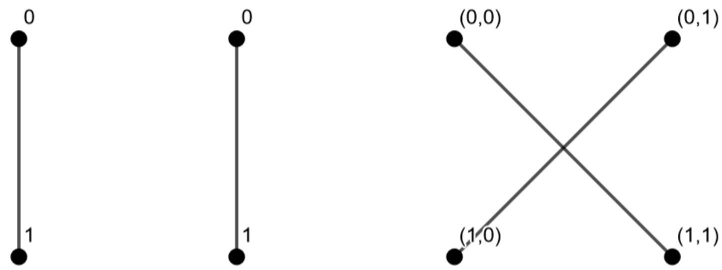

Figure 3: $K_2 \times K_2$

Example 2.2.3. Our second example is the product of $G = K_2$ and $H = K_3$.

Since the vertices in tensor products are all the possible pairs of vertices, we can observe that $|V(G\times H)| = |V(G)|\cdot|V(H)|$. We also know that $|E(G\times H)| = 2\cdot|E(G)|\cdot|E(H)|$. This is because, given any two edges $\{v_0, v_1\} \in E(G)$ and $\{u_0, u_1\} \in E(H)$, we have that $\{(v_0, u_0), (v_1, u_1)\}$ is an

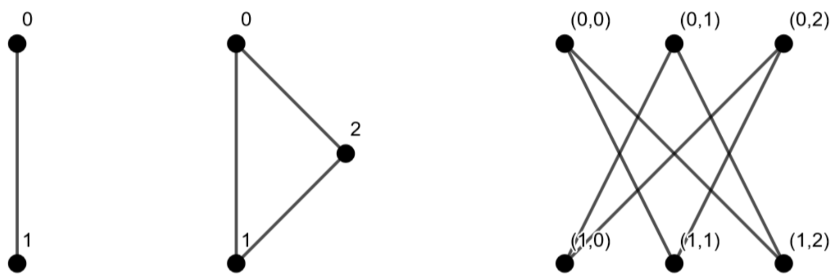

Figure 4: $K_2 \times K_3$

edge in $G \times H$. However, then we also have that $\{(v_1, u_0), (v_0, u_1)\}$ is an edge in $G \times H$. So for every edge in G paired with an edge from H, we get 2 unique edges in $G \times H$, thus giving us $2 \cdot |E(G)| \cdot |E(H)|$ edges.

The examples we have looked at so far have been relatively small in size. Now we will look at a slightly larger example to show how quickly products of graphs can "blow up".

Example 2.2.4. The product of the following two graphs (visible on the next page) has been computed in Python, as it is too large to do so by hand.

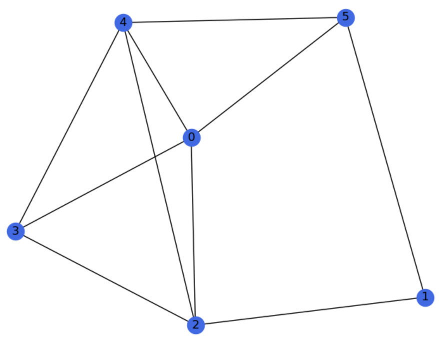

Figure 5: First graph with 6 vertices and 10 edges

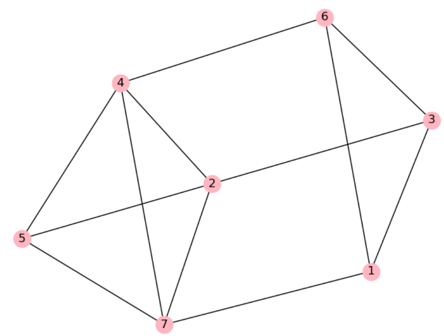

Figure 6: Second graph with 7 vertices and 12 edges

From this example, we can already see how difficult of a problem Hedetniemi's conjecture is. The product of two graphs grows large quite easily, and this leaves us with only a few graphs that we could compute manually within reason. This means that if we wanted to find a counterexample, we would already have to be looking at graphs that bring forth a rather large product.

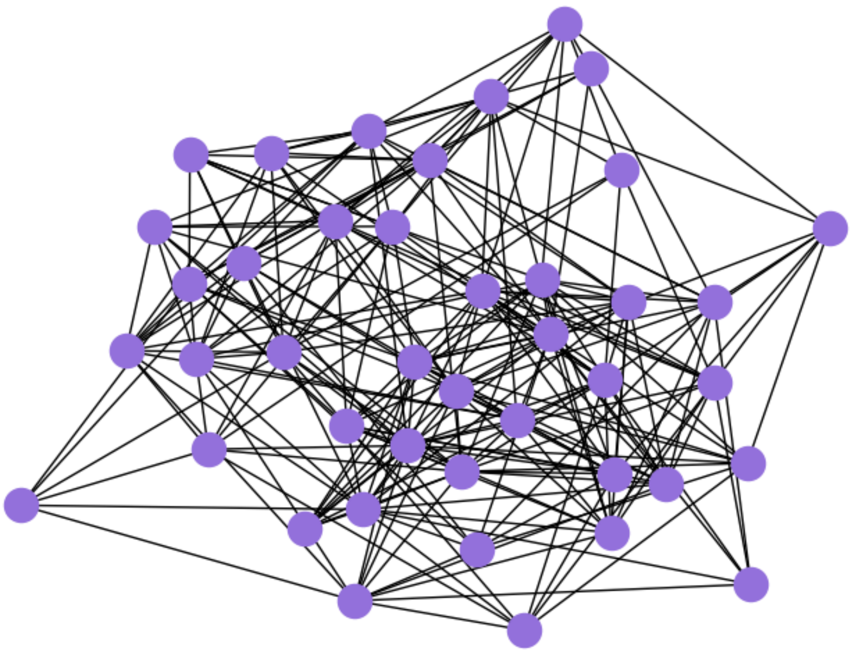

Figure 7: The product of the two graphs, with 42 vertices and 240 edges

2.3 Hedetniemi's conjecture

Now that we have gone over all the necessary background to understand Hedetniemi's conjecture, we will explore it in more detail. As a reminder, Hedetniemi's conjecture states, for graphs G and H,

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

In other words, the chromatic number of the product of two graphs is equal to the least of the chromatic numbers of the two graphs separately. At first hand, this statement seems rather plausible. In fact, this equality is true for all the examples we have looked at so far, including Figure 7. As we will see later, the conjecture is actually true for any pair of graphs that both have chromatic numbers less than or equal to 4. At the hand of the following lemma, we will see that if we wanted to disprove the conjecture, we could only do so in one direction.

Lemma 2.3.1. *For two graphs G and H, we know that $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.*

Proof. Assume that $\min\{\chi(G), \chi(H)\} = \chi(G)$ without loss of generality. Then, we know that $\chi(G \times H) \leq \chi(G)$ because we can lift the optimal coloring Ψ of G to our coloring of $G \times H$. Namely, we can color every vertex of $G \times H$ by $(g, h) \to \Psi(g)$. In this way, we group every vertex of the product into sets with other vertices that contain the same g from $V(G)$. Since G has no self-loops, we know there are no edges within these sets. Thus, this is a proper coloring of $G \times H$. Then we know that $\chi(G \times H) \leq \chi(G) = \min\{\chi(G), \chi(H)\}$. □

Example 2.3.2. The following is an example of such a lifted coloring with two graphs we have seen earlier.

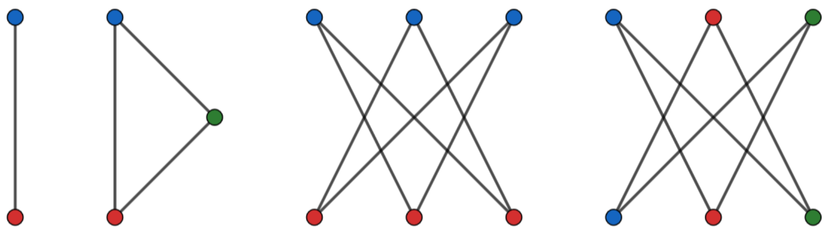

Figure 8: Two proper colorings of $K_2 \times K_3$ lifted from the colorings of K_2 and K_3 respectively.

So if we wanted to prove that Hedetniemi's conjecture is false, we would have to construct a specific case such that for certain G and H, $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$.

Before we discuss how the conjecture turned out to be false in general, we cover some specific cases of graphs or graph colorings for which the conjecture is either true, or for which a counterexample exists.

3 (Non-standard) colorings of (non-standard) graphs

As Hedetniemi had formulated his conjecture back in 1966, a lot of progress had been made on the subject of tensor products of graphs and their chromatic numbers even before a general refutation had been found. Before we discuss Hedetniemi's conjecture in its entirety, we will look at some of the efforts which have been made prior to the release of the general refutation. These are specific types of graphs or graph colorings in which the conjecture has been proven to be true or false. These types include Kneser, fractional colorings of graphs, and directed graphs.

3.1 Kneser graphs

In this section, we will be looking at Kneser graphs. These are a specific type of graph whose vertices are the unique subsets of cardinality n of a set of cardinality m, and its edges are determined by whether or not two of these subsets are disjoint. The definition is as follows.

Definition 3.1.1. For $n, m \in \mathbb{N}$ such that $m \geq 2n$, the *Kneser graph* K_n^m is the graph whose vertices are the n-subsets of $\{0, 1, \ldots, m-1\}$, where two vertices are adjacent if and only if they are disjoint.

For Kneser graphs, we use the condition that $m \geq 2n$, since if this weren't the case, we would have a graph without any edges. These cases are trivial and uninteresting to us, so we only look at Kneser graphs with $m \geq 2n$.

Example 3.1.2. A well-known example is the Petersen graph, which is the Kneser graph K_2^5.

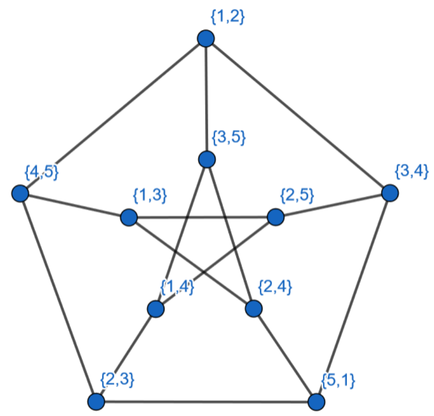

Figure 9: The Petersen graph. The vertices are all the 2-subsets of {1,2,3,4,5}. There is an edge between two vertices if and only if the subsets that these vertices represent are disjoint.

Now that we have our definition, we can rewrite Hedetniemi's conjecture in a fashion tailored to Kneser graphs. In this section, we aim to prove that

$$\chi(K_{n_i}^{m_i} \times K_{n_j}^{m_j}) = \min\{\chi(K_{n_i}^{m_i}), \chi(K_{n_j}^{m_j})\} \tag{3.1}$$

for all $m_i, m_j, n_i, n_j \in \mathbb{N}$ with $m_i \geq 2n_i$, $m_j \geq 2n_j$. The proof of this equality is adapted from [15]. Our strategy is to prove some useful properties about the chromatic numbers of products of graphs when one of the two graphs is an induced subgraph of the other, and to utilize a certain homomorphism between Kneser graphs. We start by looking at a general property of tensor products of graphs.

Definition 3.1.3. A *homomorphism* $\phi : V(G) \to V(H)$ is a function from G to H that preserves edges. If there exist homomorphisms $\Phi : V(G) \to V(H)$ and $\Psi : V(H) \to V(G)$, we say that G and H are *homomorphically equivalent.*

In other words, ϕ is a homomorphism from G to H if, given that $\{v_0, v_1\}$ is an edge in G, then $\{\phi(v_0), \phi(v_1)\}$ must also be an edge in H. Notice that if there is a homomorphism from G to H, then $\chi(G) \leq \chi(H)$. Thus, if G and H are homomorphically equivalent, $\chi(G) = \chi(H)$. We use this in our proof of the following lemma.

Lemma 3.1.4. *If H is an induced subgraph of G, then $\chi(G \times H) = \chi(H)$.*

Proof. Note that since H is an induced subgraph of G, this means that $G \times H$ and H are homomorphically equivalent. Namely, $\Phi : V(G \times H) \to V(H)$ is given by $\Phi(n, m) = m$, for vertices $n \in G$ and $m \in H$. For $\Psi : V(H) \to V(G \times H)$, we say that H has vertices $\{1, 2, \ldots, i\}$, and we call the vertices of G that form H, $\{j + 1, j + 2, \ldots, j + i\}$ (i.e 1 in H is equivalent to $j + 1$ in G, 2 is equivalent to $j + 2$, and so on). Then we define $\Psi(m) = (m + j, m)$. This shows that $G \times H$ and H are homomorphically equivalent, and thus $\chi(G \times H) = \chi(H)$. □

Regarding homomorphisms of Kneser graphs, we will utilize the following result later in our proof.

Theorem 3.1.5 (Stahl [12])**.** *For $m, n \in \mathbb{N}$ such that $n > 1$ and $m \geq 2n$, there is a homomorphism from K_n^m to K_{n-1}^{m-2}.*

Proof. In the proof, every vertex v is denoted as $\{v_1, v_2, \ldots, v_n\}$ where every v_i is a unique integer from $[m] = \{1, \ldots, m\}$, and where $v_i < v_{i+1}$. Then, we say that a vertex is k-regular if there exists a k such that $v_k = k$ and $v_{k+1} > k + 1$. Since the convention is set that $v_i < v_{i+1}$ for all $1 \leq i < n$, if a vertex $v = \{v_1, v_2, \ldots, v_n\}$ is k-regular, there can only be one such k. If a vertex is not regular for any k, we say it is irregular. An example of an irregular vertex would be $\{2, 3, 5\}$. We define the mapping η from K_n^m to K_{n-1}^{m-2} as follows:

$$\eta(v) = \{v_2 - 1, v_3 - 1, \ldots, v_k - 1, v_{k+1} - 2, \ldots, v_n - 2\}$$

if the vertex v is k-regular, and

$$\eta(v) = \{v_2 - 2, v_3 - 2, \ldots, v_n - 2\}$$

if the vertex v is irregular. Now we have to prove that this is indeed a homomorphism via case distinction, mainly that if v and u are adjacent vertices in K_n^m, then $\eta(u)$ and $\eta(v)$ must also be adjacent in K_{n-1}^{m-2}. In other words,

$$u \cup v = \emptyset \implies \eta(u) \cup \eta(v) = \emptyset.$$

Note that any regular vertex must contain the integer 1, so two vertices u, v cannot both be regular and adjacent. This reduces the proof to two cases.

Case 1. Assume both u and v are irregular. Then $\eta(u) = \{u_2 - 2, u_3 - 2, \ldots, u_n - 2\}$ and $\eta(v) = \{v_2 - 2, v_3 - 2, \ldots, v_n - 2\}$. Then, if $\eta(u) \cup \eta(v) = \emptyset$ were not true, this would mean we could find i and j such that $u_i - 2 = v_j - 2$, or $u_i = v_j$. This contradicts the adjacency of u and v.

Case 2. Assume u to be k-regular, and v to be irregular.

So $\eta(u) = \{u_2 - 1, u_3 - 1, \ldots, u_k - 1, u_{k+1} - 2, \ldots, u_n - 2\}$ and $\eta(v) = \{v_2 - 2, v_3 - 2, \ldots, v_n - 2\}$. If $\eta(u) \cup \eta(v) \neq \emptyset$, then for some $j > 1$, $v_j - 2 \in \eta(u)$. As before, if $v_j - 2 = u_i - 2$ for some $i > k$, we get a contradiction. So we assume that $v_j - 2 = u_i - 1$ for some $2 \leq i \leq k$. This means that $v_j = a_i + 1 = i + 1 \leq k + 1$. But, since u is k-regular it follows that $\{1, \ldots, k\} \subset u$. Since $u \cup v = \emptyset$, we get that $v_s \geq k + 1$ for all $v_s \in v$. Combining this with our earlier conclusion that $v_j \leq k + 1$, we get that $v_j = k + 1$, thus v_j must be the smallest element of v, so $j = 1$. But this contradicts $j > 1$.

We make a small addition to this proof to clear up the definition for η. Namely, in the case of the *arithmetic* vertex $v = \{1, 2, \ldots, n\}$ which has no proper image under this definition yet. Technically, this vertex would be considered irregular, and the function η would send the arithmetic vertex to $\{0, 1, \ldots, n - 2\}$. But $\{0\}$ is not in the image of η. Thus we define

$$\eta(\{1, 2, \ldots, n\}) = \{v_2 - 1, v_3 - 1, \ldots, v_n - 1\}.$$

This is somewhat implied but not explicitly stated in the article, which is why we make this addition. Since we have made this addition, we now also have to prove for the arithmetic vertex v, that if v and another vertex u are adjacent in K_n^m, that they are also adjacent under η. We start by observing that any vertex adjacent to v must be irregular, as it must be a subset of $\{n + 1, n + 2, \ldots, m\}$. For irregular vertices, we skip the first element of the vertex and subtract 2 from all the others. Thus, $\eta(u)$ must be a subset of $\{n, n + 1, \ldots, m - 2\}$ for any u adjacent to v. However, $\eta(v) = \{1, 2, \ldots, n - 1\}$ by our expanded definition of η. We can easily see that $\{1, 2, \ldots, n - 1\} \cap \{n, n + 1, \ldots, m - 2\} = \emptyset$, so $\eta(v)$ and $\eta(u)$ must be adjacent.

Altogether, this shows that η is a homomorphism. $\square$

Another thing to mention about the proof, is that in the article [12], $m > 2n$ is stated as the condition rather than $m \geq 2n$. However, the case for this theorem when $m = 2n$ is trivial, as this gives a Kneser graph where every vertex has only one edge, so every vertex is paired with a unique vertex. (i.e in the graph K_3^6, $\{1, 2, 3\}$ is solely adjacent to $\{4, 5, 6\}$). This case is thus rather trivial, though the construction of η in the proof works for this case as well. So this theorem works in the case $m = 2n$, too.

We now look at the chromatic number of Kneser graphs.

Theorem 3.1.6 (Kneser's Combinatorial Theorem)**.** *The chromatic number of the Kneser graph K_n^m is $m - 2n + 2$.*

This was a conjecture by Kneser, and it has later been proven by Lovász using graph theory. In fact, the original conjecture was formulated not at all like a graph theory problem. Kneser originally posed it as such: if you divide the n-subsets of a $(2n + k)$-set into $k + 1$ classes, then two disjoint sets must end up in the same class. We refer to Lovász's article for the proof of this theorem [9]. We formulate another lemma before combining some of our results thus far.

Lemma 3.1.7. *Let $n, r \in \mathbb{N}$ and $m_1 \leq m_2 \leq \ldots \leq m_r$ be positive integers such that $m_i \geq 2n$, for all $i \in [r]$. Then, $(K_n^{m_1})$ is an induced subgraph of the graph $K_n^{m_1} \times K_n^{m_2} \times \ldots \times K_n^{m_r} = \prod_i K_n^{m_i}$.*

Proof. Define $\phi : K_n^{m_1} \to \prod_i K_n^{m_i}$ by $\phi(A) = (A, A, \ldots, A)$ for all sets $A \in V(K_n^{m_1})$. Since this map is an injective homomorphism, $K_n^{m_1}$ is indeed an induced subgraph of $\prod_i K_n^{m_i}$. $\square$

Using Kneser's Combinatorial Theorem, Lemma 3.1.4 and Lemma 3.1.7, we can deduce the following.

Corollary 3.1.8. *Let $n, r \in \mathbb{N}$ and $m_1, m_2, \ldots, m_r$ be positive integers such that $m_i \geq 2n$, for all $i \in [r]$. Then,*

$$\chi(\prod_i K_n^{m_i}) = \chi(K_n^{m_1} \times K_n^{m_2} \times \cdots \times K_n^{m_r}) = \min_i\{\chi(K_n^{m_i})\} = \min_i\{m_i\} - 2n + 2.$$

Now the next step is to prove a similar result but for products with differing n_i's as well, which the following lemma will assist us in.

Lemma 3.1.9. *Let $r \in \mathbb{N}$ and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$ for all $i \in \{1, \ldots, r\}$. Assume that $n_1 \leq n_2 \leq \ldots \leq n_r$, with $n_r > 1$. Then, there exists a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$.*

Proof. By Theorem 3.1.5, we know that for each $i \in [r]$, there is a graph homomorphism $\phi_i : K_{n_r}^{m_i+2(n_r-n_i)} \to K_{n_i}^{m_i}$. This means that there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$ as well. □

This then allows us to prove our last theorem, which will directly imply that 3.1 is true.

Theorem 3.1.10. *Let $r \in \mathbb{N}$ and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$ for all $i \in [r]$. Then,*

$$\chi(\prod_i K_{n_i}^{m_i}) = \min_i\{\chi(K_{n_i}^{m_i})\}$$

Proof. We can assume that $n_1 \leq n_2 \leq \ldots \leq n_r$ without loss of generality, and we can assume that $n_r > 1$. Then we can see from Lemma 3.1.9 that there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$, which implies that $\chi(\prod_i K_{n_i}^{m_i}) \geq \chi(\prod_i K_{n_r}^{m_i+2(n_r-n_i)})$. Then, by Corollary 3.1.8 we have that

$$\chi(\prod_i K_{n_r}^{m_i+2(n_r-n_i)}) = \min_i\{m_i + 2(n_r - n_i) - 2n_r + 2\} = \min_i\{m_i - 2n_i + 2\} = \min_i\{\chi(K_{n_i}^{m_i})\},$$

thus giving us $\chi(\prod_i K_{n_i}^{m_i}) = \min_i\{\chi(K_{n_i}^{m_i})\}$. □

Using this theorem, we can show that the conjecture is true in the case of Kneser graphs

Corollary 3.1.11. *Hedetniemi's conjecture in the case of Kneser graphs (3.1) is true.*

Proof. Take $r = 2$ with the previous theorem. □

3.2 Fractional coloring number

Fractional colorings of graphs are a specific type of graph coloring where we assign more than one color to every vertex. There are multiple, equivalent ways to define and give the most optimal fractional coloring of a graph, but the definitions we use here are derived from [4]. We begin this section by going over these definitions, and then giving some examples of fractional coloring. Finally we will discuss some lemmas and then a part of the proof that

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$$

holds for all graphs G and H, where $\chi_f(G)$ denotes the fractional coloring number of the graph G. The full proof of this is given in [19], but due to the complexity of the proof we will not discuss it in its entirety in this section.

We use $\mathcal{I}(G)$ to denote all the independent sets of G, where independent sets are sets consisting of vertices of a graph, where all the vertices in an indepent set do not share any edges amongst themselves. We also use $\mathcal{I}(G, x)$ to denote all the independent sets that contain the vertex x.

Definition 3.2.1. A *fractional coloring* of a graph G is a nonnegative real-valued function f on $\mathcal{I}(G)$ such that for any vertex x of G,

$$\sum_{S \in \mathcal{I}(G,x)} f(S) \geq 1.$$

Definition 3.2.2. We say that the *weight* of a fractional coloring is the sum of all its values $f(S)$ for $S \in \mathcal{I}(G)$, and the *fractional chromatic number* $\chi_f(G)$ of the graph G is the minimum possible weight of a fractional coloring.

In other words, a fractional coloring function f can assign values to every independent set of G. But, for any vertex $x \in V(G)$, the sum of all these values over all the independent sets that contain x cannot exceed 1. The weight is simply the total sum of all the values assigned to every independent set of G.

Example 3.2.3. Below is an example showing the independent sets for a single vertex of the graph C_5. These include the vertex itself (shown with red) and two other sets both consisting of another additional vertex (shown with blue and green). If we assign the independent sets that consist of a single vertex (such as the red one) a value of 0, and the others consisting of two vertices (such as the blue and green ones) a value of $\frac{1}{2}$, we get that a total weight of $frac52$, since there are 5 such independent sets with 2 vertices total.

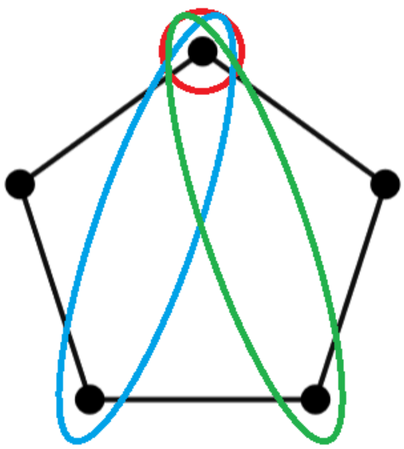

Figure 10: The graph C_5 with ovals outlining the independent sets for one vertex of the graph.

Proposition 3.2.4. *For any graph G,*

$$\chi_f(G) \leq \chi(G).$$

Proof. Say that $\chi(G) = k$. Then the color classes of this coloring of G form k pairwise disjoint independent sets $V_1, \cdots, V_k$ with $\bigcup_{i=1}^{k} V_i = V(G)$. Then the function f such that $f(V_i) = 1$ and $f(S) = 0$ for all other independent sets is a valid fractional coloring. For this particular fractional coloring, we have that the sum of the total weights is equal to k. Since we do not know if this is the minimum possible weight, we know that $k \geq \chi_f(G)$. And thus we have $\chi_f(G) \leq \chi(G) = k$. □

Example 3.2.5. We now show an example of what a fractional coloring looks like. In this case, the graph C_5, the most optimal way to assign values to every independent set is by giving $\frac{1}{2}$ to each independent set of 2 vertices, and 0 to every singleton set and the empty set. Since every vertex is part of 2 such independent sets, so note that there are thus 5 such independent sets total, so $\chi_f(C_5) = \frac{5}{2}$.

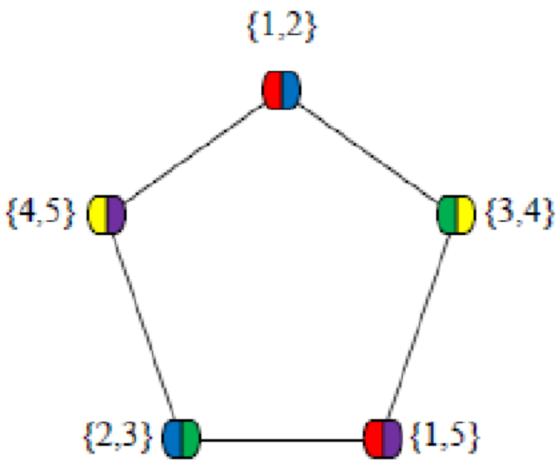

Figure 11: The optimal fractional coloring of C_5. Image sourced from [1].

Definition 3.2.6. A *fractional clique* of a graph G is a nonnegative real-valued function f on $V(G)$ such that, for any independent set S in G,

$$\sum_{x \in S} f(x) \leq 1.$$

This is to say that for fractional cliques we assign a real number to every vertex in G, as opposed to fractional colorings, where we assign a real number to every independent set. Then, for fractional cliques, the sum of all the vertices in any given independent set is not allowed to exceed 1. We refer to the sum of all the values given to every vertex as the *weight* yet again. We denote the maximum weight of a fractional clique of G by $\omega_f(G)$, and the regular clique number of G by $\omega(G)$, which is the largest clique in G. For this definition, we have a similar lemma to that of the fractional coloring number.

Lemma 3.2.7. *For any graph G,*

$$\omega(G) \leq \omega_f(G).$$

Proof. Suppose that $\omega(G) = k$, so G contains a k-clique. We assign a value of 1 to all the vertices within this k-clique, and 0 to all the others. Since there are no independent sets amongst the vertices within the k-clique aside from the vertices on their own, we do indeed have that sum of the values for every independent set is at most 1, namely when one of the vertices of the k-clique is included. This means that $\omega_f(G) = k$, since there are k vertices with a value of 1. Thus we have $\omega(G) \leq \omega_f(G)$. □

We will now go over the outline of the proof.

Proposition 3.2.8. *For graphs any G and H,*

$$\chi_f(G \times H) \leq \min\{\chi_f(G), \chi_f(H)\}.$$

Proof. Assume without loss of generality that $\chi_f(G) \leq \chi_f(H)$. Given a fractional coloring f of the graph G, we can construct a fractional coloring f' of $G \times H$ by setting $f'(I \times V(H)) = f(I)$ for all independent sets I of G, and setting $f'(U) = 0$ for all the other independent sets of $G \times H$. Note that $I \times V(H)$ for all independent sets I does indeed cover all the vertices of $G \times H$. Since f' then meets all the requirements, we have that f' is a fractional coloring of $G \times H$ with the same weight as f. Therefore we get our desired result. □

Example 3.2.9. We show an example of such a coloring f' based on a fractional coloring f, in this case that of the graph K_3.

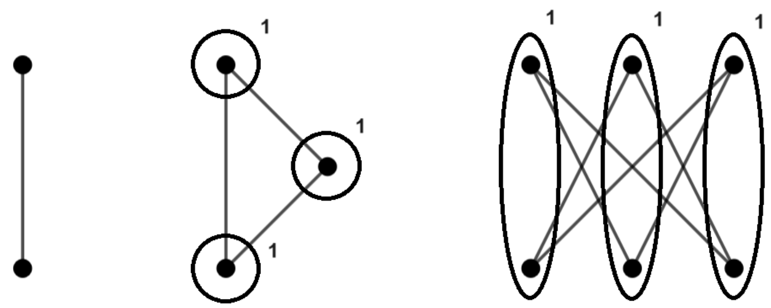

Figure 12: Fractional coloring of $K_2 \times K_3$ (right) based on the fractional coloring of K_3 (left).

Thus, all we would now need to do to prove Hedetniemi's conjecture is true in the case of fractional colorings, is to show that

$$\chi_f(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}$$

for any graphs G and H. The fractional chromatic number and the fractional clique number can be seen as two linear programming problems which are dual to each other. Thus, by the duality of linear programming we know that for any graph G, $\chi_f(G) = \omega_f(G)$ [4]. Thus, to prove Hedetniemi's conjecture, it would suffice to prove that

$$\omega_f(G \times H) \geq \min(\omega_f(G), \omega_f(H)).$$

In order to prove this, we look at the mapping $\varphi_{g,h}$. For a maximum fractional clique g of G and a maximum fractional clique h of H, $\varphi_{g.h}$ is defined as $\varphi_{g,h}((x, y)) = g(x)h(y)$. Then, the mapping

$$\frac{\varphi_{g,h}}{\max\{\omega_f(G), \omega_f(H)\}} \tag{3.2}$$

is a candidate for being a fractional clique of $G \times H$. If it is indeed a fractional clique, then its total weight would be equal to

$$\frac{\omega_f(G)\omega_f(H)}{\max\{\omega_f(G), \omega_f(H)\}} = \min\{\omega_f(G), \omega_f(H)\},$$

and then we would have that $\omega_f(G \times H) \geq \min(\omega_f(G), \omega_f(H))$. Thus we aim to prove that the aforementioned mapping 3.2 is indeed a fractional clique of $G \times H$.

The following definition is utilized in the proof.

Definition 3.2.10. We define the *neighborhood* of a vertex x in a graph G by

$$N_G(x) = \{x' \in G \mid \{x, x'\} \in G\}.$$

We define the *closed neighborhood* of a vertex x in a graph G by

$$N_G[x] = \{x' \in G \mid \{x, x'\} \in G\} \cup \{x\}.$$

Lemma 3.2.11. *For f a maximal fractional clique of the graph G, and X an independent set of G, we have that*

$$f(X) \leq \frac{f(N_G[X])}{\omega_f(G)},$$

where $f(X) = \sum_{x \in X} f(x)$.

Proof. We let $G' = G - N_G[X]$. For any Y an independent set of G', we know that $X \cup Y$ is an independent set of G. Thus we have that $f(X) + f(Y) \leq 1 \implies f(Y) \leq 1 - f(X)$. Suppose $f(X) = 1$, then we have that $f(Y) \leq 0$, so $f(y) = 0$ for all $y \in G'$. This means that $f(N_G[X]) = \omega_f(G)$ and this lemma is true. Now suppose that $f(X) < 1$. Let $f' : V(G') \to [0, 1]$ be defined by $f'(x) = f(x)/(1 - f'(X))$. Then for Y an independent set of G', we have that $f'(Y) = f(Y)/(1 - f(X)) \leq 1$. Thus, f' is a fractional clique of G'. Recall that $\omega_f(G)$ denotes the maximum weight of a fractional clique of G, so then we get that $f'(G') \leq \omega_f(G') \leq \omega_f(G)$ since $G' \subset G$. This implies $f(V(G')) = \sum_{x \in G'} f(x) = f'(V(G'))(1 - f(X)) \leq \omega_f(G)(1 - f(X))$. We know that $f(V(G')) + f(N_G[X]) = \omega_f(G)$, so inserting the inequality for $f'(V(G'))$ yields

$$\begin{aligned}(1 - f(X))\omega_f(G) + f(N_G[X]) &\geq \omega_f(G)\\ 1 - f(X) + \frac{f(N_G[X])}{\omega_f(G)} &\geq 1\\ \frac{f(N_G[X])}{\omega_f(G)} &\geq f(X).\end{aligned}$$

This proves the lemma in both cases. □

In the article [19] this lemma is then used to prove the following lemma.

Lemma 3.2.12. *Assume G and H are graphs, g is a maximum fractional clique of G and h is a maximum fractional clique of H. Let $\varphi_{g,h} : G \times H \to [0, 1]$ be defined as $\varphi_{g,h}((x, y)) = g(x)h(y)$. Then for any independent set U of $G \times H$, we have $\varphi_{g,h}(U) \leq \max\{\omega_f(G), \omega_f(H)\}$.*

This would then in turn prove that 3.2 is indeed a fractional clique of $G \times H$ with total weight $\min\{\omega_f(G), \omega_f(H)\}$. This then implies that

$$\omega_f(G \times H) \geq \min\{\omega_f(G), \omega_f(H)\}$$

and thus we would get the equality, proving Hedetniemi's conjecture is true in the case of fractional colorings. For brevity's sake, the proof for this lemma is omitted from this thesis, but can be found in [19]. All definitions utilized in the proof have already been discussed in this section, so a read-through is recommended if you are interested in this topic.

3.3 Directed graphs

Directed graphs are a specific kind of graph where edges are typically depicted as pointed arrows, indicating that the edges are not unordered pairs such as in regular graphs. For example, in a regular graph, the edges $\{v_1, v_2\}$ and $\{v_2, v_1\}$ refer to the same edge. This is not the case for directed graphs, where all edges are ordered pairs, i.e going *from* v_1 *to* v_2 or vice versa.. As is to be expected, this affects the product of directed graphs as well. In this section we will go over some definitions specific to directed graphs, and prove that Hedetniemi's conjecture is false for these graphs. The definitions and proof are adapted from [10].

Definition 3.3.1. A *digraph* (directed graph) D is a pair $D = (V, A)$, where $V = V(D)$ is the set of vertices, and $A = A(D)$ is the set of *arcs*, edges with an initial and a terminal point. $A(D)$ is a set of ordered pairs of vertices.

One can view regular graphs $G(V, E)$ as *symmetric* digraphs, i.e digraphs where, if $(v_1, v_2) \in A(D)$, then automatically also $(v_2, v_1) \in A(D)$. Conversely, a digraph is called *antisymmetric* if it is not symmetric. In this section we aim to prove that

$$\chi(D_i \times D_j) \neq \min\{\chi(D_i), \chi(D_j)\} \tag{3.3}$$

in general for directed graphs. Note that we define the chromatic number of a digraph $\chi(D)$ as the chromatic number of its symmetric modification, constructed by adding a reversed edge for every pair of vertices which only have an edge going one way. We denote the symmetric modification of a digraph D by $\tilde{D}$. We will also make the distinction between regular edges and arcs by using $\{v_1, v_2\}$ to indicate an edge, and (v_1, v_2) for arcs to avoid confusion.

As an aside, we briefly define arc-colorings on digraphs. We will not utilize this definition for our proof, but on the topic of colorings of directed graphs, it is interesting enough to mention.

Definition 3.3.2. We say two arcs are *consecutive* if the start point of one arc is the end point of the other.

Definition 3.3.3. An *arc-coloring* is a coloring of all arcs in a digraph such that no two consecutive arcs have the same color. The least number of colors it takes to do this is called the *arc-chromatic number* $c(D)$.

While not necessarily relevant for the remainder of this section, [6] discusses this type of coloring more in-depth and also presents an interesting link between $c(D)$ and $\chi(D)$ for a digraph D, namely that if $c(D) \leq n$, then $\chi(D) \leq 2^n$.

We now go back to discussing products of directed graphs. For these graphs, our initial definition of products of graphs 2.2.1 still holds.

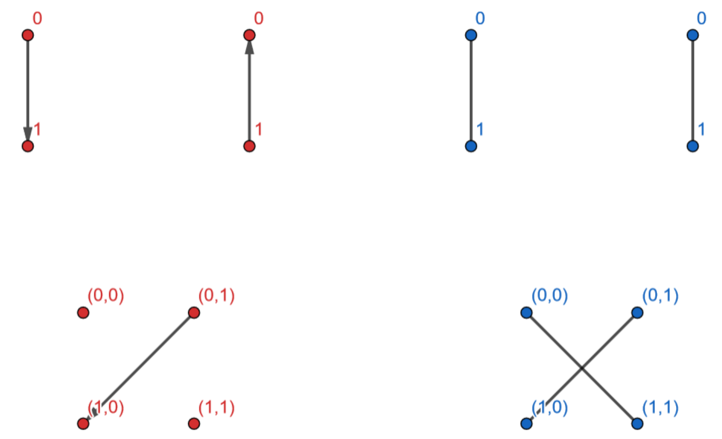

Figure 13: On the left are two directed graphs and their product, on the right are the symmetric modifications of the same two graphs and their product. Note the importance of the direction of the arcs in the former case.

Example 3.3.4. We show this point with an example of what a product of two digraphs might look like.

We can see that for the product of two digraphs, an edge $((v_0, u_0), (v_1, u_1))$ only exists if the arcs (v_0, v_1) and (u_0, u_1) exist in the first and second graph, respectively. If the first graph only had the arc (v_1, v_0), the edge $((v_0, u_0), (v_1, u_1))$ would not exist in the product.

Before we show that there does indeed exist such a counterexample to Hedetniemi's conjecture in the case of digraphs, we will first show when this is not the case.

Theorem 3.3.5. *The product of two 3-chromatic digraphs is 3-chromatic.*

Proof. Note that any 3-chromatic digraph has an antisymmetric cycle of odd length (as proven in 2.1.12. Thus we aim to prove that the product of any two antisymmetric cycles C_p and C_q (with p, q odd and C_p, C_q constructed such that $\tilde{C_p}$ and $\tilde{C_q}$ are cycles) is not 2-colorable.

We first show that the number of vertices with degree 2 in $C_p \times C_q$ is odd. By the degree of a vertex v of a digraph we refer to the sum of the outdegree (arcs that have v as its initial point) and the indegree (arcs that have v as its terminal point) of a vertex. We know for C_p and C_q that the number of vertices with outdegree 2 and indegree 2 are equal. Thus, in the product $C_p \times C_q$, the number of vertices with degree 4 and degree 0 are equal as well, and thus even. This is because a vertex (u, v) in the product $C_p \times C_q$ has degree 4 if u, v are either both vertices with indegree 2, or outdegree 2 in C_p and C_q respectively, and it has degree 0 if u has outdegree 2 and v has indegree 2, or vice versa. In all other cases, u, v has degree 2. Note that since q and p are both odd, the number of vertices in the product is odd. Since the number of vertices with degree 4 or 0 is even, the remainder, so the number of vertices with degree 2, must be odd.

Assume that we can color $C_p \times C_q$ with only 2 colors, 0 and 1. We use $n(i, k)$ to denote the number of vertices with degree i colored by k. Then the number of arcs of the product is equal to

$$2n(2,0) + 4n(4,0) = 2n(2,1) + 4n(4,1)$$

since every arc contributes a value of 1 to each of the two colors. We can reduce this to

$$n(2,0) + 2n(4,0) = n(2,1) + 2n(4,1)$$

which shows us that $n(2,0) + n(2,1)$ must be even. However, this contradicts the fact that the number of vertices in $C_p \times C_q$ with degree 2 is odd. Thus, $C_p \times C_q$ is not 2-colorable. □

This theorem proves to us the existence of a lower bound for the conjecture. We next show that Hedetniemi's conjecture does indeed fail when we increase the chromatic number for the two initial digraphs at the hand of a counterexample.

Theorem 3.3.6. *Hedetniemi's conjecture is false for digraphs with chromatic number > 3.*

Proof. We define two graphs D_r and D_r^*. $D_r = (\{1, 2, \cdots, r\}, \{(i,j)|1 \leq i < j \leq r\})$ with $r > 3$, so a digraph where every vertex only has arcs in the direction of vertices with a greater value than its own. D_r^* is derived from D_r with the only difference between the two being that the arc $(1, r)$ is reversed in D_r^*. It can then be checked that $\chi(D_r \times D_r^*) = r - 1$, while $\chi(D_r) = \chi(D_r^*) = r$. □

Example 3.3.7. See the figure below for an example of this construction.

This proves 3.3, thus showing that Hedetniemi's conjecture is false in the case of directed graphs.

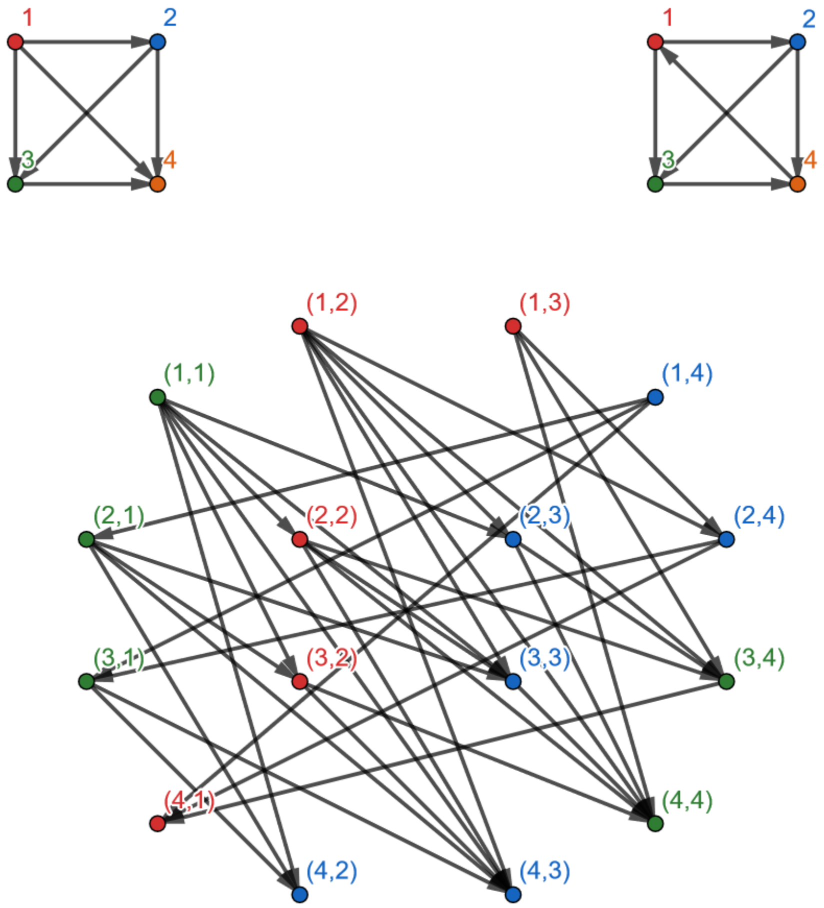

Figure 14: Counterexample of Hedetniemi's conjecture for digraphs in the case of $r = 4$ with respect to the construction in 3.3.6.

4 General refutation of Hedetniemi's conjecture

Now that we have looked at some cases in which the conjecture is either true or false, we move on to discuss the general case of Hedetniemi's conjecture. The question of whether this conjecture is true remained unanswered for quite a long time, and several strategies for tackling the problem had been thought of over the years [18], including Hajós' construction and exponential graphs. But in 2019, a paper was released [11], titled "Counterexamples to Hedetniemi's Conjecture" disproving this conjecture using the latter of the two mentioned strategies, along with strong products and results from other branches of graph theory, including fractional coloring among a few others. Said paper in which this refutation is given is rather brief and densely packed with definitions that can be quite complex for those who are not yet intimately familiar with this topic in graph theory. Thus, in this section the paper will be expanded on and every step will be handled with more attention to detail to make the proof more graspable.

4.1 Exponential graphs

As the counterexample uses a lot of complicated concepts and definitions, we will start by going over the most important and recurrent ones in his paper. Namely, exponential graphs. Many parts behind the paper's method to disprove Hedetniemi's conjecture require the use of exponential graphs, as these types of graphs are utilized to ultimately build up to a construction of a product of two graphs that serves as a "counterexample" to the conjecture. This is why it is important that we take the necessary care to ensure this definition is properly discussed. In this section, we will discuss exponential graphs, what a suited coloring is on an exponential graph, and some basic results both to help us better understand exponential graphs and the strategy behind the refutation.

Definition 4.1.1. For a (finite) graph G, we call $\mathcal{E}_c(G)$ the *exponential graph* of G with respect to c colors. This means that every *vertex* of $\mathcal{E}_c(G)$ is a different map $V(G) \to \{1, \cdots, c\}$. We say there is an *edge* between two mappings φ and ψ if and only if $\varphi(x) \neq \psi(y)$ for every $\{x, y\} \in E(G)$.

In other words, the exponential graph of G with respect to c colors is a graph where every vertex is a differently c-colored version of G itself. Every vertex in the exponential graph is a unique way to color G, and every possible c-coloring of G is a vertex in $\mathcal{E}_c(G)$ as well. Note that in this case we also include colorings which are not proper, meaning for a vertex in $\mathcal{E}_c(G)$ (or a coloring of G), we allow two adjacent vertices in that mapping to share the same color.

Proposition 4.1.2. *If* $|V(G)| = n$*, then* $|V(\mathcal{E}_c(G))| = c^n$*.*

Proof. This can be seen by counting all the possible options for maps from $V(G)$ to $\{1, \cdots, c\}$. For every vertex in G, we get c colors to assign to it, and we have n vertices in G, so this gives us c^n different maps. □

One can also deduce from this why it is called an *exponential* graph.

Example 4.1.3. Below is an example of a graph and the vertices in its exponential graph.

From this point onward, we will use φ to denote a (possibly not proper) coloring of G (i.e. a mapping $V(G) \to \{1, \cdots, c\}$). This means that every φ is a vertex of $\mathcal{E}_c(G)$. We use $\mathrm{Im}(\varphi)$ to denote the set of the colors assigned to all the vertices in G by the mapping φ. For example, if G has 3 vertices, and φ assigns them the colors 1, 2, and 1 respectively, then $\mathrm{Im}(\varphi) = \{1, 2\}$.

Additionally, we use Ψ to denote a c-coloring of $\mathcal{E}_c(G)$ (i.e. a mapping $\mathcal{E}_c(G) \to \{1, \cdots, c\}$).

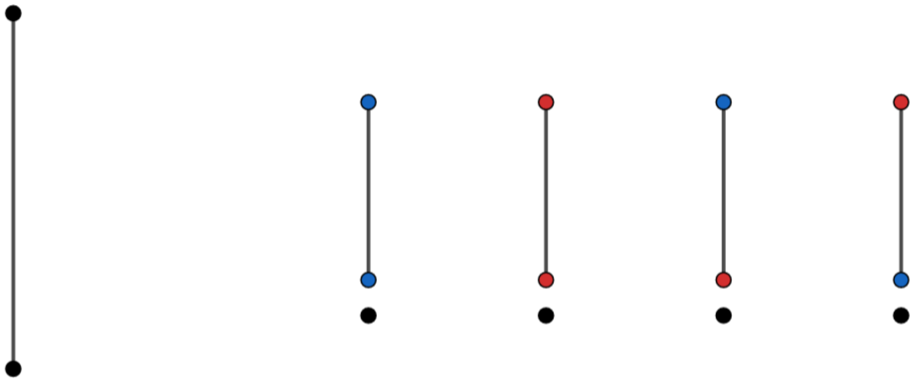

Figure 15: On the left is the graph K_2, and on the right are all the *vertices* of the graph $\mathcal{E}_2(K_2)$ are shown as black dots with their respective mappings of K_2 above.

Definition 4.1.4. We say a coloring Ψ of $\mathcal{E}_c(G)$ is *suited* if, for every color i, i is assigned to the constant mapping $\varphi^{(i,i,\cdots,i)}$, which sends every vertex of G to i.

Example 4.1.5. We show an example of a proper and suited coloring to illustrate this definition.

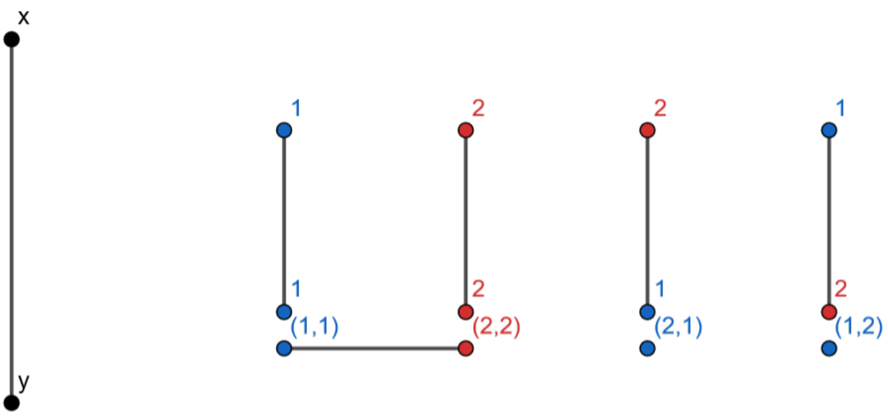

Figure 16: A proper suited coloring of $\mathcal{E}_c(K_2)$ with $c = 2$. On the left is K_2, and on the right is the graph $\mathcal{E}_c(K_2)$. As it is a suited coloring, the constant mappings (1,1) and (2,2) are given the colors 1 (blue) and 2 (red), respectively.

Note that every vertex in $\mathcal{E}_c(K_2)$ represents a different mapping of colors onto K_2, and is labeled accordingly. For example, the label (1,1) indicates that both vertices of K_2 are assigned the color 1 on this vertex of $\mathcal{E}_c(K_2)$. (2,1) and (1,2) are assigned the color 1, but assigning them the color 2 would have also given us a proper suited coloring, or assigning 1 to either and 2 to the other. There is an edge between $\varphi^{(1,1)}$ and $\varphi^{(2,2)}$ as they do not share any colors, so $\varphi^{(1,1)}(x) \neq \varphi^{(2,2)}(y)$ for all edges $\{x, y\} \in K_2$, where $\varphi^{(1,1)}$ is the constant mapping $\{v \to 1\}$ and $\varphi^{(2,2)}$ is the constant mapping $\{v \to 2\}$. However, for $\varphi^{(1,2)}$ and $\varphi^{(2,1)}$, we have that $1 = \varphi^{(1,2)}(x) = \varphi^{(2,1)}(y) = 1$, thus we cannot have an edge between $\varphi^{(1,2)}$ and $\varphi^{(2,1)}$. Note that there cannot be any other edges besides $\{\varphi^{(1,1)}, \varphi^{(2,2)}\}$ for the same reason.

We will now prove the following lemma on suited colorings of $\mathcal{E}_c(G)$.

Lemma 4.1.6. *If Ψ is a proper and suited c-coloring of $\mathcal{E}_c(G)$, then for all $\varphi \in \mathcal{E}_c(G)$, we have $\Psi(\varphi) \in \mathrm{Im}(\varphi)$.*

Proof. Recall that there is an edge between two mappings φ and ψ if and only if $\varphi(x) \neq \psi(y)$ for every $\{x, y\} \in E(G)$. This means that an arbitrary mapping φ will always share an edge with the constant mapping $\{v \to j\}$ for any j *not* in $\mathrm{Im}(\varphi)$. Since Ψ is a proper coloring, this means that φ can not be colored with any j not in its image. Thus, $\Psi(\varphi) \in \mathrm{Im}(\varphi)$. □

This lemma was the first of the three main building blocks of the refutation. Now that we went over the definitions of exponential graphs, proper colorings, and given some examples, we will discuss the rest of the proof.

4.2 Refutation of Hedetniemi's conjecture

In this section, we will go over the remainder of the steps of the proof in depth.

Definition 4.2.1. Take a graph G and a coloring Ψ of $\mathcal{E}_c(G)$. For any color $b \in [c]$ and any vertex $u \in V(G)$, we define $I(u, b)$ as the set of all $\varphi \in \Psi^{-1}(b)$ such that $\varphi(u) = b$. We denote a mapping which is part of the set $I(u, b)$ by ψ_{ub} at times.

In other words, $I(u, b)$ is the set of all φ that are assigned the color b under Ψ, but are also assigned the color b in the vertex $u \in V(G)$, so $\varphi(u) = b$.

Example 4.2.2. In Figure 16, $\varphi^{(1,1)}$ and $\varphi^{(1,2)}$ are both elements of $I(x, 1)$, as both of these mappings have the color 1 (blue) under the mapping Ψ, but are also blue in the vertex x in their own respective colorings of G. Further we have that $\varphi^{(2,1)} \in I(y, 1)$, $\varphi^{(2,2)} \in I(x, 2), I(y, 2)$, and $\varphi^{(1,1)} \in I(y, 1)$ as well.

We now prove a simple result about the sets $I(u, b)$.

Lemma 4.2.3. *For every vertex $\varphi \in \mathcal{E}_c(G)$, there exists a vertex $u \in V(G)$ and a color $b \in [c]$ such that $\varphi \in I(u, b)$.*

Proof. This follows from 4.1.6. Since every vertex in $\mathcal{E}_c(G)$ gets assigned a color $b \in [c]$, this implies that $b \in \mathrm{Im}(\varphi)$. Meaning, there is at least one $u \in G$ such that $\varphi(u) = b$. □

We discuss another definition before proving the second step in this refutation. By a color class b we refer to the set of elements (in our case mappings φ) that are colored with the color b.

Definition 4.2.4. We say that a color class b is v-robust if for every $\varphi \in \Psi^{-1}(b)$, there is a $w \in N[v]$ such that $\varphi(w) = b$, where $N[v]$ is the closed neighborhood of the vertex v. So w is either v itself or any vertex directly adjacent to v.

Now we have all the necessary definitions and lemmas to prove the following theorem.

Theorem 4.2.5. *Take a graph G with $|V(G)| = n$, and a suited, proper c-coloring Ψ of $\mathcal{E}_c(G)$. Then, there is a vertex $v \in V(G)$ such that there are $\geq c - \sqrt[n]{n^3 c^{n-1}}$ color classes $\Psi^{-1}(b)$ that are v-robust.*

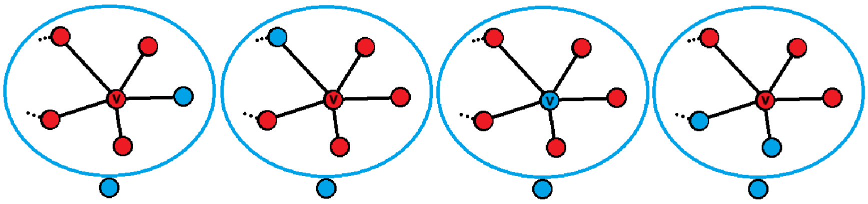

Figure 17: A small example of what v-robustness might look like. Here, the color class b (blue) is v-robust, so every mapping of G which gets assigned the color b under Ψ, must have a vertex w in the closed neighborhood of v. (These are just a few samples of mappings of a graph G and is not a comprehensive example.)

Proof. We use the classes $I(u, b)$ in this proof. Recall that we have that every $\varphi \in \mathcal{E}_c(G)$ belongs to at least one of the classes $I(u, b)$ according to 4.2.3, for some $u \in V(G)$ and some $b \in [c]$.

Suppose that $I(u, b)$ contains more than n^2c^{n-2} elements. We call $I(u, b)$ a *large* class when this is the case. Take an arbitrary mapping $\varphi_b \in \Psi^{-1}(b)$, meaning a mapping φ_b that is assigned the color b under the mapping Ψ. Now we assume that every element ψ_{ub} of the set $I(u, b)$ admits a vertex $u' \neq u$, with $\psi_{ub}(u') \in \text{Im}(\varphi_b)$. Note that we would then have at most $n - 1$ ways to pick u', and n ways to pick what the color $\psi_{ub}(u')$ is, because the image of φ_b can only contain n different elements at most. The other remaining vertices not equal to u' or u would contribute at most a factor of c^{n-2} to the amount of mappings ψ_{ub} meeting these requirements, as there are $n - 2$ vertices left and these can be any of c colors.

Then, since we know that $(n-1) \cdot n < n^2$, we can say that there are at most n^2 ways to pick u' and ψ_{ub}. Thus we have *less than* n^2c^{n-2} elements of $I(u, b)$ that meet these requirements. However, $I(u, b)$ is assumed to be a large class with *over* n^2c^{n-2} elements, so there must be at least one $\psi'_{ub} \in I(u, b)$ such that u is the *only* vector such that $\psi'_{ub}(u) = b$, and also $\text{Im}(\varphi_b) \cap \text{Im}(\psi'_{ub}) = b$, as in both cases there would otherwise be a $u' \neq u$ such that $\psi'_{ub}(u') \in \text{Im}(\varphi_b)$.

Keep in mind that every $\psi_{ub} \in I(u, b)$ is also assigned the color b by Ψ. This means that, if $I(u, b)$ is a *large* class, all ψ_{ub} are not allowed to be adjacent to any other $\varphi_b \in \Psi^{-1}(b)$. Having these two mappings be adjacent would contradict Ψ being a proper coloring, as they would be adjacent and share a color. Since we know that $\psi'_{ub}(u) = b$ by definition, this means that for every φ_b there must be at least one $w \in N(u)$ such that $\varphi_b(w) = b$, as this would not allow φ_b and ψ'_{ub} to be adjacent according to our definition on adjacency in exponential graphs. Thus, by Definition 4.2.4, we have that a color class b is u-robust if $I(u, b)$ is large.

Then, if we have a vertex $v \in V(G)$ such that $I(v, b)$ is large for at least $c - \sqrt[n]{n^3c^{n-1}}$ color classes, we are done. If this is not the case, we could define more than n^3c^{n-1} mappings $\varphi : V(G) \to \{1, \cdots, c\}$ for which the value of φ on any vertex w does not equal the colors b for which $I(w, b)$ is large, so φ is not part of any large class. Note that there are nc classes $I(u, b)$ total, n for each vertex u we can choose, and c for each color b we can choose. This means that all the non-large classes in total only cover n^3c^{n-1} mappings at most, since a large class has to have n^2c^{n-2} elements at least. Since we can define *more* than n^3c^{n-1} mappings φ, this means that one of these mappings φ must be a part of a large class, giving us a contradiction. Thus, such a vertex $v \in v(G)$ must exist. $\square$

Definition 4.2.6. For graphs G and H, the strong product $G \boxtimes H$ is the graph with vertices $V(G) \times V(H)$ like the tensor product, but with an edge between (v_0, u_0) and (v_1, u_1) iff $\{v_0, v_1\} \in E(G)$ or $(v_0 = v_1)$ and $\{u_0, u_1\} \in E(H)$.

The only difference between the *strong* product of two graphs and the *tensor* product of two graphs being that only one of the two edges has to be present in G or H for there to be an edge in $G \boxtimes H$, rather than both. We include an example of this to show what this difference looks like in practice.

Example 4.2.7. Below is an example of a strong product between two graphs.

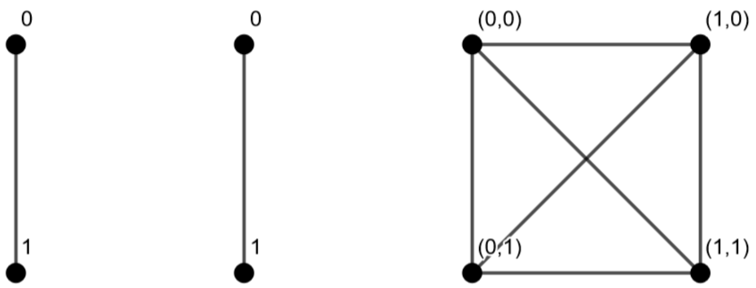

Figure 18: Strong product $K_2 \boxtimes K_2$

We will use the strong product to construct a counterexample for the conjecture. Mainly, we will use the product $G \boxtimes K_q$ for a sufficiently large q. We already know that

$$\chi((G \boxtimes K_q) \times \mathcal{E}_c(G \boxtimes K_q)) \leq c$$

as the mapping $(u, \varphi) \to \varphi(u)$ is a proper c-coloring of any graph of the form $G \times \mathcal{E}_c(G)$. Additionally, the paper [3] proves the existence of graphs with arbitrarily large girth and fractional coloring number, which we will use to prove that $\chi(G \boxtimes K_q) > c$. Namely it states that for all $r \in \mathbb{N}$ there exists a constant c such that for n sufficiently large there exists an r-chromatic graph with n vertices which has a girth $\geq c \log n$ (result from (6) in the paper), which also applies to fractional coloring. Then all that is left is to show that $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ for q large enough. This is would then be a counterexample to Hedetniemi's conjecture.

We will now move on to prove that $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ for sufficiently large q.

Lemma 4.2.8. *Let G be a graph with finite girth ≥ 6, and a vertex $v \in G$ as in Theorem 4.2.5. For an integer q and $c = \lceil 3.1q \rceil$, there is a clique $\mathcal{M}$ in the proper suited c-coloring of $\mathcal{E}_c(G \boxtimes K_q)$ of size $c - q$.*

Proof. Firstly, we find a vertex v as in 4.2.5. We define the vertices of the clique $\mathcal{M} = \{\mu_{q+1}, \ldots, \mu_c\}$ as follows for $i \in \{1, \ldots, q\}$ and $t \in \{q+1, \ldots, c\}$,

(1.1) $\mu_t(g, i) = i$ for all $g \in V(G)$ satisfying $\text{dist}(v, g) \in \{0, 2\}$,

(1.2) $\mu_t(g, i) = q + i$ for all $g \in V(G)$ satisfying $\text{dist}(v, g) = 1$,

(1.3) $\mu_t(g, i) = t$ for all $g \in v(G)$ satisfying $\text{dist}(v, g) \geq 3$.

We illustrate this in the figure on the next page.

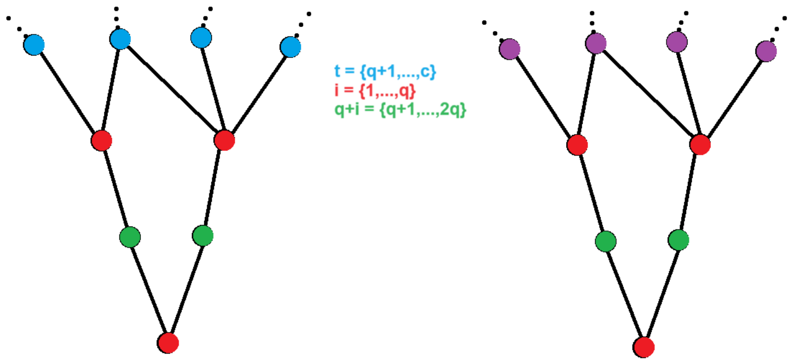

Figure 19: Two simplified elements of the clique $\mathcal{M}$, but where every vertex represents $\mu_t(v,i)$ for a single $v \in V(G)$, and for all $i \in [q]$. The bottom red vertex is $\mu_t(v,i)$ for v the vertex as in 4.2.5. So all the red vertices represent (1.1), the green vertices represent (1.2), and the blue and purple vertices represent (1.3).

Notice that all $\mu_t(g,i)$ with $\text{dist}(v,g) \leq 2$ get assigned colors independent of t, so they are the same for every t. Due to the assumption of girth on G, we know that we cannot have two adjacent vertices that both have a distance of 1 or 2 from v, as this would give us a girth of 5 or less. Knowing this, we can see that two vertices both from (1.1) or both from (1.2) cannot be both adjacent in $\mathcal{E}_c(G \boxtimes K_q)$ and have the same color. Additionally, since $i \in \{1,\ldots,q\}$ and $q+i \in \{q+1,\ldots 2q\}$ are distinct, there does not exist a pair of vertices as defined in (1.1) and (1.2) such that they are both adjacent in $G \boxtimes K_q$, and share the same color.

(1.3) assigns a different color for every t, so no two elements μ_t, $\mu_{t'}$ of $\mathcal{M}$ will have use the same color for their vertices in (1.3). Additionally, vertices in (1.3) could only be adjacent to other vertices in (1.3), which we have just covered, or vertices in (1.1) due to the distance. Since $i \in \{1,\ldots,q\}$ and $t \in \{q+1,\ldots,c\}$ are disjoint sets, vertices from (1.1) and (1.3) can never share a color.

This means that for $t \neq t'$, and all adjacent pairs of vertices $(g,i),(h,j) \in G \boxtimes K_q$ (where we do not necessarily require that $g \neq h$ or $i \neq j$), we have that $\mu_t(g,i) \neq \mu_{t'}(h,j)$. Following the definition of adjacency in exponential graphs, this proves that $\mathcal{M} = \{\mu_{q+1},\ldots,\mu_c\}$ is a clique in $\mathcal{E}_c(G \boxtimes K_q)$. ☐

This lemma will then allow us to show that we will have $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ with $c = \lceil 3.1q \rceil$ for a sufficiently large q.

Theorem 4.2.9. *Let G be a finite graph with finite girth ≥ 6. Then, for sufficiently large q, one has $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ with $c = \lceil 3.1q \rceil$.*

Proof. Assume that $\mathcal{E}_c(G \boxtimes K_q)$ has a proper suited c-coloring Ψ. Additionally, we define the graph Γ_G by adding a self-loop to every vertex of G. The restriction of Ψ to mappings that are constant on the cliques $\{g\} \times K_q \subset G \boxtimes K_q$ is a proper coloring $\Lambda : \mathcal{E}_c(\Gamma_G) \to \{1,\ldots,c\}$, and we find a vertex v as in Theorem 4.2.5 with respect to the coloring Λ. The reason we define such a graph Γ_G and this mapping is because the clique $\mathcal{M}$ we have defined earlier also consists of proper colorings $\mathcal{E}_c(\Gamma_G) \to \{1,\ldots,c\}$. The second mapping we will construct will also be a proper coloring of $\mathcal{E}_c(\Gamma_G)$ to $\{1,\ldots,c\}$, and we also later use the mapping Λ to find a v-robust color class to suit our needs.

By Lemma 4.2.8 we know that the clique $\mathcal{M}$ on $\mathcal{E}_c(G \boxtimes K_q)$ with respect to the vertex v has $c-q \geq 2.1q$ colors. We use the same notation as in said lemma for all the vertices μ_t of this clique. Then, by the pigeonhole principle, we can find a $\tau \in \{q+1,\ldots,c\}$ such that $\Psi(\mu_\tau) \notin \{1,\ldots,2q\}$. Since $\text{Im}(\mu_\tau) \subset \{1,\ldots,2q\} \cup \{\tau\}$, and because of Lemma 4.1.6, we know that $\tau = \Psi(\mu_\tau)$. Further, there are only $o(q)$ color classes that are not v-robust with respect to Λ in the terminology of Theorem 4.2.5. This means that we can find a v-robust class $\sigma \notin \{1,\ldots,2q\} \cup \{\tau\}$.

With all of this in mind, we can define a mapping $\nu : G \boxtimes K_q \to \{1,\ldots,c\}$ which is constant on the cliques $\{g\} \times K_q \subset G \boxtimes K_q$ by, for all i,

$$\nu(g,i) = \begin{cases} \tau & \text{for all } g \in V(G) \text{ in the closed neighborhood } N[v] \\ \sigma & \text{for all } v \in V(G) \text{ satisfying } \text{dist}(v,g) \geq 2 \end{cases}$$

We illustrate this in the figure below as well.

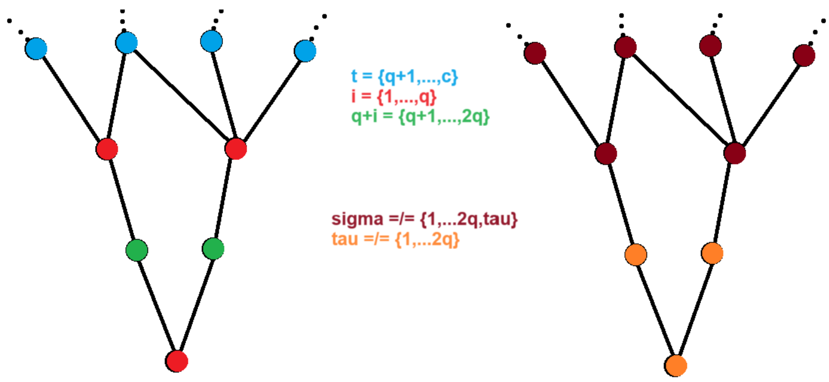

Figure 20: An element of the clique $\mathcal{M}$ (left) and the mapping ν (right), where every vertex represents the mapping for a single $v \in V(G)$, and for all $i \in [q]$.

Then, this mapping is adjacent to μ_τ, since $\text{Im}(\mu_\tau) = \{1,\ldots,2q\} \cup \{\tau\}$ and $\sigma \notin \{1,\cdots,2q\} \cup \{\tau\}$, and the vertices of ν that are assigned a value of τ can only ever be adjacent to vertices of μ_τ that get assigned a value within $\{1,\ldots,2q\}$. Since σ is v-robust with respect to Λ, we cannot have that $\Psi(\nu) = \sigma$ by Theorem 4.2.5, since there is no $w \in N[v]$ such that $\nu(w,i) = \sigma$ by construction of ν. Since the only other color in the image of ν is τ, this would imply by Lemma 4.1.6 that $\Psi(\nu) = \tau$. However, this would mean that $\Psi(\nu) = \Psi(\mu_\tau) = \tau$, which is a contradiction given they are adjacent. Thus, no such proper suited c-coloring of $\mathcal{E}_c(G \boxtimes K_q)$ can exist, and we conclude that $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$. $\square$

By the aforementioned paper [3], we can find a graph G that satisfies $\chi_f(G) > 3.1$. If we set $c = \lceil 3.1q \rceil$ and choose q sufficiently large, we will get $\chi(G \boxtimes K_q) \geq q * \chi_f(G) > c$. Then, we have both $\chi((G \boxtimes K_q) \times \mathcal{E}_c(G \boxtimes K_q)) = c$ as we had shown earlier, and $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ as we have just proven in Theorem 4.2.9. Thus, we have found a two graphs $G \boxtimes K_q$ and $\mathcal{E}_c(G \boxtimes K_q)$ such that $\min\{(G \boxtimes K_q), \mathcal{E}_c(G \boxtimes K_q)\} > c$, but $\chi((G \boxtimes K_q) \times \mathcal{E}_c(G \boxtimes K_q)) = c$. This means that Hedetniemi's conjecture is false in general.

Though this does disprove Hedetniemi's conjecture at the hand of a counterexample, it would be quite a challenge to illustrate this at the hand of an actual example, even using a computer. This is because the construction of this proof shows us that the conjecture fails for some c about

$p^3 3^p$ and with about $c^{p^3 2^{p-1}}$ vertices, with $p \leq 83$. Thus this result shows us that it fails for some c about 3^{95} and for graphs with about $(3^{95})^{3^{99}}$ vertices [20].

5 Progress on the conjecture

Though the conjecture was proven false in general in 2019 as we saw in the last section, there were still aspects of it that were in need of closure. Namely, for what $n = \min\{\chi(G), \chi(H)\}$ is the conjecture true and false? In this section we will show that the chromatic number of the product of two 4-chromatic graphs is always 4, that the conjecture is asymptotically false, and lastly we will show that the conjecture can be false if $\min\{\chi(G), \chi(H)\} = 5$.

5.1 Chromatic number at most 4

One can see that the conjecture is trivially true in the case of 1 and 2-chromatic graphs. In this section, we will look at 3 and 4-chromatic graphs which are less trivial, and show that the conjecture is true in those cases, i.e if $\min\{\chi(G), \chi(H)\} = 3$ or 4. We do this at the hand of exponential graphs. Definitions, lemmas and theorems are adapted from [2].

The outline of the proof is as follows. We first want to prove an equivalency between two statements:

Conjecture 5.1.1. *Let $n \in \mathbb{N}$. For graphs G and H, $\chi(G) > n$ and $\chi(H) > n$ implies that $\chi(G \times H) > n$.*

Note that this is Hedetniemi's conjecture but rephrased, as we know that $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.

Conjecture 5.1.2. *Let $n \in \mathbb{N}$. $\chi(G) > n$ implies that $\chi(\mathcal{E}_n(G)) = n$.*

If these two statements are equivalent for the same n, then to prove Hedetniemi's conjecture to be true for 4-chromatic graphs, all we would have to do is show that Conjecture 5.1.2 holds for $n = 3$.

Proposition 5.1.3. *Conjecture 5.1.1 and Conjecture 5.1.2 are equivalent for any $n \in \mathbb{N}$.*

Proof. Assume Conjecture 5.1.1 to not be true for a certain n. That is to say, we can find graphs G, H that are $(n+1)$-chromatic but $\chi(G \times H) = n$. for $f : G \times H \to \{1, \cdots, n\}$ a proper n-coloring, then we define for each vertex $v \in H$ the mapping f_v of G by $f_v(x) = f(x, v)$, $x \in G$. Then the mapping $\alpha : H \to \mathcal{E}_n(G)$ which sends each $v \in H$ to f_v is an edge-preserving map. Thus we get that $n+1 = \chi(H) \leq \chi(\alpha(H))$. Thus we have that $\chi(\mathcal{E}_n(G)) > n$, which contradicts Conjecture 5.1.2.

Conversely, we now assume Conjecture 5.1.2 to be false. Namely, we assume $\chi(G) = n+1$ and $\chi(\mathcal{E}_n(G)) > n$. Then, $G \times \mathcal{E}_n(G)$ would be a counterexample to Conjecture 5.1.1, as we can obtain a proper n-coloring by the mapping $(u, \varphi) \to \varphi(u)$, $u \in G$ and $\varphi \in \mathcal{E}_n(G)$. This proves the equivalency. ☐

From this equivalency we can also immediately prove the conjecture to be true in the case of $n = 3$:

Corollary 5.1.4. *Hedetniemi's conjecture 5.1.1 is true for $n = 2$, i.e. for $\min\{\chi(G), \chi(H)\} = 3$.*

Proof. If $\chi(G) \geq 3$ and G is connected, then $\mathcal{E}_2(G)$ contains exactly one edge, namely the one between the constant mappings. Thus $\chi(\mathcal{E}_2(G)) = 2$. By equivalency of Conjecture 5.1.1 and Conjecture 5.1.2, we have that Hedetniemi's conjecture is true for $n = 2$. ☐

Now we only need to prove Hedetniemi's conjecture in the case of $n = 3$. To do this, we first must go over a few lemmas on to 3-colorings of cycles of odd length, and their relation to exponential graphs.

Definition 5.1.5. Let $\varphi \in \mathcal{E}_3(C_n)$. We say a vertex $v_i \in C_n$ is *fixed* if its neighbors have different colors assigned to them, so if $\varphi(v_{i+1}) \neq \varphi(v_{i-1})$. We say that φ has *odd parity*, or is simply an *odd coloring* if it has an odd number of fixed vertices. Even parity is defined analogously.

The reason for the term *fixed* is that if a vertex v_i is fixed under a mapping φ, then any mapping adjacent to φ also assigns the same color to v_i.

Lemma 5.1.6. *Let $\varphi \in \mathcal{E}_3(C_n)$. Then the number of triples of consecutive vertices v_{i-1}, v_i, v_{i+1} which get three different colors by φ has the same parity as φ. If there are an odd number of such triples then φ has odd parity and vice versa.*

Proof. We partition C_n into monochromatic intervals of consecutive vertices. Then every interval $\{v_i, \cdots, v_{i+k}\}$ for $k \geq 1$ will contribute two to the number of fixed vertices, since the only fixed vertices in this interval are v_i and v_{i+k}. An interval with only a single vertex, $\{v_i\}$, can only contribute at most one if and only if v_{i+1} and v_{i-1} have different colors. This would make v_{i-1}, v_i, v_{i+1} a triplet, and thus the number of triplets determines whether φ has odd or even parity. □

Lemma 5.1.7. *A proper coloring of an odd (resp. even) cycle with at most three colours is odd (resp. even).*

Proof. We prove this lemma via induction on the length of the cycle. A proper coloring of C_3 has three fixed vertices. A proper coloring of C_4 has no fixed vertices if two colors are used, and two fixed vertices if three colors are used. Take f a proper coloring of C_n for $n \geq 5$. The statement is true if every vertex of C_n is fixed, so assume that there is a vertex v_i which is not fixed, and we let $f(v_i) = 2$ and $f(v_{i+1}) = f(v_{i-1}) = 1$. If we were to remove v_i and "merge" the vertices v_{i+1} and v_{i-1}, we would get a $(n-2)$-cycle which still has f as a proper coloring. Note that the number of fixed vertices decreases by two if $f(v_{i+2}) = f(v_{i-2}) = 3$, since that would mean that v_{i+1} and v_{i-1} were fixed vertices. In any other case the number does not change. Therefore the parity of f is the same as the parity of the resulting coloring of C_{n-2}, proving the lemma. □

Lemma 5.1.8. *Let φ_1 and φ_2 be connected by an edge in $\mathcal{E}_3(C_n)$. Then, f_1 and f_2 have the same parity.*

Proof. We can construct every pair of mappings φ_1, φ_2 in $\mathcal{E}_3(C_n)$ by looking at the product $C_n \times K_2$. The graph $C_n \times K_2$ consists of solely a cycle of length $2n$ if n is odd, and two cycles of length n if n is even. Figure 21 visualizes this. Denote by a_1, a_2 the vertices of K_2. We define a proper coloring φ of this product by $\varphi(v_i, a_j) = \varphi_j(v_i)$ for $i = 1, \cdots, n$, $j = 1, 2$. Note that by the construction of φ, that this defines φ_1 and φ_2 such that they would be adjacent in $\mathcal{E}_3(C_n)$. We now show that they have the same parity. Due to their adjacency in the exponential graph, a vertex v_i is fixed by φ_1 if and only if v_i is fixed by φ_2, as we have remarked earlier in this section. This tells us that the sum of the number of vertices fixed by φ_1 and φ_2 is equal to the number fixed by φ. By Lemma 5.1.7, this must be even since φ is proper. We conclude that φ_1 and φ_2 must have the same parity. □

From this lemma we can conclude that all vertices in a connected subset of $\mathcal{E}_3(C_n)$ have the same parity.

For the proof of the following proposition, we refer to the article [2].

Proposition 5.1.9. *Let C_n with vertices $v_1, \cdots, v_n$ and C_m with vertices $u_1, \cdots, u_m$ be two odd cycles. Then for any proper 3-coloring f of $C_n \times C_m$, the parity of the induced colorings f_{v_i} is different from the parity of f_{u_j}.*

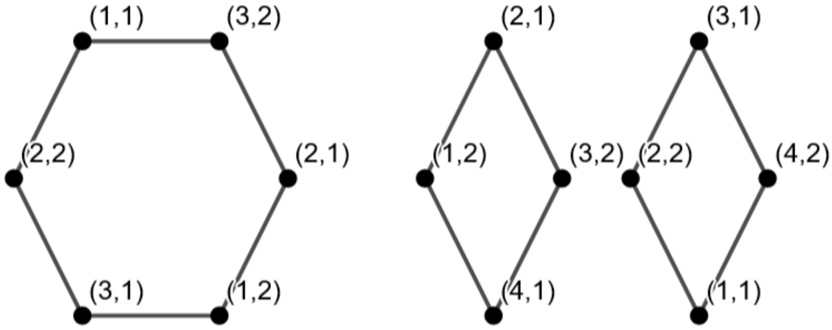

Figure 21: $C_3 \times K_2$ and $C_4 \times K_2$, respectively, where $\{1,2\}$, $\{2,3\}$, and so forth are edges in C_n. $C_3 \times K_2$ has a single cycle of length 6, and $C_4 \times K_2$ has two cycles of length 4.

We will now prove the main result that $\mathcal{E}_3(G)$ is 3-chromatic for all 4-chromatic graphs G at the hand of a few additional propositions and theorems. To do this, we consider the restriction of mappings $\varphi \in \mathcal{E}_3(G)$ to odd cycles of G.

It is likely that a coloring $\varphi \in \mathcal{E}_3(G)$ restricted to each odd cycle of G such that it has odd parity does not exist. But in the case that it does, the following proposition serves to prove that it is then an isolated vertex of $\mathcal{E}_3(G)$.

Proposition 5.1.10. *Let G be a 4-chromatic graph. Suppose there is a mapping $\varphi \in \mathcal{E}_3(G)$ whose restriction to each odd cycle of G has odd parity. Then, φ is an isolated vertex of $\mathcal{E}_3(G)$.*

Proof. We assume that $\{\varphi, \psi\}$ is an edge of $\mathcal{E}_3(G)$. We define:

$$X = \{x \in V(G) \mid \exists y \in V(G) \text{ such that } \{x,y\} \in E(G), \varphi(x) = \varphi(y)\}.$$

We claim that the induced subgraph $G(X)$ of G has chromatic number at least 3. Trivially, $\chi(G(X)) \geq 2$ since $G(X)$ has at least two vertices which share an edge. Suppose X is 2-chromatic and separate it into its two respective color classes, $X = X_1 \cup X_2$. We then get a proper f of G defined by

$$f(v) = \begin{cases} \varphi(v) & \text{for } v \in V(G) - X_1 \\ \psi(v) & \text{for } v \in X_1 \end{cases}$$

which is a 3-coloring since φ, ψ are 3-colorings. This contradicts G being 4-chromatic. Thus X cannot be 2-chromatic, so $\chi(G(X)) \geq 3$ and G contains an odd cycle which we denote by C. Thus by Lemma 5.1.6 and by φ having odd parity, there exists at least one consecutive triplet of vertices v_1, v_2, v_3 on C with $\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} = \{1,2,3\}$. Without loss of generality assume $\varphi(v_i) = i$, then since v_2 is adjacent to both v_1 and v_3 and by adjacency of φ and ψ, we require that $\varphi(v_1) \neq \psi(v_2) \neq \varphi(v_3)$. So, $\psi(v_2) = 2$. By the definition of X, there is a vertex $u \in G$ adjacent to v_2 such that $\varphi(u) = \varphi(v_2) = 2$. Therefore $\varphi(u) = \psi(v_2) = 2$, which contradicts their adjacency since we require $\varphi(x) \neq \psi(y)$ for all edges $\{x,y\} \in \mathcal{E}_3(G)$. Thus φ cannot have any adjacent vertices. □

Theorem 5.1.11. *Let C_n be an odd cycle. Then each component of $\mathcal{E}_3(C_n)$ with even parity is at most 3-chromatic.*

Proof. Let T be an even-parity component of $\mathcal{E}_3(C_n)$ and assume that H is a connected subgraph of T which is 4-chromatic. We define a 3-coloring ϕ of the graph $C_n \times H$ by $\phi(v,h) = h(v)$. Then ϕ is a proper coloring of $C_n \times H$ and for each $h \in H$, the induced coloring ϕ_h is simply the coloring h itself. Thus, by Proposition 5.1.9, each induced coloring ϕ_v, $v \in C_n$ has odd parity on every odd cycle of H, since every coloring ϕ_h must have even parity. Additionally, two such colorings $\phi_v, \phi_{v'}$ are adjacent in $\mathcal{E}_3(H)$ whenever v, v' are adjacent in C_n due to ϕ being a proper coloring. This contradicts Proposition 5.1.10. □

Now we have proven everything we need to show that the conjecture is true in the case $n = 4$.

Theorem 5.1.12. *$\mathcal{E}_3(G)$ is 3-chromatic for all 4-chromatic graphs G.*

Proof. Let H be a 4-chromatic connected subgraph of $\mathcal{E}_3(G)$, and $h_1 \in H$. From 5.1.10, we know that there must exist an odd cycle C in G such that the restriction of h_1 to C has even parity. Define a mapping $\alpha : H \to \mathcal{E}_3(C)$ by mapping each coloring $h \in H$ to its restriction on C. It is clear that α is edge-preserving. Therefore α maps H into a component T of $\mathcal{E}_3(C)$ with even parity. Therefore $\chi(H) \leq \chi(T)$, in contradiction to Theorem 5.1.11, since T is a component with even parity with a chromatic number higher than 3. Thus, a 4-chromatic subgraph of $\mathcal{E}_3(G)$ cannot exist. □

This proves that Conjecture 5.1.2 is true for $n = 3$, namely that $\chi(G) > 3$ implies that $\chi(\mathcal{E}_3(G)) = 3$. This in turn proves that Conjecture 5.1.1 is true for $n = 3$, so if $\min\{\chi(G), \chi(H)\} = 4$, then $\chi(G \times H) = 4$.

5.2 Hedetniemi's conjecture is asymptotically false

In section 4, we thoroughly discussed the refutation of Hedetniemi's conjecture by showing that, for q sufficiently large, $\chi(\mathcal{E}_c(G \boxtimes K_q)) \geq c+1$ for $c = \lceil 3.1q \rceil$, in turn proving that $\chi((G \boxtimes K_q) \times \mathcal{E}_c(G \boxtimes K_q)) = c$ while $\min\{\chi(G \boxtimes K_q), \chi(\mathcal{E}_c(G \boxtimes K_q))\} \geq c+1$. In this section we are interested in this "gap" in value between $\chi(G \times H)$ and $\min\{\chi(G), \chi(H)\}$. We can phrase this in terms of the *Poljak-Rödl function*.

Definition 5.2.1. The Poljak-Rödl function is a function $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \min\{\chi(G \times H) \mid \min\{\chi(G), \chi(H)\} = n\}.$$

In some pieces of literature this function is also often defined as

$$f(n) = \min\{\chi(G \times H) \mid \chi(G), \chi(H) \geq n\}$$

or

$$f(n) = \min\{\chi(G \times H) \mid \chi(G) = \chi(H) = n\}.$$

However, note that these ultimately all mean the same thing. For its similarity to Hedetniemi's conjecture, we use Definition 5.2.1.

In terms of this function, in section 4 we essentially proved that $f(n) \leq n-1$. In this section, we want to analyze what happens with $f(n)$ as n approaches infinity. Namely, we want to prove the following in this section:

Proposition 5.2.2. $\lim_{n\to\infty}(n - f(n)) = \infty$.

To do this, we again look at exponential graphs along with strong products, similar to the proof in section 4.

In fact, the proof uses a small subgraph of $\mathcal{E}_c(G \boxtimes K_q)$, and thus it is possible that this construction already provides examples such that show $\lim_{n\to\infty} f(n)/n = 0$. At the same time, we know that $\chi_f(G \boxtimes K_q) > c$, and the fractional version of Hedetniemi's conjecture is true, this would imply that $\chi_f(\mathcal{E}_c(G \boxtimes K_q)) = c$. Thus we might assume $\chi(\mathcal{E}_c(G \boxtimes K_q))/c$ to be bounded, and we would have to prove $\lim_{n\to\infty} f(n)/n = 0$ via a different construction.

We will now prove Proposition 5.2.2.

Proof. Choose a positive integer d. We will prove that for n sufficiently large, we have $f(n+d) \leq n$. Let G_d be a graph with girth at least 6 and $\chi_f(G_d) \geq 8d$. Then by Theorem 4.2.9 we have that $\chi(\mathcal{E}_c(G_d \boxtimes K_q)) \geq c+1$ while $\chi(G_d \boxtimes K_q) \geq 2cd$. Now we consider the graph $\mathcal{E}_{cd}(G_d \boxtimes K_q)$. For $i = 0, 1, \cdots, d-1$, we let Q_i be the subgraph of $\mathcal{E}_{cd}(G_d \boxtimes K_q)$ induced by the functions with the image in $\{ic+1, ic+2, \cdots, ic+c\}$, so

$$Q_i = \{\varphi \in \mathcal{E}_{cd}(G_d \boxtimes K_q) \mid \text{Im}\varphi \in \{ic+1, ic+2, \cdots, ic+c\}\}.$$

Note that each Q_i is isomorphic to $\mathcal{E}_c(G_d \boxtimes K_q)$ and hence at least $c+1$ colors are needed to color each Q_i. For each $i \neq j$, each function in Q_i is adjacent to each function in Q_j, since their images are disjoint. Hence, $\chi(\mathcal{E}_{dc}(G_d \boxtimes K_q)) \geq d(c+1)$. As $\chi((G_d \boxtimes K_q) \times \mathcal{E}_d c(G_d \boxtimes K_q)) = dc$ and $\chi(G_d \boxtimes K_q) \geq 2cd \geq cd + d$, it follows that $f(dc+d) \leq dc$.

This tells us that for every d there are infinitely many values of n in the form of $dc+d$ such that $n - f(n) \geq d$. Now we only have to show that the gap between $f(n)$ and n does not close going from one value of c to the next. Note that $c = \lceil 3.1q \rceil$, where q is any value above a fixed threshold, and $\lceil 3.1(q+1) \rceil - \lceil 3.1q \rceil \leq 4$. Thus we only need to examine the values $n = dc+d+i$ where $i \leq 4d$, and we can suppose that $c \geq 5$. The graph $\mathcal{E}_{cd+i}(G_d \boxtimes K_q)$ has $\mathcal{E}_{cd}(G_d \boxtimes K_q)$ as a subgraph, namely all the mappings with image $\{1, \cdots, cd\}$. For $j = cd+1, cd+2, \cdots, cd+i$ the constant mappings φ_j with image $\{j\}$ are all adjacent to each other, and each is adjacent to all the mappings in $\mathcal{E}_{cd}(G_d \boxtimes K_q)$. Hence

$$\chi(\mathcal{E}_{cd+i}(G_d \boxtimes K_q) \geq \chi(\mathcal{E}_{cd}(G_d \boxtimes K_q) + i \geq cd + d + i.$$

For $i \leq d(c-1)$, we also have $\chi(G_d \boxtimes K_q) \geq cd+d+i$, so that $f(cd+d+i) \leq cd+i$. Altogether, the inequality $f(n+d) \leq n$ is established for all but finitely many values of n. Thus we can conclude that $\lim_{n\to\infty} n - f(n) = \infty$. □

5.3 Product of 5-chromatic graphs can be 4

In section 4 we saw that, at the hand of that construction, the conjecture would fail for some c around 3^{95}, which is rather large and makes it near impossible for us to illustrate in the form of a counterexample. However, the year after the refutation, it was proven that the conjecture also fails with $\min\{\chi(G), \chi(H)\} = 126$ [20]. Shortly thereafter this number was lowered down to 14 in [13]. In early 2023 a proof was published showing that conjecture can also fail if at least one of the two graphs has a chromatic number of 5 [14]. Earlier in this section we discussed the proof on the conjecture always being true in the case of a chromatic number of 4 or less, so this does in fact answer the last remaining question to do with the general case of Hedetniemi's conjecture. In this section we will go over the outline of this proof, but we omit quite a lot details due to the complexity and length of the proof.

The proof uses a definition which we can use to construct a statement similar to saying that Hedetniemi's conjecture is true for a certain $n = \min\{\chi(G), \chi(H)\}$, namely the following definition.

Definition 5.3.1. We say a graph K is *multiplicative* if it satisfies the following property: if $G \times H$ admits a homomorphism to K, then either G or H admits a homomorphism to K.

A proper n-coloring of a graph G is a homomorphism from G to K_n, the complete graph on n vertices. The homomorphism is simply given by sending every vertex of G to the i-th vertex of K_n, where i is the color assigned to said vertex. Thus, if K_n is multiplicative for a certain $n \in \mathbb{N}$, this is equivalent to saying $G \times H$ is n-colorable $\implies$ G or H is n-colorable. Hedetniemi's conjecture is thus equivalent to saying that K_n is multiplicative for every $n \in \mathbb{N}$. In section 5.1, we showed that K_1, K_2, and K_3 are multiplicative. The article we will discuss in this section [14] proves that K_4 is non-multiplicative, so if $\chi(G \times H) = 4$, it does not necessarily imply that $\min\{\chi(G), \chi(H)\} = 4$. Note that in section 5.1 we *did* prove the opposite implication, namely that $\min\{\chi(G), \chi(H)\} = 4 \implies \chi(G \times H) = 4$.

The article even gives a counterexample showing K_4 is non-multiplicative, with $\min\{\chi(G), \chi(H)\} \geq 5$ while $\chi(G \times H) = 4$. We will go over the construction of such G and H, namely for $G = \mathcal{E}_4(\Omega_{13}(K_8))$, and H a subgraph of G. We show a depiction of H on the next page, and discuss the construction of the graph $\Omega_{13}(K_8)$.

Definition 5.3.2. The family of graphs $\Omega_w(K_m)$, named the *universal graphs for wide colorings*, is defined as follows.

We write m as a power of 2, i.e $m = 2^k$ for some $k \in \mathbb{N}$, and we let $v \geq 2k$ and $w = 2v+1$. Then the vertices of $\Omega_w(K_m)$ are $(v+1)$-tuples $(X_0, X_1, \ldots, X_v)$ such that $|X_0| = 1$ and $X_1, \ldots, X_v$ are nonempty subsets of $V(K_m)$ which satisfy the following two requirements:

$X_i \cap X_{i+1} = \emptyset$ for $i = 0, \ldots, v-1$,

$X_i \subset X_{i+2}$ for $i = 0, \ldots, v-2$.

Then we say there is an edge between two vertices $(X_0, X_1, \ldots, X_v)$ and $(Y_0, Y_1, \ldots, Y_v)$ if the following two requirements are met:

$X_i \subset Y_{i+1}$ and $Y_i \subset X_{i+1}$ for $i = 0, \ldots, v-1$,

$X_i \cap Y_i = \emptyset$ for all $i = 0, \ldots, v$.

So for our graph G we use $m = 8$, $v = 6$ and $w = 13$. We use the following theorem to show that the chromatic number of G is 8.

Theorem 5.3.3 ([5],[17]). *For $m \geq 2$ and an odd w, $\chi(\Omega_w(K_m)) = m$.*

Recall that we have $G = \Omega_{13}(K_8)$ and $H = \mathcal{E}_4(\Omega_{13}(K_8))$. Since we can color $G \times H$ properly with 4 colors with the coloring $(u, \varphi) \to \varphi(u)$, all that remains to prove is that $\chi(\mathcal{E}_4(\Omega_{13}(K_8)) \geq 5$.

To do this, the articles utilizes the following definition, where S is a subset of $V(K_w)$, and a, b are two distinct vertices of K_n, and $i \in \{0, \ldots, v\}$. We define the function $\sigma_{a,b}^{i,S} \in V(\mathcal{E}_n(\Omega_w(K_m)))$:

$$\sigma_{a,b}^{i,S} = \begin{cases} a & \text{if } S \cap X_i \neq \emptyset \\ b & \text{otherwise} \end{cases}$$

and for R, T two disjoint subsets of $V(K_m)$ and a, b, c three distinct vertices of K_n. Then the function $\tau_{a,b,c}^{i,R,T} \in V(\mathcal{E}_n(\Omega_w(K_m)))$ is defined by:

$$\tau_{a,b,c}^{i,R,T} = \begin{cases} a & \text{if } R \cap X_i \neq \emptyset \\ b & \text{if } R \cap X_i = \emptyset \text{ and } T \cap X_i \neq \emptyset \\ c & \text{otherwise} \end{cases}$$

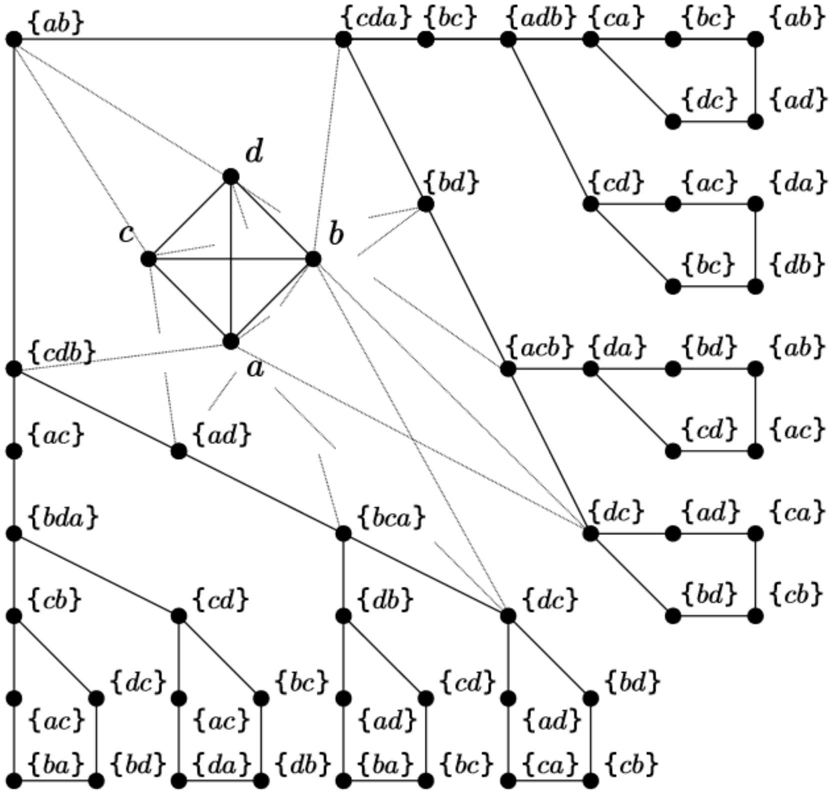

Figure 22: A subgraph of $\mathcal{E}_4(\Omega_{13}(K_8))$. The depicted graph is 5-chromatic, and if the full graph is multiplied with the 8-chromatic graph $\Omega_{13}(K_8)$ itself, gives a 4-chromatic product. A copy of K_4 is present in the graph, given by $\{a, b, c, d\}$. Vertices are labeled with a, b, c, and d to indicate which of the vertices of the K_4 they are *not* adjacent to. Not all edges are shown for clarity. Figure from the article [14].

Then, in terms of these functions, the top line in figure 22 is the sequence

$$\sigma_{a,b}^{0,\{1,2,3,4\}},\ \tau_{c,d,a}^{1,\{1,2\},\{3,4\}},\ \sigma_{b,c}^{2,\{1,2\}},\ \tau_{a,d,b}^{3,\{1\},\{2\}},\ \sigma_{c,a}^{4,\{1\}},\ \sigma_{b,c}^{5,\{1\}},\ \sigma_{a,b}^{6,\{1\}}.$$

In the article there are 4 different lemmas that altogether prove that $\sigma_{a,b}^{0,\{1,2,3,4\}}$ is indeed adjacent to $\tau_{c,d,a}^{1,\{1,2\},\{3,4\}}$, and that $\tau_{c,d,a}^{1,\{1,2\},\{3,4\}}$ is indeed adjacent to $\sigma_{b,c}^{2,\{1,2\}}$, and so forth. The last of these 4 lemmas also allows us to show that $\sigma_{a,b}^{6,\{1\}}$ is adjacent to $\sigma_{a,d}^{6,\{1\}}$, the vertex right below it in the figure. For now we only discuss the first and the last of these lemmas as we need these to construct a set of vertices on $\Omega_{13}(K_8)$ that show that the chromatic number of this graph is at least 5.

Lemma 5.3.4. *For S a subset of $V(K_m)$, a, b, c distinct vertices of K_n, and all $i \in \{0, \ldots, v-1\}$, we have that $\sigma_{a,b}^{i,S}$ and $\sigma_{c,a}^{i+1,S}$ are adjacent to each other.*

Proof. We let $X = (X_0, X_1, \ldots, X_v)$ and $Y = (Y_0, Y_1, \ldots, Y_v)$ be adjacent vertices in $\Omega_w(K_m)$. We want to prove that $\sigma_{a,b}^{i,S}(X) \neq \sigma_{c,a}^{i+1,S}(Y)$. By our definition, $\sigma_{a,b}^{i,S}(X) \in \{a, b\}$ and $\sigma_{c,a}^{i+1,S}(Y) \in \{c, a\}$. If $\sigma_{a,b}^{i,S}(X) = b$ then the functions will not be equal. If we have $\sigma_{a,b}^{i,S} = a$, then we must have that $S \cap X_i \neq \emptyset$. Since X and Y are adjacent in $\Omega_w(K_m)$, we must have that $X_i \subset Y_{i+1}$. This means that Y_{i+1} also shares an element with S, and $S \cap Y_{i+1} \neq \emptyset$ as well, meaning that $\sigma_{c,a}^{i+1,S} = c$. So in both cases, $\sigma_{a,b}^{i,S}(X) \neq \sigma_{c,a}^{i+1,S}(Y)$. □

This lemma shows us that $\sigma_{c,a}^{4,\{1\}}$ is adjacent to $\sigma_{b,c}^{5,\{1\}}$, and that $\sigma_{b,c}^{5,\{1\}}$ is adjacent to $\sigma_{a,b}^{6,\{1\}}$. The next lemma we prove is similar to the one above, but it will allow is to prove that $\sigma_{a,b}^{6,\{1\}}$ is adjacent to $\sigma_{a,d}^{6,\{1\}}$, the vertex right below it in the figure. Most importantly, this lemma will also allow us to point out a segment in our graph $H = \mathcal{E}_4(\Omega_{13}(K_8))$ that is 5-chromatic.

Lemma 5.3.5. *Let x be a single vertex of $V(K_m)$, and let a, b, c be three distinct vertices of K_n. For $i \in \{0, \ldots, v\}$, $\sigma_{a,b}^{i,\{x\}}$ and $\sigma_{a,c}^{i,\{x\}}$ are adjacent.*

Proof. We let $X = (X_0, X_1, \ldots, X_v)$ and $Y = (Y_0, Y_1, \ldots, Y_v)$ be adjacent vertices in $\Omega_w(K_m)$. If $\sigma_{a,b}^{i,\{x\}} = b$ then $\sigma_{a,b}^{i,\{x\}} \neq \sigma_{a,c}^{i,\{x\}}$ since $\sigma_{a,c}^{i,\{x\}}$ is either a or c. If $\sigma_{a,b}^{i,\{x\}} = a$, then since $S = \{x\}$, we must have that $x \in X_i$ since $\{x\} \cap X_i$ is nonempty. Since for two adjacent vertices X and Y in $\Omega_w(K_m)$ we require that $X_i \cap Y_i =$ for all $i \in \{0, \cdots, v\}$, we must conclude that $x \notin Y_i$ and that $\sigma_{a,c}^{i,\{x\}} = c$. Thus, in both cases, we get that $\sigma_{a,b}^{i,\{x\}} \neq \sigma_{a,c}^{i,\{x\}}$. □

This lemma does indeed show us that $\sigma_{a,b}^{6,\{1\}}$ is adjacent to $\sigma_{a,d}^{6,\{1\}}$, since a, b, d are three distinct vertices of K_4. It also assists in our construction of a counterexample. The next lemma serves to prove that one of the vertices in our construction must be assigned a specific color.

Lemma 5.3.6 ([14])**.** *Let $i = 0, 1, \cdots, k-1$ and recall that $2^k = m$. There exists a set $S \subset V(K_m)$ of size 2^{k-i-1} and distinct vertices a, b of K_n such that the color of $\sigma_{a,b}^{2i,S}$ is a.*

We can use this lemma to construct our counterexample. Namely, we now know that in our graph $\mathcal{E}_4(\Omega_{13}(K_8))$, there exists a vertex $x \in K_8$ and two distinct vertices a, b in K_4 such that the color of $\sigma_{a,b}^{2k-2,\{x\}}$ is a. We let c, d be the other two distinct vertices in K_4 that are not a or b. By Lemma 5.3.4, we have that the sequence

$$\sigma_{b,c}^{2k,\{x\}},\ \sigma_{c,a}^{2k-1,\{x\}},\ \sigma_{a,b}^{2k-2,\{x\}},\ \sigma_{d,a}^{2k-1,\{x\}},\ \sigma_{b,d}^{2k,\{x\}}.$$

is such that $\sigma_{b,c}^{2k,\{x\}}$ is adjacent to $\sigma_{c,a}^{2k-1,\{x\}}$, $\sigma_{c,a}^{2k-1,\{x\}}$ is adjacent to $\sigma_{a,b}^{2k-2,\{x\}}$, and so forth. The central vertex $\sigma_{a,b}^{2k-2,\{x\}}$ has the color a, and since $\sigma_{c,a}^{2k-1,\{x\}}$ and $\sigma_{d,a}^{2k-1,\{x\}}$ are adjacent, they must have the color c and d respectively. This must mean that $\sigma_{b,c}^{2k,\{x\}}$ and $\sigma_{b,d}^{2k,\{x\}}$ must both have the color b due to their adjacency as well. However, by Lemma 5.3.5 we know that $\sigma_{b,c}^{2k,\{x\}}$ and $\sigma_{b,d}^{2k,\{x\}}$ are adjacent, so they cannot have the same color. This gives us a contradiction, and thus $\chi(\mathcal{E}_4(\Omega_{13}(K_8)) \geq 5$.

This disproves Hedetniemi's conjecture in general in the case of $\chi(G \times H) = 4$, namely if $G = \Omega_{13}(K_8)$ which has a chromatic number of 8 as we have seen at the hand of Theorem 5.3.3, and $H = \mathcal{E}_4(G)$ which, as we have just proven, has a chromatic number of at least 5.

6 References

[1] G. Caroline and Dr. M. Siva. A study on colouring and chromatic number of graphs. *International Journal of Food and Nutritional Sciences*, 2022.

[2] M. El-Zahar and N. Sauer. The chromatic number of the product of two 4-chromatic graphs is 4. *Combinatorica*, 5:121–126, 1985.

[3] P. Erdös. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

[4] Chris Godsil and Gordon Royle. *Algebraic Graph Theory.* Graduate texts in mathematics. Springer, New York, NY, 2001 edition, April 2001.

[5] Hossein Hajiabolhassan. On colorings of graph powers. *Discrete Mathematics*, 309(13):4299–4305, 2009.

[6] C.C Harner and R.C Entringer. Arc colorings of digraphs. *Journal of Combinatorial Theory, Series B*, 13(3):219–225, Dec 1972.

[7] Stephen T. Hedetniemi. *Homomorphisms of graphs and automata.* PhD thesis, University of Michigan, 1966.

[8] Stephen T Hedetniemi. My top 10 graph theory conjectures and open problems. In *Graph Theory*, Problem books in mathematics, pages 109–134. Springer International Publishing, Cham, 2016.

[9] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319–324, Nov 1978.

[10] S. Poljak and V. Rödl. On the arc-chromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 31(2):190–198, Oct 1981.

[11] Yaroslav Shitov. Counterexamples to Hedetniemi's conjecture. *Annals of Mathematics*, 190(2):663 – 667, 2019.

[12] Saul Stahl. N-tuple colorings and associated graphs. *Journal of Combinatorial Theory, Series B*, 20(2):185–203, Apr 1976.

[13] Claude Tardif. The chromatic number of the product of 14-chromatic graphs can be 13. *Combinatorica*, 42(2):301–308, 3 2022.

[14] Claude Tardif. The chromatic number of the product of 5-chromatic graphs can be 4. *Combinatorica*, 43(6):1067–1073, 6 2023.

[15] Mario Valencia-Pabon and Juan Vera. Independence and coloring properties of direct products of some vertex-transitive graphs. *Discrete Mathematics*, 306(18):2275–2281, Sep 2006.

[16] Robin J. Wilson. *Introduction to Graph Theory.* Pearson, 2010.

[17] Marcin Wrochna. On inverse powers of graphs and topological implications of hedetniemi's conjecture. *Journal of Combinatorial Theory, Series B*, 139:267–295, 2019.

[18] Xuding Zhu. A survey on Hedetniemi's conjecture. *Taiwanese Journal of Mathematics*, 2(1):1 – 24, 1998.

[19] Xuding Zhu. The fractional version of hedetniemi's conjecture is true. *European Journal of Combinatorics*, 32(7):1168–1175, 2011. Homomorphisms and Limits.

[20] Xuding Zhu. Relatively small counterexamples to hedetniemi's conjecture. *Journal of Combinatorial Theory, Series B*, 146:141–150, 2021.