



MASTER MATHEMATICS

Electrical Vehicle Charging Under Two Types of Steering Signals With a Lower Bound

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Abstract

The increasing use of electrical vehicles and other large electric loads these days is causing a substantial increase of power consumption within the local distribution grid. Conventional reinforcements are expensive, so to alleviate the expected problems within the distribution network, demand side management (DSM) is considered. Due to long parking hours, the charging of electrical vehicles is often shiftable in time. So, electrical vehicles offer a great opportunity for the use of demand side management.

In this thesis we consider the problem of constructing an optimal schedule for charging an electrical vehicle under two types of steering signals: time-varying price signals and a target profile. We want to minimize an objective function based on these two steering signals. As an additional constraint we assume that on every time interval we can only charge a positive amount between two bounds, X_{\min} and X_{\max} , or there is no charging to be done at all on that time interval.

We construct an optimal schedule by slowly adding the charging that has to be done over the given time intervals, while keeping the (partial) schedule optimal. We see that a lot of different corner cases have to be considered and most of them are leading to very large expressions. To this end, we only show that it is theoretically possible to determine the partial schedules without explicitly constructing the required expressions.

Furthermore, we consider a special case of the problem, such that several corner cases no longer need to be considered. For this problem we can exactly determine the required expressions, to solve the problem to optimality. In the end we give an algorithm that combines all the expressions to construct an optimal schedule for a given EV in polynomial time.

Contents

1	Introduction	4
1.1	Background information	4
1.1.1	Demand Side Management	4
1.2	Outline of the thesis	4
2	Problem statement	6
2.1	Problem definition	6
2.2	Properties of solutions	7
3	General case	9
3.1	The case without bounds	9
3.2	The case with a lower bound	10
3.2.1	Lower the amount charged in time intervals equally	11
3.2.2	Not lowering the amount charged in time intervals equally	13
3.3	The case with a lower bound and an upper bound	19
3.3.1	Not taking b into account	20
3.3.2	Taking both b and c into account	22
4	Practical case	26
4.1	Determining the borders	26
4.2	Algorithm for the CPLBS	33
4.2.1	Activating time intervals t with $\alpha_t = \alpha_n$	33
4.2.2	Making the standard solution	35
4.2.3	Activating a time interval $n + 1$ with $\alpha_{n+1} > \alpha_n$	37
4.2.4	Activating the last time interval	40
4.2.5	Charge the remaining $r(x, C)$ on the already activated time intervals	41
4.2.6	Finding the optimal solution	42
5	Conclusion	44
5.1	Future work	44
	References	45

1 Introduction

1.1 Background information

Today we see an increasing use of electrical vehicles (EVs) and the amount of EVs used is expected to keep on growing in the coming years [3]. These EVs use a battery powered engine. So, for this engine to keep on working, the battery needs to be charged regularly. Most of the batteries of these cars are charged at home, using a local grid connection. Because of the limited charging capabilities of these grid connections, charging an EV takes a long time. Furthermore, the increasing use of electrical vehicles and the frequent need for charging causes a huge increase of the domestic energy requirement.

The supply of energy is increasingly coming from non-controllable sources, such as wind and sun. Due to their uncontrollable nature, these sources cannot always provide all the energy required within the grid. Also, the energy demand in part of the domestic grid, e.g. a neighbourhood or city, is usually not equally divided over time. There are time windows during the day in which the demand for energy is very high, these are called the peak hours. Because energy suppliers always have to meet the consumers demand, the energy demand that cannot be produced by the non-controllable sources has to be provided by peak generators. Since peak generators are generally expensive, inefficient and rely on carbon based fuels, their use is undesirable. A method to tackle this problem is derived in [3].

1.1.1 Demand Side Management

In demand side management (DSM), we try to modify the consumers demand for energy through various methods. We want to turn the consumers into active customers. To obtain this we use some steering signals. An often used example is the use of time-varying price signals, this way consumers are tempted into using more energy during the off-peak hours by offering them lower energy prices. The main goal of using demand side management is not to lower the total energy demand, but to shift the energy demand over time to off-peak hours to lower the peaks. Doing so is called peak shaving. Peak shaving can lead to less energy waste and reduce the need for peak generators.

In this thesis we use DSM, to charge electrical vehicles during off-peak hours. EVs offer great potential for using DSM, because the energy demand is easily shiftable in time due to the long parking hours at night. In this thesis we use two types of steering signals. In [2] it is shown that only using time-varying price signals as steering signals can cause even bigger peaks. To solve these issues, Gerards et al. developed a DSM methodology called profile steering in [2], where EV-specific target profiles are used. More about these steering signals and the reason why we use these specific steering signal is explained in [2].

1.2 Outline of the thesis

In this thesis we consider the problem of charging electrical vehicles in an optimal way, i.e. optimal with regard to a certain objective function. This objective function and some constraints are given in Subsection 2.1. After that, in Subsection 2.2, we give some properties of optimal solutions. The properties make it easier to construct optimal solutions later on.

We construct optimal schedules, using a method in which we increase the total charging done by the schedule in every step, while keeping the (partial) schedule optimal. To this end, we construct borders for the total amount charged by a schedule to decide when another time

interval is activated, i.e. used for a positive amount of charging, in Section 3. In Subsection 3.2 we first consider the case where only a lower bound on the amount that has to be charged on each time interval is present. In this case we already get very large expressions for the borders, so when we consider the case with a lower bound and an upper bound in Subsection 3.3, we do not exactly determine these borders anymore. Instead we argue that they exist and explain how we can find them.

Because of the large expressions and the many different corner cases that appeared in Section 3, we consider a special case of the problem in Section 4. We assume that the steering signals used in the objective value are sufficiently different, causing some corner cases to disappear. This leads to more manageable expressions for the above mentioned borders for the charging to be done. In Subsection 4.1 we calculate all the borders and determine some other properties, such that, in Subsection 4.2, we can develop an algorithm that solves the special case of the problem to optimality. The algorithm developed consists of many small steps, each shown in detail. Finally, each of these steps runs in polynomial time, resulting in the algorithm running in polynomial time.

2 Problem statement

In this section we first give a definition of the problem we are considering, we also give the mathematical formulation of the problem. After that, we give some properties of optimal solutions, which we can use when we are constructing an optimal schedule for charging the EV.

2.1 Problem definition

We consider a problem where an EV has to be charged. We want to find an optimal schedule for charging this EV, within a given time frame and taking given steering signals into account. We have two types of steering signals: a time-varying price and a target profile that gives the desired amount of energy charged into the EV for each time interval.

Mathematically we formulate the problem as follows. We have a set of time intervals of equal length $\mathcal{T} = \{1, \dots, T\}$ and for each time interval $t \in \mathcal{T}$, we have two steering signals: a price a_t per unit of power consumption and a desired amount to be charged, p_t . Furthermore, we add a positive weight factor β between the consumption cost and the squared deviation from the desired power consumption. In the basic problem introduced in [1] the consumption per interval has to be positive and cannot exceed a given maximum charging power X_{\max} . The amount of energy the vehicle requires over the whole planning period, i.e. the depth of discharge of the battery at the moment the vehicle arrives, is denoted by C . If we denote a schedule for this problem by a vector $x \in \mathbb{R}^T$, where x_t denotes the amount of energy charged on a certain time interval $t \in \mathcal{T}$, we can formalize the problem as follows:

$$\begin{aligned} \min \quad & f(x) = \sum_{t=1}^T a_t x_t + \beta(x_t - p_t)^2 \\ \text{s.t.} \quad & 0 \leq x_t \leq X_{\max} \quad 1 \leq t \leq T \\ & \sum_{t=1}^T x_t = C \end{aligned} \tag{1}$$

We call the problem formulated in (1) the basic charging problem (BCP).

In this thesis we consider a variant of the BCP where, if some energy is charged on a certain time interval $t \in \mathcal{T}$, at least a minimum of X_{\min} has to be charged. So, for every $t \in \mathcal{T}$ we require either $X_{\min} \leq x_t \leq X_{\max}$ or $x_t = 0$. To model this, we add extra variables y_t for every time interval $t \in \mathcal{T}$, which denote whether the EV is charging on this time interval or not. So $y_t \in \{0, 1\}$ for all $t \in \mathcal{T}$, such that $y_t = 0$ if $x_t = 0$ and $y_t = 1$ if $x_t > 0$. We can replace the constraints $0 \leq x_t \leq X_{\max}$ in the basic charging problem by $y_t X_{\min} \leq x_t \leq y_t X_{\max}$. The resulting formulation of this variant of BCP is as follows:

$$\begin{aligned} \min \quad & f(x) = \sum_{t=1}^T a_t x_t + \beta(x_t - p_t)^2 \\ \text{s.t.} \quad & y_t X_{\min} \leq x_t \leq y_t X_{\max} \quad 1 \leq t \leq T \\ & \sum_{t=1}^T x_t = C \\ & y_t \in \{0, 1\} \quad 1 \leq t \leq T \end{aligned} \tag{2}$$

This variant of BCP, formulated in (2), is called the charging problem with lower bounds (CPLB). The CPLB is a variant of resource allocation problems. A general introduction on resource allocation problems is given in [4] and [5].

2.2 Properties of solutions

To characterize optimal solutions of BCP and CPLB, we define the vector $\alpha \in \mathbb{R}^T$, with $\alpha_t = \frac{a_t}{2\beta} - p_t$. If we order the time intervals $1, \dots, T$, such that α is non-decreasing. It is proven in [1] that there exists an optimal solution x^* which is non-increasing with respect to this ordering. In this proof, the constraints $0 \leq x_t \leq X_{\max}$ for $t \in \mathcal{T}$, are not used, so this holds for both BCPs and CPLBs. Therefore, in this paper we only consider these non-increasing solutions.

As a consequence, for every ordered (optimal) solution we can determine three time intervals $0 \leq l \leq m \leq n \leq T$, such that:

$$\begin{cases} x_t = X_{\max} & \text{if } 1 \leq t \leq l; \\ X_{\min} < x_t < X_{\max} & \text{if } l < t \leq m; \\ x_t = X_{\min} & \text{if } m < t \leq n; \\ x_t = 0 & \text{if } n < t. \end{cases} \quad (3)$$

For a non-increasing solution of a CPLB to be optimal, all the time intervals $l < t, s \leq m$ have to satisfy the following balance equation:

$$x_t + \alpha_t = x_s + \alpha_s \quad (4)$$

I.e., in an optimal solution x , equation (4) holds for all time intervals $t, s \in \mathcal{T}$, with $X_{\min} < x_t, x_s < X_{\max}$. Again, in [1] this is only proven for BCPs. However, this proof too, does not use the constraints $0 \leq x_t \leq X_{\max}$ for $t \in \mathcal{T}$, so this also holds for CPLBs.

To derive a further property of optimal solution of BCPs an CPLBs we use a specific property of the objective function $f(x)$, namely that $f(x)$ is separable, i.e. $f(x) = \sum_{t=1}^T f_t(x_t)$ with $f_t(x_t) = a_t x_t + \beta(x_t - p_t)^2$. Let x be a solution of a BCP or a CPLB and suppose we add δ extra charging to a certain time interval $t \in \mathcal{T}$. Because of the separability of the objective value we can easily calculate the change in the objective value, which we denote by $d_t(x_t, \delta)$.

$$\begin{aligned} d_t(x_t, \delta) &= f(x_t + \delta) - f_t(x_t) \\ &= a_t(x_t + \delta) + \beta(x_t + \delta - p_t)^2 - a_t x_t - \beta(x_t - p_t)^2 \\ &= 2\beta\delta(x_t + \alpha_t + \frac{\delta}{2}). \end{aligned} \quad (5)$$

Now we can derive another property for optimal non-increasing solutions of CPLBs.

Lemma 2.1. *Let x be an optimal solution of a CPLB with time intervals $0 \leq l \leq m \leq n$ as in equation (3). Then all time intervals $1 \leq s \leq t \leq n$ satisfy the equation:*

$$x_s + (\alpha_t - \alpha_s) \leq x_t. \quad (6)$$

Proof. We distinguish between two cases. First we assume $x_s = X_{\min}$, because x is non-increasing and α is non-decreasing we know $x_t = X_{\min}$ and $\alpha_t \geq \alpha_s$. So $(\alpha_t - \alpha_s) \geq 0$ and thus $x_s - (\alpha_t - \alpha_s) \leq X_{\min} = x_t$.

In the second case we assume $x_s > X_{\min}$ and also assume $x_s - (\alpha_t - \alpha_s) > x_t$. Set $\delta := x_s - (\alpha_t - \alpha_s) - x_t$, then $\delta > 0$. Because $x_s \leq X_{\max}$ and $x_t \geq X_{\min}$ we get $X_{\min} < x_t + \frac{\delta}{2} = x_s - \frac{\delta}{2} < X_{\max}$. So if we add $\frac{\delta}{2}$ extra charging to time interval t and $-\frac{\delta}{2}$ to time interval s , we still have a feasible solution and the total amount charged on all time intervals together

does not change. With equation (5) we can calculate the change of the objective value:

$$\begin{aligned}\frac{d_s(x_s, -\frac{\delta}{2}) + d_t(x_t, \frac{\delta}{2})}{2\beta} &= -\frac{\delta}{2}(x_s + \alpha_s - \frac{\delta}{4}) + \frac{\delta}{2}(x_t + \alpha_t + \frac{\delta}{4}) \\ &= \frac{\delta}{2}(x_t - x_s + (\alpha_t - \alpha_s) + \frac{\delta}{2}) \\ &= \frac{\delta}{2}(-\frac{\delta}{2}) \\ &< 0.\end{aligned}$$

So, the change of the charging on time intervals t and s gives us a feasible solution with a smaller objective value. This contradicts with the assumption that x is an optimal solution. \square

3 General case

In this section we consider CPLBs and we construct optimal solutions for these problems. We start with the easiest case without any bounds on the amount that can or has to be charged on each time interval, i.e. we set $X_{\min} = 0$ and $X_{\max} = \infty$. After that in Subsection 3.2 we add a lower bound, i.e. we consider arbitrary lower bounds X_{\min} . Finally in Subsection 3.3 we also add the upper bound and thus consider arbitrary CPLBs.

Throughout this whole section we assume the time intervals are ordered such that the sequence α is non-decreasing, i.e. $\alpha_t \leq \alpha_s$ for all $1 \leq t \leq s \leq T$. Because of this, by the non-increasing property, we know that there is a non-increasing optimal solution $x \in \mathbb{R}^T$, i.e. $x_t \geq x_s$ for all $1 \leq t \leq s \leq T$. Our goal is to derive a method which finds such a non-increasing optimal solution for a CPLB. For this, we first introduce the notion of partial (optimal) solutions.

Definition 3.1. *Let P be a CPLB with T time intervals and let C be the required amount of energy. For every $x \in \mathbb{R}^T$, we define $C(x) := \sum_{t=1}^T x_t$ and we say x is a partial solution for P , with respect to $C(x)$, if x is a solution for the CPLB P' where C is replaced by $C(x)$ and all the other parameters are the same as in P .*

Furthermore, we say x is a partial optimal solution, if there is no other partial solution x' with $C(x') = C(x)$ such that $f(x') < f(x)$.

From now on we only look at non-increasing (partial) solutions. So if we are talking about a (partial) solution, we mean a non-increasing (partial) solution.

Definition 3.2. *For a (partial) solution x , a time interval t is called active when $x_t > 0$, and inactive when $x_t = 0$. We write x^n for a (partial) solution with n active time intervals. Note that for a (partial) solution x^n to be optimal, exactly the first n intervals need to be active.*

To find an optimal solution for a given CPLB we proceed in an iterative way. Assume that we already have a partial optimal solution $x = x^n$. Now we want to determine a border B for $C(x)$, such that when we cross this border it is cheaper to activate the next time interval (i.e. charge a positive amount on the next time interval). More precisely, B is determined such that if $C > B$, there is no partial optimal solution x with $C(x) = C$ on less than $n + 1$ active time intervals.

To determine the border we have to distinguish between different cases that can occur. For all the cases we use some figures to clarify the situation. In the figures there is a bar for every time interval t that begins at height α_t . Each bar can be filled up to a certain height, which represents the amount charged on that time interval. Because all bars begin at height α_t , the bars corresponding to all the time intervals t with $X_{\min} < x_t < X_{\max}$, have to be filled up to the same height, to satisfy the balance equation (4). Because of Lemma 2.1, in an optimal solution, no bar corresponding to a time interval t can be filled higher than a bar corresponding to a time interval $s > t$. Furthermore, we use blue dotted lines and red dotted lines to indicate the bounds X_{\min} and X_{\max} , respectively.

3.1 The case without bounds

We start with the easiest case, where there are no bounds on the amount that has to be (or can be) charged on the time intervals, i.e. set $X_{\min} = 0$ and $X_{\max} = \infty$. We know all active time intervals in an optimal (partial) solution have to satisfy the balance equation (4) and equation (6). In this case we obviously have to activate the next interval when $x_t = (\alpha_{n+1} - \alpha_t)$, $\forall 1 \leq t \leq n$. So the border B for $C(x)$ is $\sum_{t=1}^n (\alpha_{n+1} - \alpha_t)$. If x^n is the

partial optimal solution with $C(x) = B$, see Figure 1a, and we want to charge $\delta > 0$ extra, we have to activate time interval $n + 1$ and we have to add $\frac{\delta}{n+1}$ to all time intervals $1, \dots, n + 1$ to make sure the new (partial) solution still satisfies the balance equation (i.e. to keep the (partial) solution optimal). This situation is shown in Figure 1b.

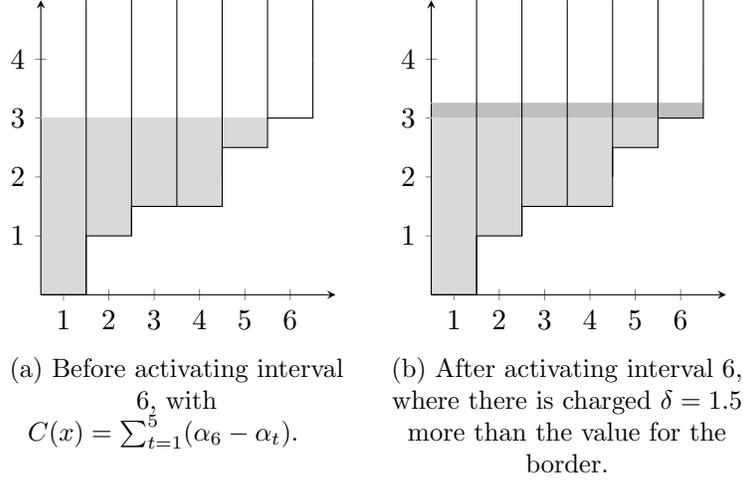


Figure 1: Activate a new interval in the case without bounds. The darker gray area represents δ .

3.2 The case with a lower bound

In this subsection we consider the problem, where only a lower bound X_{\min} on the amount that has to be charged on all the active time intervals is added, i.e. we consider CPLBs with arbitrary X_{\min} and $X_{\max} = \infty$. Again we assume we already have a partial optimal solution $x = x^n$ and our goal is to find the border for $C(x)$, from which it is cheaper to activate time interval $n + 1$, so from which no (partial) solution x^n can be optimal.

The problem with the lower bound X_{\min} is that if we activate a new time interval, we directly have to add an amount of X_{\min} . Thus, activating interval $n + 1$ now means that we have to lower $\sum_{t=1}^n x_t$ with a total amount of X_{\min} , so that we can add this to x_{n+1} . However, it can be that there are active time intervals t with $x_t = X_{\min}$. Lowering in these time intervals can only be done by decreasing the charging directly with X_{\min} , which causes the obtained (partial) solution to be no longer non-increasing and hence it cannot be optimal anymore.

Definition 3.3. For every (partial) solution $x = x^n$, we define the time interval $n'_x \leq n$ as the time interval such that:

$$\begin{cases} x_t > X_{\min} & \text{if } 1 \leq t \leq n'_x, \\ x_t = X_{\min} & \text{if } n'_x < t \leq n. \end{cases} \quad (7)$$

When we change the charging on a certain time interval t with δ , we can calculate $d_t(x_t, \delta)$, the change of the objective value, with equation (5). For a given (partial) solution x we denote by $\delta \in \mathbb{R}^T$ a vector, where δ_t gives the amount of charging added to time interval t . Let x and δ be given, then we define $\Delta(x, \delta) := \sum_{t=1}^T d_t(x_t, \delta_t)$ to be the total change of the objective value. To activate the next time interval we define a special activating vector.

Definition 3.4. Let $x = x^n$ be a (partial) solution. To activate time interval $n + 1$ we define an activating vector $\delta \in \mathbb{R}^T$, such that:

- $\sum_{t=1}^n \delta_t = -X_{\min}$, i.e. we lower the time intervals $1, \dots, n$ with a total of X_{\min} ;
- $\delta_{n+1} = X_{\min}$, i.e. we activate the next time interval $n + 1$ with a minimal amount;
- $\delta_t = 0$ for all $n + 1 < t \leq T$.

Note that these three conditions imply, that the total amount charged is not changed.

3.2.1 Lower the amount charged in time intervals equally

First we consider the case that we can activate the next time interval by lower the amount charged on some time intervals with an equal amount. We determine the border from which it is optimal to activate the next time interval as well as the time intervals in which the amount charged has to be lowered, in order to activate the next time interval.

Let $n' = n'_x$, meaning that we can only lower on time intervals $1 \leq t \leq n'$. The easiest way to activate time interval $n + 1$, is when we lower all time intervals $1, \dots, n'$ equally, i.e. when we use activating vector δ with:

$$\delta_t = \begin{cases} -\frac{X_{\min}}{n'} & \text{if } 1 \leq t \leq n', \\ X_{\min} & \text{if } t = n + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

This situation is shown in Figure 2, where we added a blue dotted line to indicate the bound X_{\min} .

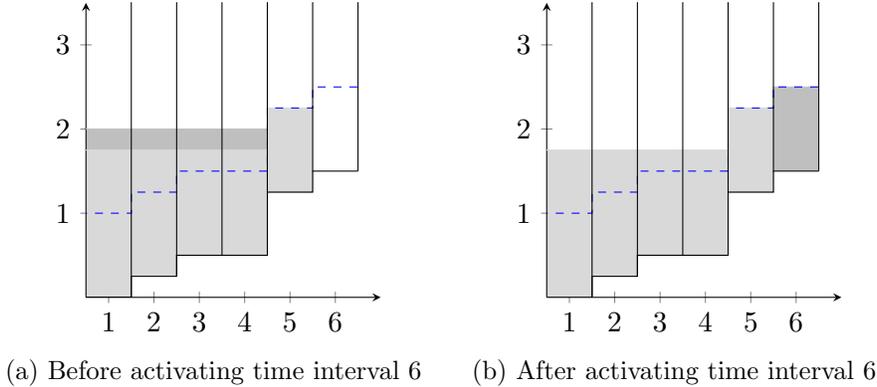


Figure 2: Activate a new interval, when lowering all time intervals $1 \leq t \leq n' = 4$ equally.

Remark 3.1. Let $x = x^n$ be an optimal (partial) solution for some $n \in \mathcal{T}$. Suppose we charge a bit more on the first n time intervals, while maintaining a (partial) solution that satisfies the balance equation and equation (6). This (partial) solution x is still optimal until $C(x)$ crosses some border C for $C(x)$, from which it is optimal to activate the next time interval. Note that, we do not consider activating other time intervals, because we are only considering non-increasing (optimal) solutions.

We determine this border C , such that, for the (partial) solution x with $C(x) = C$, there exists an activating vector δ with $\Delta(x, \delta) = 0$ and for which $x' = x + \delta$ is feasible. Then, for the (partial) solution x' we have $f(x') = f(x) + \Delta(x, \delta) = f(x)$, so this (partial) solution is optimal too. Because optimal (partial) solutions have to satisfy the balance equation and equation (6), we only need to consider activating vectors δ , such that the feasible (partial) solution $x' = x + \delta$ satisfies these equations.

We wish to find an optimal (partial) solution x^n , such that if we activate time interval $n + 1$ using activating vector δ as in (8), the (partial) solution after activating time interval $n + 1$ is still optimal. So we want $\Delta(x^n, \delta) = 0$ and we want $x' = x + \delta$ to satisfy the balance equation and (6). Note that if we find such a (partial) solution x^n , then $C(x^n)$ is the border, from which we activate time interval $n + 1$.

Lemma 3.1. *Let $x = x^n$ be a (partial) solution that satisfies the balance equation and let δ be the activating vector as in (8) and let $n' := n'_x$. Then $\Delta(x, \delta) = 0$ if and only if $C(x) = \frac{(n'+1)X_{\min}}{2} + \sum_{i=1}^{n'}(\alpha_{n+1} - \alpha_i) + (n - n')X_{\min}$.*

Proof. We calculate the total change of the objective value using equation (5).

$$\begin{aligned} \frac{\Delta(x, \delta)}{2\beta} &= \frac{\sum_{t=1}^{n'} d_t(x_t, -\frac{X_{\min}}{n'}) + d_{n+1}(0, X_{\min})}{2\beta} \\ &= -\sum_{t=1}^{n'} \frac{X_{\min}}{n'} \left(x_t + \alpha_t - \frac{X_{\min}}{2n'} \right) + X_{\min} \left(\alpha_{n+1} + \frac{X_{\min}}{2} \right) \\ &= \frac{X_{\min}}{n'} \left(n' \alpha_{n+1} - \sum_{t=1}^{n'} \alpha_t + \frac{(n'+1)X_{\min}}{2} - \sum_{t=1}^{n'} x_t \right) \\ &= \frac{X_{\min}}{n'} \left(\sum_{t=1}^{n'} (\alpha_{n+1} - \alpha_t) + \frac{(n'+1)X_{\min}}{2} - (C(x) - (n - n')X_{\min}) \right) \end{aligned}$$

Now $\Delta(x, \delta) = 0$ if and only if $\sum_{t=1}^{n'} (\alpha_{n+1} - \alpha_t) + \frac{(n'+1)X_{\min}}{2} - (C(x) - (n - n')X_{\min}) = 0$, i.e. if and only if

$$C(x) = \sum_{t=1}^{n'} (\alpha_{n+1} - \alpha_t) + \frac{(n'+1)X_{\min}}{2} + (n - n')X_{\min} \quad (9)$$

Note that, because x has to satisfy the balance equation, $x_t = (\alpha_{n+1} - \alpha_t) + \frac{(n'+1)X_{\min}}{2n'}$ for $1 \leq t \leq n'$. \square

With Lemma 3.1 we can determine the border for $C(x)$, such that activating the next time interval, while lowering the charging done in the time intervals $1, \dots, n'_x$ all equally, does not change the objective value. In the next step, we determine some properties such that we have an optimal (partial) solution x satisfying (9) and such that the solution $x' = x + \delta$ after activating the next time interval, with δ as in (8), is feasible and optimal.

If x^n is a non-increasing solution with $n'_x = n'$ satisfying (9) and the balance equation, it can be that $x_t > x_{n'+1} + (\alpha_{n'+1} - \alpha_t)$ for some time interval $1 \leq t \leq n'$, as shown in Figure 3a. However, this solution can not be optimal by Lemma 2.1. It thus remains to determine a time interval n' where this does not happen, i.e. such that there is a (partial) solution x^n with $n'_x = n'$, $x_t = (\alpha_{n+1} - \alpha_t) + \frac{(n'+1)X_{\min}}{2n'}$ and $x_t \leq x_{n'+1} + (\alpha_{n'+1} - \alpha_t)$ for all $1 \leq t \leq n'$. Because for all $1 \leq t \leq n'$ we have

$$x_t > x_{n'+1} + (\alpha_{n'+1} - \alpha_t) \iff (\alpha_{n+1} - \alpha_{n'+1}) > \left(1 - \frac{n'+1}{2n}\right) X_{\min},$$

we wish to find the first time interval $1 \leq n' \leq n$ such that $(\alpha_{n+1} - \alpha_{n'+1}) \leq \left(1 - \frac{n'+1}{2n}\right) X_{\min}$. If there's no such time interval, we can take $n' = n$. Summarized, we are looking for a time interval with the following property:

$$n' := \min \left(\left\{ 1 \leq t \leq n \mid (\alpha_{n+1} - \alpha_{n'+1}) \leq \left(1 - \frac{n'+1}{2n}\right) X_{\min} \right\} \cup \{n\} \right). \quad (10)$$

Another problem that can occur is that, when we lower the amounts charged in the time intervals $1, \dots, n'$ all equally with $\frac{X_{\min}}{n'}$, some time intervals might come below X_{\min} . This situation is shown in Figure 3b. Again, we wish to find a property, such that this does not happen.

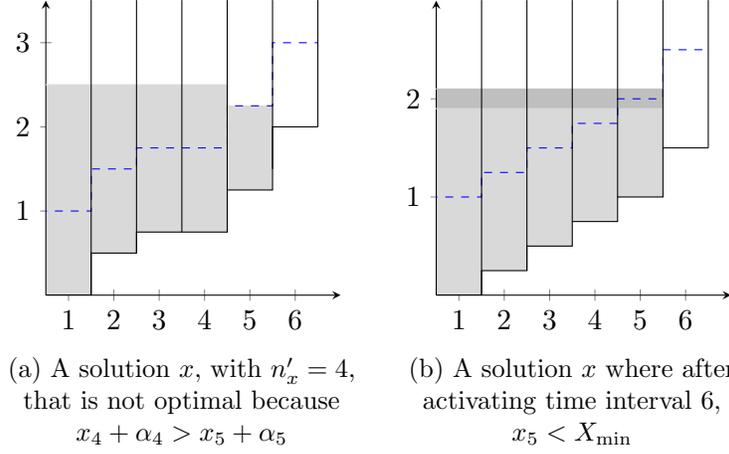


Figure 3: Situations where activating the next time interval using Lemma 3.1 does not give us an optimal solution

Lemma 3.2. *Let $x = x^n$ be an optimal (partial) solution satisfying equation (9), let δ be the activating vector as in (8) and assume $n'_x = n'$, with n' as in equation (10). We can lower all time intervals $1, \dots, n'$ equally, i.e. $x' = x + \delta$ is feasible, if and only if $(\alpha_{n+1} - \alpha_{n'}) \geq \frac{n'+1}{2n'} X_{\min}$.*

Proof. We need to make sure that when we lower all time intervals $1, \dots, n'$ equally with $\frac{X_{\min}}{n'}$, they do not come below X_{\min} . This is the case if and only if $x_t - \frac{X_{\min}}{n'} \geq X_{\min}$ for all $1 \leq t \leq n'$. We also know that $x_t \geq x_s$ for all $t \leq s$, so $x_{n'} - \frac{X_{\min}}{n'} \leq x_t - \frac{X_{\min}}{n'}$ for all $1 \leq t \leq n'$. For the (partial) solution x^n we know $x_t = (\alpha_{n+1} - \alpha_t) + \frac{n'+1}{2n'} X_{\min}$ for $1 \leq t \leq n'$. So we can lower all time intervals $1, \dots, n'$ equally if and only if

$$x'_{n'} - \frac{X_{\min}}{n'} = (\alpha_{n+1} - \alpha_{n'}) + \frac{n'+1}{2n'} X_{\min} - \frac{X_{\min}}{n'} \geq X_{\min}.$$

Note that, because we lower the charging done in all time intervals $1, \dots, n'$ equally, x' still satisfies the balance equation and equation (6), so x' is optimal. \square

3.2.2 Not lowering the amount charged in time intervals equally

Up to now, we know how we can find the border for the charging done in an optimal (partial) solution x , such that the solution after activating the next time interval is also optimal and while lowering all time intervals $1, \dots, n'_x$ equally. In Lemma 3.2 we saw this is only possible when the α 's satisfy some property. Next, we consider the case where the α 's do not satisfy that property and we show how to activate the next time interval in this case.

Let n' be as in equation (10) and assume that $(\alpha_{n+1} - \alpha_{n'}) < \frac{n'+1}{2n'} X_{\min}$. Furthermore, let $x = x^n$ be a solution satisfying (9). In this situation we cannot lower all time intervals $1, \dots, n'$ equally, because we cannot lower time interval n' with more than $x_{n'} - X_{\min} < \frac{X_{\min}}{n'}$. To deal with this situation, for every solution $x = x^n$ satisfying the balance equation, we define $b_x := x_{n'} - X_{\min}$, where $n' = n'_x$. So we can lower each time interval $1 \leq t \leq n'$ with a

maximum of $b + (\alpha_{n'} - \alpha_t)$.

A situation where not all time intervals are equally lowered to activate a new time interval is shown in Figure 4, where the height of the darkest gray area in Figure 4a represents b .

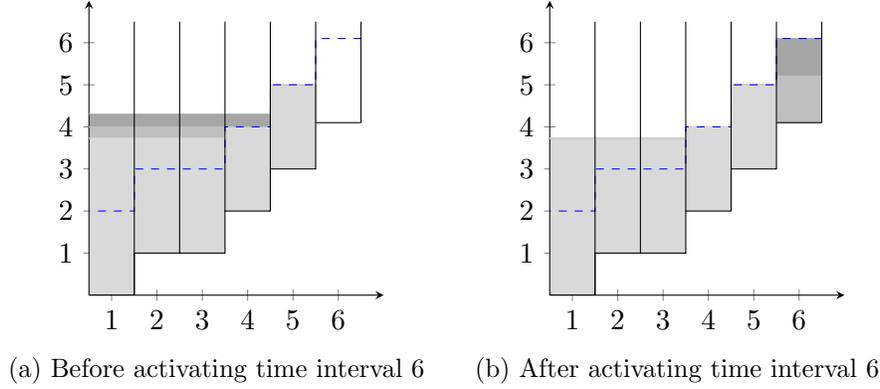


Figure 4: Situation where we cannot lower all time intervals equally to activate a new time interval

Based on the discussion up to now, we can distinguish for the time intervals $1, \dots, n'$ between two types of intervals:

1. The time intervals t , that we have to lower with the maximum amount $b + (\alpha_{n'} - \alpha_t)$;
2. The time intervals that we can lower with less than the maximum amount. Since after activating the next time interval these time intervals have to satisfy the balance equation, we have to lower them all equally.

The aim is to achieve after activating time interval $n + 1$ a (partial) solution x' , which is optimal. So, from Lemma 2.1, we know that we have to determine a time interval $1 \leq m \leq n$, so that all time intervals $m+1, \dots, n'$ are of the first type and the time intervals $1, \dots, m$ are of the second type. We lower all the time intervals of the second type with $\frac{X_{\min} - \sum_{t=m+1}^{n'} b + (\alpha_{n'} - \alpha_t)}{m}$, in order to get a total lowering of X_{\min} .

For x' to satisfy equation (6), we also need to make sure that $\frac{X_{\min} - \sum_{t=m+1}^{n'} b + (\alpha_{n'} - \alpha_t)}{m} \geq b + (\alpha'_n - \alpha_{m+1})$, i.e. we have to lower each time interval of the second type more than we lower each time interval of the first type. So, for a (partial) solution $x = x^n$ we define the time interval m_x , such that all time intervals $1, \dots, m_x$ are of the second type and are lowered more than all time intervals $m_x + 1, \dots, n'$, that are of the first type.

Definition 3.5. Let x be an optimal (partial) solution with n active time intervals and let $n' = n'_x$. We define

$$m_x := \max \left\{ 1 \leq s \leq n' \mid \frac{X_{\min} - \sum_{t=s+1}^{n'} b + (\alpha_{n'} - \alpha_t)}{s} < x_s - X_{\min} \right\}, \quad (11)$$

to be the last time interval which we don't have to lower with the maximal amount $b + (\alpha_{n'} - \alpha_{m_x})$, i.e. m_x is the last time interval of the second type.

For an optimal (partial) solution x^n we define the activating vector $\delta \in \mathbb{R}^T$ as follows:

$$\delta_t = \begin{cases} -\frac{X_{\min} - \sum_{t=m+1}^{n'} (b + (\alpha_{n'} - \alpha_t))}{m} & \text{if } 1 \leq t \leq m_x, \\ -b - (\alpha_{n'} - \alpha_t) & \text{if } m_x < t \leq n', \\ X_{\min} & \text{if } t = n + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

For a given time interval n , we want to find the time interval m^* , for which we can find an optimal (partial) solution $x = x^n$ with $m_x = m^*$ and such that it is rewardable to activate the next time interval $n + 1$, i.e. such that $\Delta(x, \delta) = 0$ (with δ as in equation (12)). For this, we first determine the boundaries between which b must lie, such that a non-increasing (partial) solution $x = x^n$, with $b_x = b$, satisfies $m_x = m$ and $n'_x = n'$, for some time intervals $1 \leq m \leq n' \leq n \leq n$.

Lemma 3.3. *Let $1 \leq m \leq n' \leq n \leq T$ be three time intervals and $x = x^n$ a non-increasing solution satisfying the balance equation, with $n'_x = n'$ and define $b := b_x$. Now we have $m_x = m$ if and only if*

$$\frac{X_{\min} - m(\alpha_{n'} - \alpha_m) - \sum_{t=m+1}^{n'} (\alpha_{n'} - \alpha_t)}{n'} < b \leq \frac{X_{\min} - m(\alpha_{n'} - \alpha_{m+1}) - \sum_{t=m+1}^{n'} (\alpha_{n'} - \alpha_t)}{n'}.$$

Proof. By the definition of m_x we have $m_x = m$ if and only if

$$\frac{X_{\min} - \sum_{t=m+1}^{n'} (b + (\alpha_{n'} - \alpha_t))}{m} < x_m - X_{\min}; \quad (13)$$

$$\frac{X_{\min} - \sum_{t=m+2}^{n'} (b + (\alpha_{n'} - \alpha_t))}{m+1} \geq x_{m+1} - X_{\min}. \quad (14)$$

By the definition of b , we can write $x_m - X_{\min} = b + (\alpha_{n'} - \alpha_m)$ and $x_{m+1} - X_{\min} = b + (\alpha_{n'} - \alpha_{m+1})$. First, we take a look at equation (13):

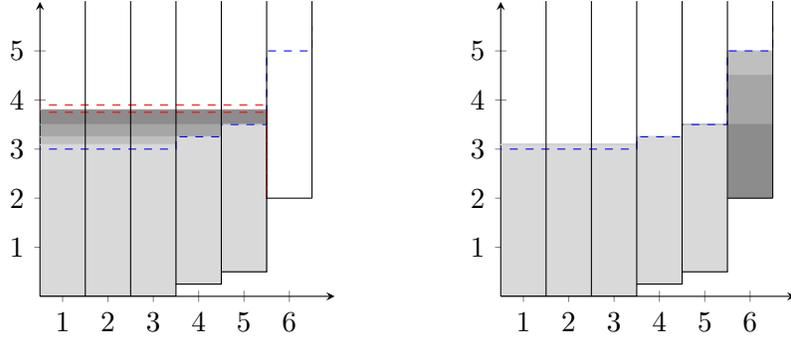
$$\begin{aligned} & \frac{X_{\min} - \sum_{t=m+1}^{n'} (b + (\alpha_{n'} - \alpha_t))}{m} < b + (\alpha_{n'} - \alpha_m) \quad \left(\text{add } \frac{(n'-m)b}{m} - (\alpha_{n'} - \alpha_m) \right) \\ \Leftrightarrow & \frac{X_{\min} - m(\alpha_{n'} - \alpha_m) - \sum_{t=m+1}^{n'} (\alpha_{n'} - \alpha_t)}{m} < b + \frac{(n'-m)b}{m} \quad \left(\text{multiply by } \frac{m}{n'} \right) \\ \Leftrightarrow & \frac{X_{\min} - m(\alpha_{n'} - \alpha_m) - \sum_{t=m+1}^{n'} (\alpha_{n'} - \alpha_t)}{n'} < b. \end{aligned}$$

Next, we take a look at (14):

$$\begin{aligned} & \frac{X_{\min} - \sum_{t=m+2}^{n'} (b + (\alpha_{n'} - \alpha_t))}{m+1} \geq b + (\alpha_{n'} - \alpha_{m+1}) \quad \left(\text{add } \frac{(n'-m-1)b}{m+1} - (\alpha_{n'} - \alpha_{m+1}) \right) \\ & \frac{X_{\min} - (m+1)(\alpha_{n'} - \alpha_{m+1}) - \sum_{t=m+2}^{n'} (\alpha_{n'} - \alpha_t)}{m+1} \geq b + \frac{(n'-m-1)b}{m+1} \quad \left(\text{multiply by } \frac{m+1}{n'} \right) \\ & \frac{X_{\min} - m(\alpha_{n'} - \alpha_{m+1}) - \sum_{t=m+1}^{n'} (\alpha_{n'} - \alpha_t)}{n'} \geq b. \end{aligned}$$

These two considerations prove the Lemma. \square

In Figure 5, we determined the bounds on b for $m = 3$ and $n' = n = 5$, in this case the solution (given by the gray area) has to be between the red dotted lines. The darkest gray area represents b .



(a) Before activating time interval 6 (b) After activating time interval 6

Figure 5: Determine the bounds on b for a given time interval m

For every fixed pair of time intervals n' and m we can now determine the bounds between which b must lie, such that x is a non-increasing solution satisfying the balance equation with $b_x = b$, $n'_x = n'$ and $m_x = m$. Next, we use these bounds to show how we can find the time intervals n' and m^* , such that there exists an optimal solution x with $n'_x = n'$, $m_x = m^*$ and $\Delta(x, \delta) = 0$, with δ as in (12).

For some fixed n' we calculate the bounds from Lemma 3.3 for every $1 \leq m \leq n'$ and we use the upper bound

$$UB_m := \frac{X_{\min} - m(\alpha_{n'} - \alpha_{m+1}) - \sum_{t=m+1}^{n'} (\alpha_{n'} - \alpha_t)}{m} \quad (15)$$

to define a solution $x(UB_m)$.

Definition 3.6. For every time interval $1 \leq m \leq n' \leq n$ calculate UB_m as in (15). We define the (partial) solution $x(UB_m)$, with

$$x(UB_m)_t = \begin{cases} UB_m + (\alpha_{n'} - \alpha_t) & \text{if } 1 \leq t \leq n', \\ X_{\min} & \text{if } n' < t \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed n' we may get the situation that

$$x(UB_m)_{n'} = X_{\min} + UB_m > x_{n'+1} + (\alpha_{n'+1} - \alpha_{n'}) = X_{\min} + (\alpha_{n'+1} - \alpha_{n'}).$$

This is not an optimal solution by Lemma 2.1. So we set $n' = n' + 1$ and recalculate UB_m with the new value of n' . We iteratively repeat this until we have found an upperbound UB_m such that $UB_m \leq (\alpha_{n'} - \alpha_n)$. This situation is shown in Figure 6.

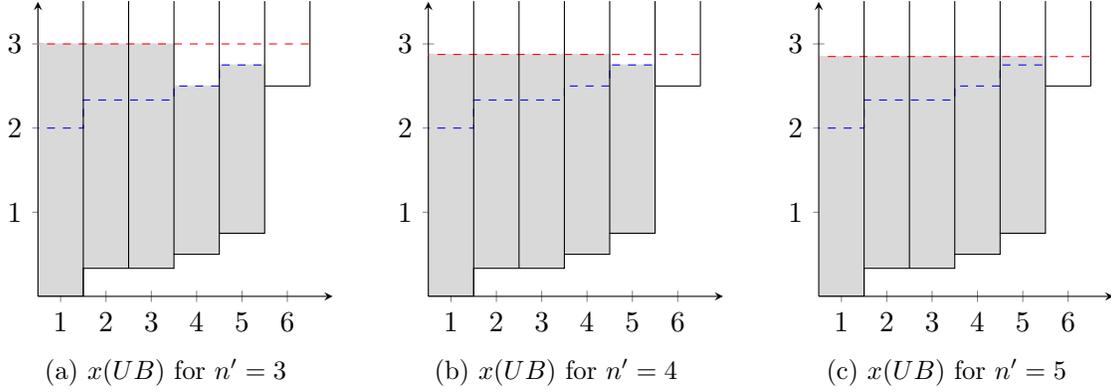


Figure 6: find the time interval n' for $m = 3$

To find the time interval m^* , such that there is an optimal (partial) solution x with $m_x = m^*$ and for which it is optimal to activate the next time interval, we define some steps. Iteratively repeating these steps will give us this time interval m^* .

Let $x = x^n$ be the optimal (partial) solution we get after activating time interval n and set $m = m_x$ and $n' = n'_x$. Because of the ordering of the time intervals, we know $m^* \geq m$. To find m^* we iteratively repeat the following two steps.

1. Repeat calculating the upper bound UB_m on b for m with Lemma 3.3, while updating n' , until we find an upper bound UB_m such that $UB_m \leq (\alpha_{n'+1} - \alpha_{n'})$.
2. Set $x = x(UB_m)$ and the activating vector δ as in equation (12). Then calculate the difference in cost $\Delta(x, \delta)$. If $\Delta(x, \delta) > 0$, set $m = m + 1$ and go back to step 1, else we set $m^* = m$ and we are done.

Let UB_{m^*-1} and UB_{m^*} be the upperbounds on b , found in the end of step 1. We know that, if $\Delta(x(UB_{m^*-1}), \delta) > 0$ and $\Delta(x(UB_{m^*}), \delta) \leq 0$, then $\Delta(x, \delta) = 0$ for an optimal (partial) solution $x = x^n$ with b_x between UB_{m^*-1} and UB_{m^*} .

The only thing which remains, is to find the time interval n' , such that we have an optimal (partial) solution $x = x^n$ with $m_x = m^*$, $n'_x = n'$, an activating vector δ as in equation 12 and $\Delta(x, \delta) = 0$.

In the end of step 1, which we used for finding the time interval m^* , we found a time interval n'_1 that belongs to m^* and a time interval n'_2 belonging to $m^* - 1$. These two time intervals don not necessarily have to be the same, see Figure 7. So, we know that the time interval n' , such that we have an optimal (partial) solution x with $n'_x = n'$ and for which it is optimal to activate the next time interval, has to be between n'_1 and n'_2 . We wish to find this time interval n' .

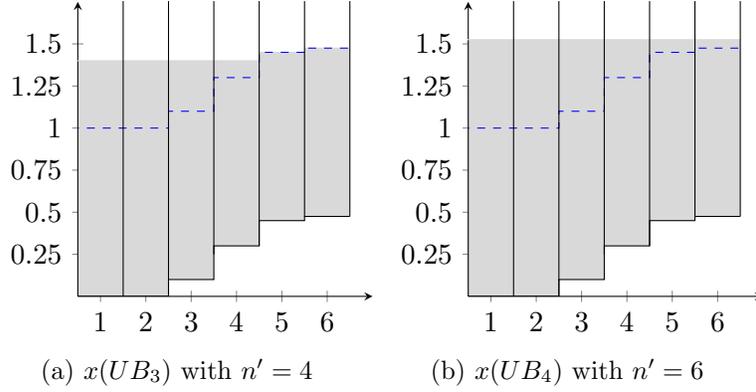


Figure 7: Different n' for two consecutive intervals m

We can find this n' in about the same way as we have found m^* . Therefore, we define the (partial) solution $x(n')$, for every $n'_1 \leq n' \leq n'_2$, with

$$x(n')_t = \begin{cases} X_{\min} + (\alpha_{n'+1} - \alpha_t) & \text{if } 1 \leq t \leq n', \\ X_{\min} & \text{if } n' < t \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Our goal is to find the smallest n' such that $\Delta(x(n'), \delta) \leq 0$.

Now we know the time intervals m^* and n' and the bounds between which b must lie, such that there is an optimal (partial) solution $x = x^n$ with $b_x = b$, $m_x = m^*$, $n'_x = n'$ and $\Delta(x, \delta) = 0$. The concrete value for b which we have to use, is specified in the following Lemma 3.4.

Lemma 3.4. *Let $1 \leq m \leq n' \leq n$ be time intervals such that there is an optimal (partial) solution $x = x^n$ with $m_x = m$ and $n'_x = n'$. Let $Y = \sum_{j=m+1}^n (\alpha_n - \alpha_j)$ and $z_i = (\alpha_n - \alpha_i)$. Then $\Delta(x, \delta) = 0$ if and only if $b = b_x = x_{n'} - X_{\min} = \frac{X_{\min} - Y}{(n-m)} - \sqrt{D}$ with*

$$D = \frac{m}{n} \left(\frac{X_{\min} - Y}{(n-m)} \right)^2 + \frac{m(n-m)X_{\min}^2}{n(n-m)^2} - \frac{2m(\alpha_{n+1} - \alpha_n)X_{\min} + m \sum_{i=m+1}^n z_i^2}{n(n-m)}$$

Proof. For a (partial) solution x , with given time intervals m and n' , we can calculate the cost difference when we activate time interval $n+1$ using activating vector δ as in equation (12):

$$\begin{aligned} \Delta(x, \delta) &= \sum_{i=1}^m d_i \left(X_{\min} + b + (\alpha_n - \alpha_i), -\frac{X_{\min} - \sum_{j=m+1}^n (b + (\alpha_n - \alpha_j))}{m} \right) \\ &+ \sum_{i=m+1}^n \left(d_i (X_{\min} + b + (\alpha_n - \alpha_i), -b - (\alpha_n - \alpha_i)) \right) + d_{n+1}(0, X_{\min}) \end{aligned}$$

Let $Y := \sum_{j=m+1}^n (\alpha_n - \alpha_j)$ and $z_i := (\alpha_n - \alpha_i)$. We now get

$$\begin{aligned}
\frac{\Delta(x, \delta)}{2\beta} &= \left(\sum_{i=m+1}^n (b + z_i) - X_{\min} \right) \left(X_{\min} + b + \alpha_n + \frac{(n-m)b + Y - X_{\min}}{2m} \right) \\
&\quad - \sum_{i=m+1}^n \left((b + z_i) \left(X_{\min} + b + \alpha_n - \frac{b + z_i}{2} \right) \right) + X_{\min} \left(\alpha_{n+1} + \frac{X_{\min}}{2} \right) \\
&= X_{\min} \left((\alpha_{n+1} - \alpha_n) - \frac{(m-1)X_{\min}}{2m} - \frac{(m+n)b}{2m} - \frac{Y}{2m} \right) \\
&\quad + \sum_{i=m+1}^n (b + z_i) \left(\frac{nb}{2m} - \frac{X_{\min}}{2m} + \frac{Y}{2m} + \frac{z_i}{2} \right) \\
&= \frac{n(n-m)}{2m} b^2 + (Y - X_{\min}) \frac{n}{m} b \\
&\quad + X_{\min} \left((\alpha_{n+1} - \alpha_n) - \frac{Y}{m} - \frac{m-1}{2m} X_{\min} \right) + \frac{Y^2}{2m} + \frac{\sum_{i=m+1}^n z_i^2}{2}
\end{aligned}$$

This gives us a second degree polynomial in b , with zeroes given by $b = \frac{X_{\min} - Y}{(n-m)} \pm \sqrt{D}$, where

$$D = \frac{m}{n} \left(\frac{X_{\min} - Y}{(n-m)} \right)^2 + \frac{m(n-m)X_{\min}^2}{n(n-m)^2} - \frac{2m(\alpha_{n+1} - \alpha_n)X_{\min} + m \sum_{i=m+1}^n z_i^2}{n(n-m)} \quad (17)$$

As we have $X_{\min} \geq Y + (n-m)b$ and thus $b \leq \frac{X_{\min} - Y}{n-m}$, the only zero between the given bounds is $b = \frac{X_{\min} - Y}{(n-m)} - \sqrt{D}$. \square

So, for a CPLB with arbitrary X_{\min} and $X_{\max} = \infty$, we determined in all cases in which time intervals the charging amount has to be lowered to activate the next time interval. Furthermore, we determined exactly the total amount $C(x)$ that has to be charged before it is optimal to activate the next time interval.

3.3 The case with a lower bound and an upper bound

In this subsection we consider the problem where also an upper bound X_{\max} on the amount we can charge on each time interval is given, i.e. we consider the CPLB in its general form. In a somehow symmetric way to Section 3.2 we define a new interval l_x for every non-increasing (partial) solution x to indicate the last time interval that is charging X_{\max} .

Definition 3.7. *Let $x = x^n$ be a non-increasing (partial) solution. We define $0 \leq l_x \leq n$ to be the time interval such that $x_t = X_{\max}$ if and only if $1 \leq t \leq l_x$.*

Let x be an optimal (partial) solution and let $l := l_x$. In the previous subsection we lowered the charging amounts in all intervals up to a certain time interval m with an equal value, when activating a new time interval. If we do the same in this case we can come in a situation that doesn't satisfy the balance equation as in Figure 8a and 8b, where in Figure 8b the first time intervals do not fulfill the balance equation. However, if we don't lower the time intervals $1, \dots, l$ at all we can get a situation where $x_l = X_{\max} \geq x_{l+1} + (\alpha_{l+1} - \alpha_l)$, which is not optimal by Lemma 2.1. Such a situation is given in Figure 8c.

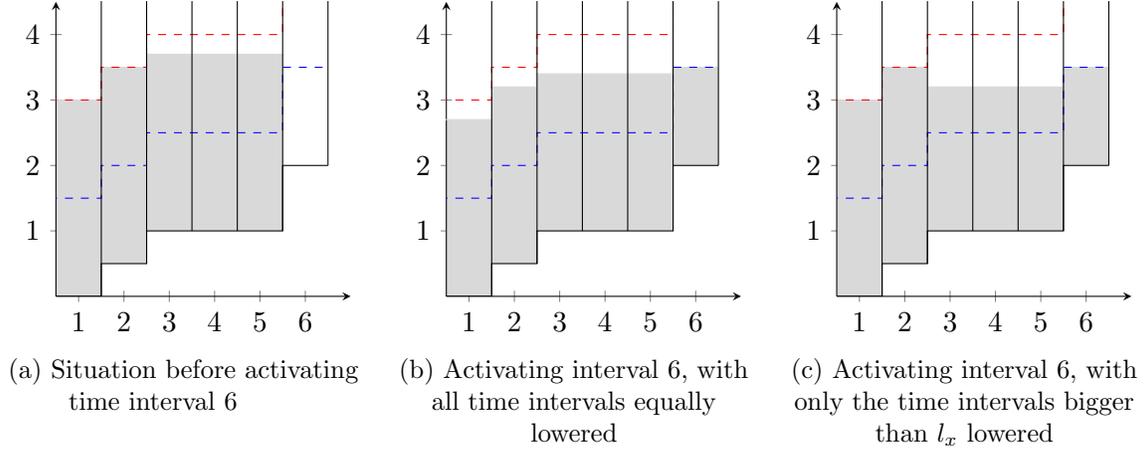


Figure 8: Activating time interval 6, with $l_x = 2$

Again, suppose we already have an optimal partial solution $x = x^n$. We want to find a border for $C(x)$ and an activating vector $\delta \in \mathbb{R}^T$, such that $\Delta(x, \delta) = 0$ and the (partial) solution after activating time interval $n+1$ is still optimal. For this, we define $c_x := x_{l+1} + (\alpha_{l+1} - \alpha_l) - X_{\max}$ for every non-increasing (partial) solution $x = x^n$ satisfying the balance equation and $l = l_x$.

3.3.1 Not taking b into account

For this moment we suppose we can lower the charging in all intervals $l+1, \dots, n'$ equally, meaning that, we do not need to take b into account. We start with lowering the time intervals $l+1, \dots, n'$. If we reach the situation, where $x_{l+1} + (\alpha_{l+1} - \alpha_l) = X_{\max}$, we also have to lower time interval l . Note, that this is the case when we lowered all time intervals $l+1 \dots n'$ with c_x . From this moment on we need to lower all time intervals l, \dots, n' equally until $x_l + (\alpha_l - \alpha_{l-1}) = X_{\max}$, and so on. This situation is shown in Figure 9.

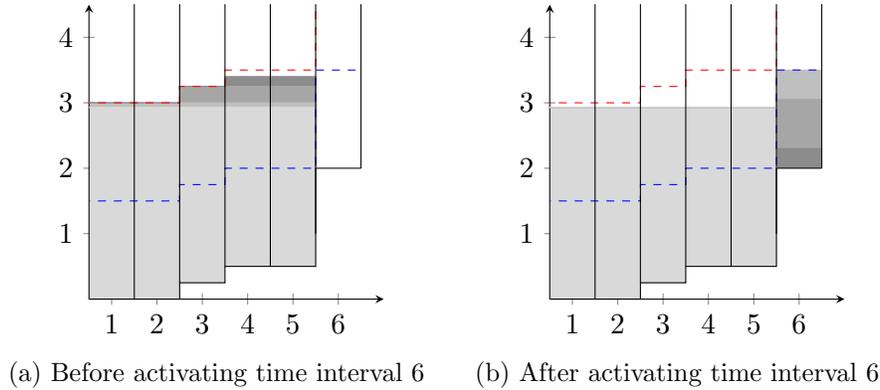


Figure 9: Activating time interval 6, with $l_x = 3$ and $c = 0.15$, denoted by the darkest gray area

It remains to determine on how many time intervals from the set $\{1, \dots, l\}$ we have to lower the charging, until we have lowered the total charging on the time intervals $1, \dots, n'$ enough (i.e. with X_{\min}) to activate time interval $n+1$. Suppose we lower all time intervals starting from a certain time interval $l' \leq n'$. The goal is to reach a (partial) solution after activating the next time interval which satisfies the balance equation and equation (6). So, we can lower interval l' with a maximum of $(\alpha_{l'} - \alpha_{l'-1})$. Suppose we lower time interval l' with δ , then we have to lower time intervals $l'+1 \leq t \leq l$ with $(\alpha_t - \alpha_{l'}) + \delta$ and the time intervals

$l + 1 \leq t \leq n'$ with $(\alpha_l - \alpha_{l'}) + \delta + c$. As we have to lower the time intervals l', \dots, n' with a total of X_{\min} , we define the time interval l' as follows.

Definition 3.8. For a given non-increasing (partial) solution $x = x^n$, satisfying the balance equation, let $c := c_x$ and $l := l_x$. We define

$$l'_x := \min\{1 \leq s \leq l \mid (n-l)c + \sum_{t=s+1}^l (n-t+1)(\alpha_t - \alpha_{s-1}) \leq X_{\min}\}, \quad (18)$$

meaning that l'_x is the first time interval which is lowered to activate the next time interval.

The concrete vector $\delta \in \mathbb{R}^T$ to activate the next time interval, for a (partial) solution $x = x^n$ with $l'_x = l'$, $l_x = l$ and $n'_x = n'$, is given by

$$\delta_t = \begin{cases} -\frac{X_{\min} - ((n-l)(c + (\alpha_l - \alpha_{l'}) + \sum_{s=l'+1}^l (\alpha_s - \alpha_{l'}))}{n-l'+1} - (\alpha_t - \alpha_{l'}) & \text{if } l' \leq t \leq l, \\ -\frac{X_{\min} - ((n-l)(c + (\alpha_l - \alpha_{l'}) + \sum_{s=l'+1}^l (\alpha_s - \alpha_{l'}))}{n-l'+1} - (\alpha_l - \alpha_{l'}) - c & \text{if } l < t \leq n', \\ X_{\min} & \text{if } t = n + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

So for every (partial) solution x^n , $n \geq 1$, we can determine the time intervals l', l and n' and the activating vector δ as in equation (19) and, based on this, we can calculate $\Delta(x^n, \delta)$. Our goal is to find a (partial) solution x^n such that $\Delta(x^n, \delta) = 0$.

Because of the ordering of the time intervals we know that when we have a (partial) solution x^n with $x_t = X_{\min}$, for all $1 \leq t \leq n$, i.e. with $C(x^n) = nX_{\min}$, there is no activating vector $\delta \in \mathbb{R}^T$ such that $\Delta(x^n, \delta) < 0$.

In the case that $C(x^n) = nX_{\max}$, i.e. $x_t = X_{\max}$ for all $1 \leq t \leq n$ and we set δ as in (19), we discuss two options:

1. It is still not rewardable to activate time interval $n + 1$, meaning that $\Delta(x^n, \delta) > 0$. Then, if we still have to charge more, we have to activate the next time interval, because we cannot charge more on the previous time intervals.
2. We have $\Delta(x^n, \delta) \leq 0$. In this case, because we have a continuous objective function, we know that there has to be an optimal (partial) solution x^n with $nX_{\min} \leq C(x^n) \leq nX_{\max}$ such that $\Delta(x^n, \delta) = 0$. This is actually true in all cases, i.e. also in the case where also X_{\min} restricts the activation of the next time interval. However, in this case we may have to use another activating vector δ to make sure that the (partial) solution $x' = x + \delta$ after activating the next time interval is still feasible.

Consider first the situation that we are in the second scenario and we know that there are time intervals $1 \leq l' \leq l \leq n' \leq n$ such that there is a (partial) solution x^n with $l'_x = l'$, $l_x = l$, $n'_x = n'$ and $\Delta(x^n, \delta) = 0$. In this situation, we can determine these time intervals, in precisely the same way as we did in the previous subsection. Using these time intervals we can calculate the value for c , such that for an optimal (partial) solution x with $c_x = c$ we have $\Delta(x, \delta) = 0$.

Remark 3.2. Let x^n be an optimal (partial) solution with $c_x = c$ and let δ be the activating vector as in equation (19). Then, for every $t \in \mathcal{T}$, δ_t and x_t are polynomials in c of degree at most 1. The change in cost, $d_t(x_t, \delta_t) = \delta_t(x_t + \alpha_t + \frac{\delta_t}{2})$, for every $t \in \mathcal{T}$ then is a polynomial in c of degree at most 2. So the total change in cost $\Delta(x, \delta)$ which is just the sum of all d_t 's, is a polynomial in c of degree at most 2.

Lemma 3.5. *Let $1 \leq l' \leq l \leq n' \leq n$ be time intervals such that there is an optimal (partial) solution $x = x^n$ with $l'_x = l'$, $l_x = l$, $n'_x = n'$ and $\Delta(x, \delta) = 0$, with δ as in equation (19). Then we can calculate $c^* \in \mathbb{R}$ such that $\Delta(x, \delta) = 0$ if and only if $c_x = c^*$.*

Proof. Let x be an optimal (partial) solution with $c_x = c$ and δ be the activating vector as in equation (19). We have already remarked that $\Delta(x, \delta)$ is a polynomial in c of degree at most 2. If we define c^* to be the zero of this polynomial that is between the already calculated bounds, then $\Delta(x, \delta) = 0$ if and only if $c = c^*$. \square

We know that for an optimal partial solution x with $c_x = c^*$ we have

$$C(x) = \sum_{t=1}^l X_{\max} + \sum_{t=l+1}^{n'} X_{\max} + c - (\alpha_t - \alpha_l) + \sum_{t=n'+1}^n X_{\min}.$$

This implies that we activate time interval $n + 1$ if $C \geq C(x)$.

3.3.2 Taking both b and c into account

In the following we consider the non-increasing (partial) solutions $x = x^n$ where both X_{\min} and X_{\max} restrict the activation of a new time interval. Again, for every (partial) solution $x = x^n$, we determine the time intervals l_x and n'_x such that:

$$\begin{cases} x_t = X_{\max} & \text{if } 1 \leq t \leq l_x, \\ X_{\min} < x_t < X_{\max} & \text{if } l < t \leq n'_x, \\ x_t = X_{\min} & \text{if } n'_x < l \leq n, \\ x_t = 0 & \text{otherwise} \end{cases} \quad (20)$$

and we set $b_x = x_{n'} - X_{\min}$ and $c_x = x_{l+1} + (\alpha_{l+1} - \alpha_l) - X_{\max}$. An example of such a solution is shown in Figure 10.

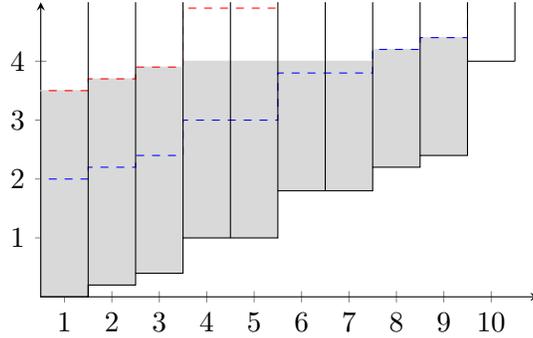


Figure 10: A solution x^9 with $l_x = 3$, $n'_x = 7$, $b_x = 0.2$ and $c_x = 0.1$.

To activate the next time interval $n + 1$, again we need to lower the time intervals $1, \dots, n$ with a total of X_{\min} . However the (partial) solution after activating the next time interval, still has to be optimal, i.e. it has to satisfy equation (6) and the balance equation. So, next to l_x and n'_x , we also need to determine time intervals l'_x and m'_x such that:

- the time intervals $1, \dots, l'_x - 1$ are not changed;
- $x_{l'_x}$ is the first time interval that changes, and we change it with $-\delta_x$ for some $\delta_x > 0$;
- the time intervals $l'_x < t \leq l_x$ are lowered with $\delta_x + (\alpha_t - \alpha_{l'_x})$;

- the time intervals $l_x + 1, \dots, m'_x$ are all lowered with the same amount $(\alpha_{l_x} - \alpha_{l'_x}) + c + \delta_x$;
- the time intervals $m'_x < t \leq n'_x$ are lowered with $b + (\alpha_{n'_x} - \alpha_t)$.

This situation is shown in Figure 11. If we have fixed all these time intervals $1 \leq l' \leq l \leq m' \leq n' \leq n$ for some (partial) solution x , we can calculate δ_x .

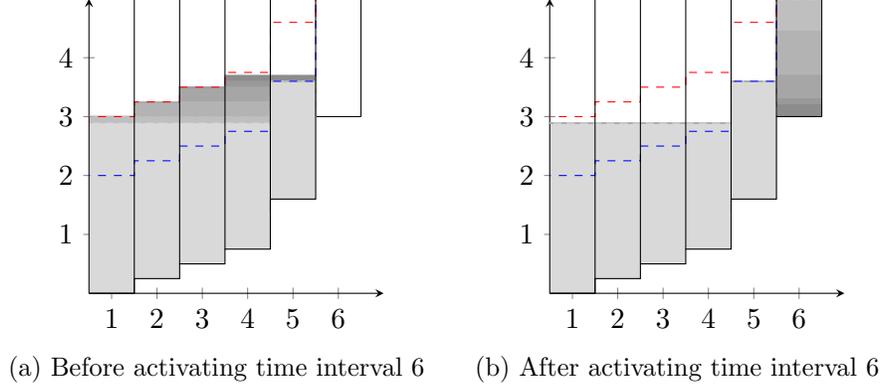


Figure 11: Solution x^5 with $l'_x = 1$, $l_x = 3$, $m'_x = 4$ and $n'_x = 5$.

Lemma 3.6. *Let $x = x^n$ be an optimal partial solution and l, n' time intervals such that $l_x = l$ and $n'_x = n'$, set $1 \leq l' \leq l$ and $l \leq m \leq n'$. Then*

$$\delta_x = \frac{X_{\min} - \sum_{t=l'}^l (\alpha_t - \alpha_{l'}) - (m' - l)((\alpha_l - \alpha_{l'}) + c) - \sum_{t=m'+1}^{n'} ((\alpha_{n'} - \alpha_t) + b)}{m' - l' + 1} \quad (21)$$

Proof. To activate the next interval we need to lower the time intervals $1, \dots, n$ with a total of X_{\min} . So we need to get

$$\begin{aligned} X_{\min} &= \sum_{t=l'}^l (\delta_x + (\alpha_t - \alpha_{l'})) + \sum_{t=l+1}^{m'} (\delta_x + (\alpha_l - \alpha_{l'}) + c) + \sum_{t=m'+1}^{n'} ((\alpha_{n'} - \alpha_t) + b) \\ &= (m' - l' + 1)\delta_x + \sum_{t=l'}^l (\alpha_t - \alpha_{l'}) + (m' - l)((\alpha_l - \alpha_{l'}) + c) + \sum_{t=m'+1}^{n'} ((\alpha_{n'} - \alpha_t) + b). \end{aligned}$$

If we subtract $X_{\min} + (m' - l' + 1)\delta_x$ from both sides and multiply both sides with $\frac{-1}{(m' - l' + 1)}$ we obtain the result of the lemma. \square

For every chosen pair of time intervals l', m' equation (21) allows us to calculate δ_x . Furthermore, the (partial) solution after activating time interval $n + 1$, which we call x^{n+1} , has to satisfy equation (6). So we need to find the pair l'_x, m'_x such that:

- $\delta_x \geq 0$ (if this is not the case we get $x_l^{n+1} > X_{\max}$, see Figure 12a);
- $\delta_x \leq (\alpha_{l'} - \alpha_{l'-1})$ (if this is not the case we get $x_{l'}^{n+1} + (\alpha_{l'} - \alpha_{l'+1}) \leq x_{l'-1}^{n+1}$, see Figure 12b);
- $(\alpha_l - \alpha_{l'}) + c + \delta_x \leq (\alpha_{n'} - \alpha_{m'}) + b$ (if this is not the case we get $x_{m'}^{n+1} < X_{\min}$, see Figure 12b);
- $(\alpha_l - \alpha_{l'}) + c + \delta_x \geq (\alpha_{n'} - \alpha_{m'+1}) + b$ (if this is not the case we get $x_{m'}^{n+1} > x_{m'+1}^{n+1} + (\alpha_{m'+1} - \alpha_{m'})$, see Figure 12c);

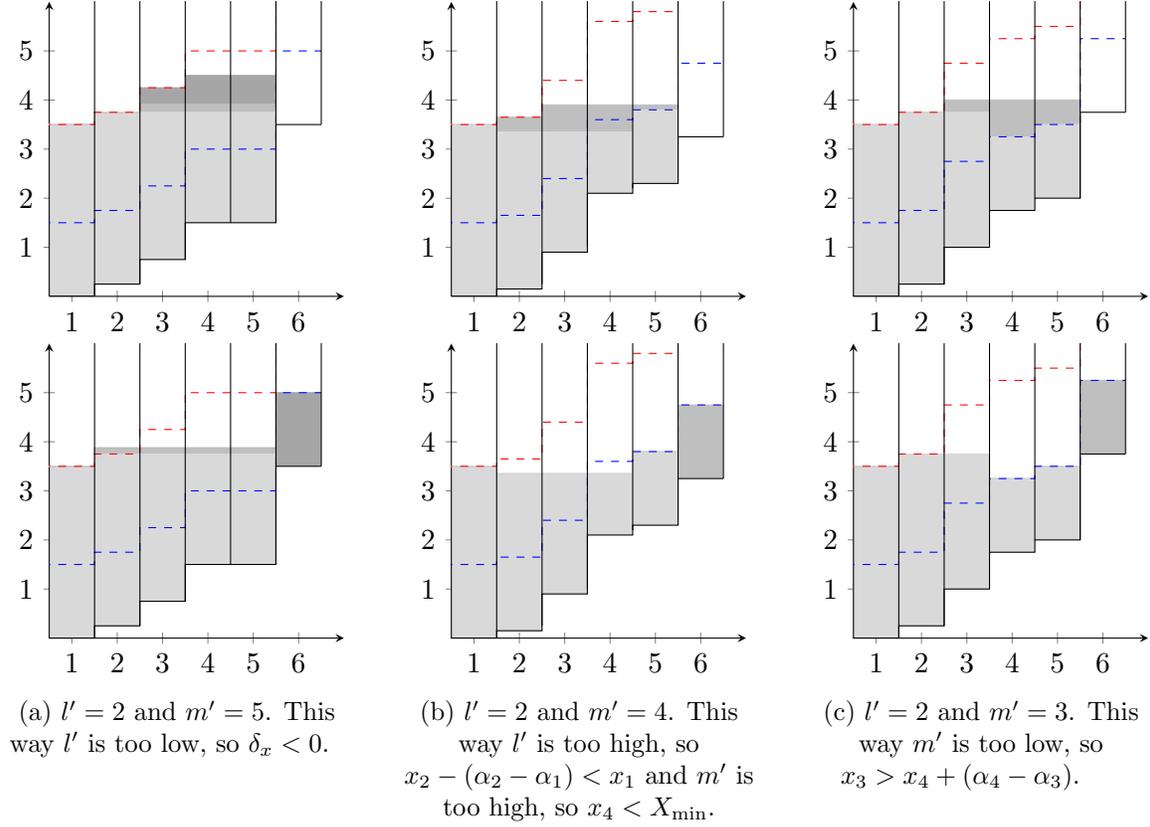


Figure 12: Wrong choices for l' and m' .

1st row: before activating time interval 6. 2nd row: after activating time interval 6

So for every (partial) solution x^n we can determine the time intervals $1 \leq l'_x \leq l_x \leq m'_x \leq n'_x \leq n$ as before and calculate δ_x using equation (21), such that these intervals and δ_x satisfy all the above. To activate time interval $n+1$ we use the activating vector δ with:

$$\delta_t = \begin{cases} -\delta_x - (\alpha_t - \alpha_{l'}) & \text{if } l' \leq t \leq l, \\ -\delta_x - (\alpha_t - \alpha_{l'}) - c & \text{if } l' < t \leq m', \\ -(\alpha_{n'} - \alpha_t) - b & \text{if } m' < l \leq n', \\ X_{\min} & \text{if } t = n+1, \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

We want to show that there exists an optimal (partial) solution $x = x^n$, such that $\Delta(x, \delta) = 0$, i.e. such that it is optimal to activate the next time interval. Therefore, again, we suppose that for $x = x^n$ with $C(x) = nX_{\max}$ we have $\Delta(x, \delta) < 0$, otherwise we still have to activate time interval $n+1$ if we have to charge more than $n \cdot X_{\max}$. Then we know there is a (partial) solution x^* with $nX_{\min} \leq C(x^*) \leq nX_{\max}$ and $\Delta(x^*, \delta) = 0$, with activating vector δ as in equation (22). For this (partial) solution we determine the time intervals $1 \leq l'^* \leq l^* \leq m'^* \leq n'^* \leq n$ such that $l'_{x^*} = l'^*$, $l_{x^*} = l^*$, $m'_{x^*} = m'^*$ and $n'_{x^*} = n'^*$. Again, we can exactly determine all these time intervals such that there is a (partial) solution x , where the time intervals l', l, m' and n' corresponding to this (partial) solution coincide with these time intervals, with $\Delta(x, \delta) = 0$, in about the same way as we did in the previous subsection. However, again, we only show that we can calculate the exact value c^* , such that $c_{x^*} = c^*$, when we have already found the time intervals l'^*, l^*, m'^* and n'^* .

Lemma 3.7. *Let $0 \leq l' \leq l \leq m' \leq n'$ be time intervals such that there is an optimal (partial) solution $x = x^n$ with $l'_x = l'$, $l_x = l$, $m'_x = m'$, $n'_x = n'$ and $\Delta(x, \delta) = 0$ using*

activating vector δ as in equation 22 and such that the resulting (partial) solution $x' = x + \delta$ after activating time interval $n + 1$ is still optimal. Then we can calculate a value $c^* \in \mathbb{R}$ such that $\Delta(x, \delta) = 0$ if and only if $c_x = c^*$.

Proof. Let x be a (partial) solution as in equation (20). We set $b = b_x := x_{n'} - X_{\min}$ and $c = c_x := x_{l+1} + (\alpha_{l+1} - \alpha_l) - X_{\max}$. From the balance equation we get $x_{n'} = x_{l+1} - (\alpha_{n'} - \alpha_{l+1})$. So we can write

$$\begin{aligned} b &= x_{l+1} - (\alpha_{n'} - \alpha_{l+1}) - X_{\min} \\ &= c - (\alpha_{l+1} - \alpha_l) + X_{\max} - (\alpha_{n'} - \alpha_{l+1}) - X_{\min} \\ &= c + X_{\max} - X_{\min} - (\alpha_{n'} - \alpha_l). \end{aligned}$$

Let δ be the activating vector as in equation 22. For all $1 \leq t \leq T$, x_t and δ_t are polynomials in c of degree at most 1. So $\Delta(x, \delta)$ is a polynomial in c of degree at most 2, as we saw in Remark 3.2. So we can calculate the zeroes of this polynomial and let c^* be the zero of this polynomial within the proper bounds, then we get $\Delta(x, \delta) = 0$ if and only if $c_x = c^*$. \square

For a partial solution x with $c_x = c^*$ we know

$$C(x) = \sum_{t=1}^l X_{\max} + \sum_{t=l+1}^{n'} (X_{\max} + c^* - (\alpha_t - \alpha_l)) + \sum_{t=n'+1}^n X_{\min}$$

So we activate a new time interval if and only if $C \geq C(x)$.

Now we showed for all the cases of the general CPLB (i.e. with a lower bound and an upper bound), how we can find borders for $C(x)$ from which it is optimal to activate the next time interval. Because of the large expressions we already got in Section 3.2, we did not exactly determine these borders and the time intervals in which the charging has to be lowered to activate the next time interval, we only argued that they exist.

4 Practical case

In the previous section we showed that it is possible to find an optimal non-increasing (partial) solution x^n , with n active intervals, and a border for $C(x)$, from which it is optimal to activate a new time interval. However, to solve the problem we had to distinguish between a lot of different cases and these cases gave us very large expressions. Therefore, we now consider a special case of the CPLB, which is easier to solve. More precisely, throughout this section we suppose

$$(\alpha_t - \alpha_s) > X_{\min} \vee (\alpha_t - \alpha_s) = 0, \quad \forall t, s \in \mathcal{T}. \quad (23)$$

We call this variant of the CPLB with the extra condition (23), the CPLBS.

Remark 4.1. *With condition (23) on the α 's, most corner cases of Section 3 disappear. For example, consider the case without the upper bound X_{\max} and suppose we already have an optimal (partial) solution x^n with $\alpha_{n+1} > \alpha_n$. Because α is a non-decreasing sequence, for all $1 \leq n' \leq n$ we have $(\alpha_{n+1} - \alpha_{n'}) \geq (\alpha_{n+1} - \alpha_n) \geq X_{\min}$. So by Lemma 3.2, we know we are in the case where we can lower all time intervals until a certain time interval $n' \leq n$ equally, to activate the next time interval.*

Furthermore, we will see later on, that in the case with an upper bound other corner cases are also excluded.

Next, in Subsection 4.1 we give the borders for $C(x)$, from which it is optimal to activate the next time interval, in all different cases. After that, in Subsection 4.2, we give an algorithm that solves the CPLBS to optimality and in polynomial time.

4.1 Determining the borders

First, we define a certain kind of (partial) solution for some time intervals $1 \leq n \leq T$. Which is also shown in Figure 13.

Definition 4.1. *For every $1 \leq n < T$ with $\alpha_{n+1} > \alpha_n$, we define the standard solution \bar{x}^n on n time intervals, as follows:*

$$\bar{x}_t^n = \begin{cases} \min\{X_{\max}, (\alpha_{n+1} - \alpha_t)\} & \text{if } 1 \leq t \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

An example of a standard solution is shown in Figure 13.

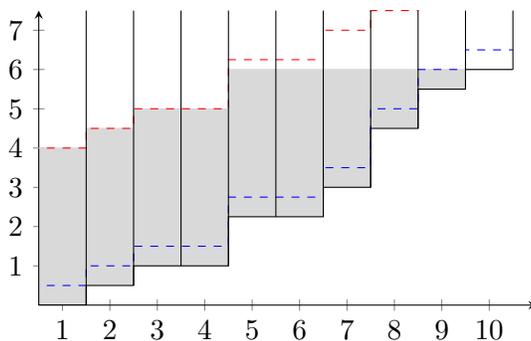


Figure 13: Standard solution for $n = 9$, with $X_{\min} = 0.5$.

In the following Lemma we show that the standard solutions are optimal (partial) solutions.

Lemma 4.1. *Let $n \in \mathcal{T}$ be a time interval, such that $\alpha_{n+1} > \alpha_n$. Then the standard solution \bar{x}^n is an optimal (partial) solution.*

Proof. Let $\bar{x} = \bar{x}^n$ be the standard solution on n time intervals. Let n^* be the number of active time intervals in an optimal partial non-increasing solution x^* with $C(x^*) = C(\bar{x})$. Because \bar{x} satisfies the balance equation and equation (6), we only need to show that $n = n^*$. First suppose $n^* > n$, so it has to be cheaper to activate time interval $n + 1$. However, for every activating vector $\delta^{n+1} \in \mathbb{R}^T$ we have:

$$\begin{aligned}
\frac{\Delta(\bar{x}, \delta^{n+1})}{2\beta} &= \sum_{t=1}^n \left(\delta_t^{n+1} \left(\bar{x}_t + \alpha_t + \frac{\delta_t^{n+1}}{2} \right) \right) + X_{\min} \left(\alpha_{n+1} + \frac{X_{\min}}{2} \right) \\
&\geq \sum_{t=1}^n \left(\delta_t^{n+1} \left((\alpha_{n+1} - \alpha_t) + \alpha_t + \frac{\delta_t^{n+1}}{2} \right) \right) + X_{\min} \left(\alpha_{n+1} + \frac{X_{\min}}{2} \right) \\
&= \sum_{t=1}^n \left(\frac{(\delta_t^{n+1})^2}{2} \right) + \alpha_{n+1} \sum_{t=1}^n \delta_t^{n+1} + \alpha_{n+1} X_{\min} + \frac{X_{\min}^2}{2} \\
&= \sum_{t=1}^n \left(\frac{(\delta_t^{n+1})^2}{2} \right) + \frac{X_{\min}^2}{2} > 0.
\end{aligned}$$

In the above, the first inequality follows from the fact that \bar{x} is a standard solution and because the change of the objective value is positive we know that $n^* \leq n$.

Next, we suppose $n^* < n$. This means it would be cheaper to lower the time interval n with \bar{x}_n and add a total of \bar{x}_n of extra charging on the time intervals $1, \dots, n-1$. Note, that we can only add extra charging on the time intervals $l+1, \dots, n-1$, where l is defined as in the previous section to be the last time interval which is charging X_{\max} . However, for every $\delta \in \mathbb{R}^T$, with $\sum_{t=l+1}^{n-1} \delta_t = \bar{x}_n$, $\delta_n = -\bar{x}_n$, $x_t \geq 0$, for all $l+1 \leq t \leq n-1$ and $\delta_t = 0$ otherwise, we get a total cost change of:

$$\begin{aligned}
\frac{\Delta(\bar{x}, \delta)}{2\beta} &= \sum_{t=l+1}^{n-1} \delta_t \left((\alpha_{n+1} - \alpha_t) + \alpha_t + \frac{\delta_t}{2} \right) - \bar{x}_n \left(\bar{x}_n + \alpha_n - \frac{\bar{x}_n}{2} \right) \\
&\geq \sum_{t=l+1}^{n-1} \delta_t \left((\alpha_{n+1} - \alpha_t) + \alpha_t + \frac{\delta_t}{2} \right) - \bar{x}_n \left((\alpha_{n+1} - \alpha_n) + \alpha_n - \frac{\bar{x}_n}{2} \right) \\
&= \alpha_{n+1} \sum_{t=l+1}^{n-1} \delta_t + \sum_{t=l+1}^{n-1} \frac{\delta_t^2}{2} - \alpha_{n+1} \bar{x}_n + \frac{\bar{x}_n^2}{2} \\
&= \sum_{t=l+1}^{n-1} \frac{\delta_t^2}{2} + \frac{\bar{x}_n^2}{2} > 0.
\end{aligned}$$

Again, the first inequality follows from the fact that \bar{x} is a standard solution. This implies $n^* = n$ and thus \bar{x} is optimal. \square

Let x^n be a (partial) solution for some time interval $n \in \mathcal{T}$. To activate a new time interval, we define some specific time intervals.

Definition 4.2. Let $1 \leq n \leq T$ be a time interval and $x = x^n$ a (partial) solution satisfying the balance equation and (6). We define l_x and l'_x to be the time intervals such that:

1. $x_t = X_{\max}$ if and only if $1 \leq t \leq l_x$;
2. $\alpha_t = \alpha_{l'_x}$ if and only if $l_x + 1 \leq t \leq l'_x$, i.e. these are the first time intervals t that charge less than X_{\max} and that have $\alpha_t = \alpha_{l_x+1}$.

Let $1 \leq n \leq T$ be a time interval such that $\alpha_{n+1} > \alpha_n$ and let x^n be an optimal (partial) solution. We want to find the border for $C(x^n)$, such that the cost difference for activating time interval $n+1$ becomes negative. First we take the easier case where all the time intervals that are lowered to activate time interval $n+1$, are all lowered equally. So to activate a new time interval, we use activating vector δ^r with:

$$\delta_t^r = \begin{cases} -\frac{X_{\min}}{n-r} & \text{if } r < t \leq n, \\ X_{\min} & \text{if } t = n+1, \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

for some $0 \leq r \leq n$, see Figure 14.

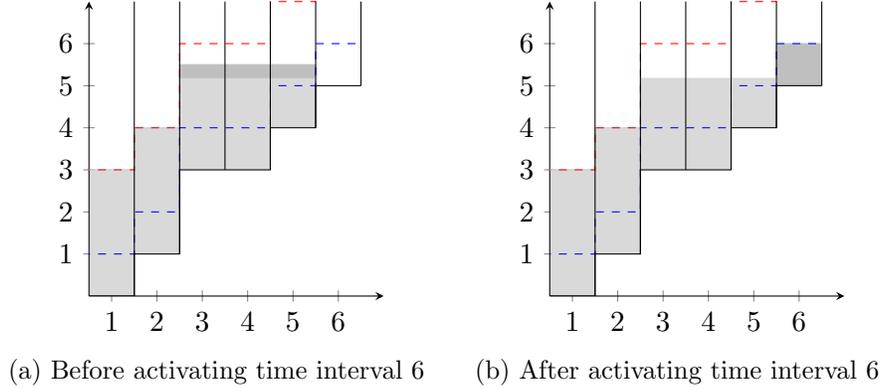


Figure 14: Activating a new time interval in a solution x with $l_x = 2$ and $l'_x = 4$, using δ^2 .

Because we want the (partial) solution after activating the time interval $n+1$ to be optimal, we can use Lemma 3.1 to determine the border for $C(x^n)$, such that $\Delta(x^n, \delta_r) = 0$, for some $1 \leq r \leq n'$.

Lemma 4.2. *Let $n \in \mathcal{T}$ be a time interval such that $\alpha_{n+1} > \alpha_n$. Let $x = \bar{x}^n$ be the standard solution on n time intervals, and set $l = l_x$ and $l' = l'_x$. Let x^* be a non-increasing partial solution satisfying the balance equation and equation (6).*

(i) *If $(\alpha_{n+1} - \alpha_{l'}) \leq X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)}$ then*

$$\Delta(x^*, \delta) = 0 \iff C(x^*) = l \cdot X_{\max} + \frac{(n-l+1)X_{\min}}{2} + \sum_{t=l+1}^n (\alpha_{n+1} - \alpha_t).$$

Here $\delta = \delta_l$ as in equation (25). Furthermore, the (partial) solution $x' = x^ + \delta$ after activating time interval $n+1$ is optimal.*

(ii) *If $(\alpha_{n+1} - \alpha_{l'}) \geq X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$ then*

$$\Delta(x^*, \delta) = 0 \iff C(x^*) = l' \cdot X_{\max} + \frac{(n-l'+1)X_{\min}}{2} + \sum_{t=l'+1}^n (\alpha_{n+1} - \alpha_t).$$

Here $\delta = \delta_{l'}$ as in equation (25). Furthermore, the (partial) solution $x' = x + \delta$ after activating time interval $n+1$ is optimal.

Proof. Let x^n be a (partial) solution. If we want to activate time interval $n+1$, we want the (partial) solution after activating time interval $n+1$ to satisfy the balance equation. Therefore, we know that all time intervals t with $X_{\min} < x_t < X_{\max}$ have to be lowered. In

Lemma 4.1 we saw that standard solutions are optimal for the amount they charge. Because of the optimality of standard solutions (Lemma 4.1), it is not yet optimal to activate the next time interval when there is charged less than there is charged in a standard solution. So, for the proofs of both (i) and (ii) we already know $C(x^*) \geq C(\bar{x}^n)$. This means we have to lower all time intervals from a certain interval $t \geq l$ to n .

Proof of (i): Assume $(\alpha_{n+1} - \alpha_l) \leq X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)}$. If we replace time interval 1 in Lemma 3.1 by $l+1$, and set $n' = n$, then we can use this lemma to see that $\Delta(x^*, \delta_l) = 0$ if and only if $C(x^*) = l \cdot X_{\max} + \frac{(n-l+1)X_{\min}}{2} + \sum_{t=l+1}^n (\alpha_{n+1} - \alpha_l)$. For all $l < t \leq n$ we have:

$$x_t^* = (\alpha_{n+1} - \alpha_t) + \frac{(n-l+1)X_{\min}}{2(n-l)};$$

$$x_t' = (\alpha_{n+1} - \alpha_t) + \frac{(n-l-1)X_{\min}}{2(n-l)}.$$

Because of the assumption that $(\alpha_{n+1} - \alpha_l) \leq X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)}$, we have $x_t^* \leq X_{\max}$ on all time intervals $1 \leq t \leq T$, so x^* is a feasible solution. Because $l = l_x$ for a standard solution x , we know $x_l = X_{\max} \leq (\alpha_{n+1} - \alpha_l)$. Therefore we have:

$$\begin{aligned} x_t^* > x_t' &= (\alpha_{n+1} - \alpha_t) + \frac{(n-l-1)X_{\min}}{2(n-l)} \\ &\geq (\alpha_{n+1} - \alpha_l) - (\alpha_t - \alpha_l) \\ &\geq X_{\max} - (\alpha_t - \alpha_l), \end{aligned}$$

for all $l < t \leq n$. So both the (partial) solutions x^* and x' satisfy equation (6). Because all time intervals $l+1, \dots, n$ are equally lowered, the (partial) solution x' also satisfies the balance equation. So x' is optimal.

Proof of (ii): Assume $(\alpha_{n+1} - \alpha_{l'}) \geq X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$. Again we use Lemma 3.1, where we replace time interval 1 by $l'+1$ and $n' = n$. Then Lemma 3.1 implies that $\Delta(x^*, \delta_l) = 0$ if and only if $C(x^*) = l' \cdot X_{\max} + \frac{(n-l'+1)X_{\min}}{2} + \sum_{t=l'+1}^n (\alpha_{n+1} - \alpha_{l'})$. For all $l' < t \leq n$ we get:

$$x_t^* = (\alpha_{n+1} - \alpha_t) + \frac{(n-l'+1)X_{\min}}{2(n-l')};$$

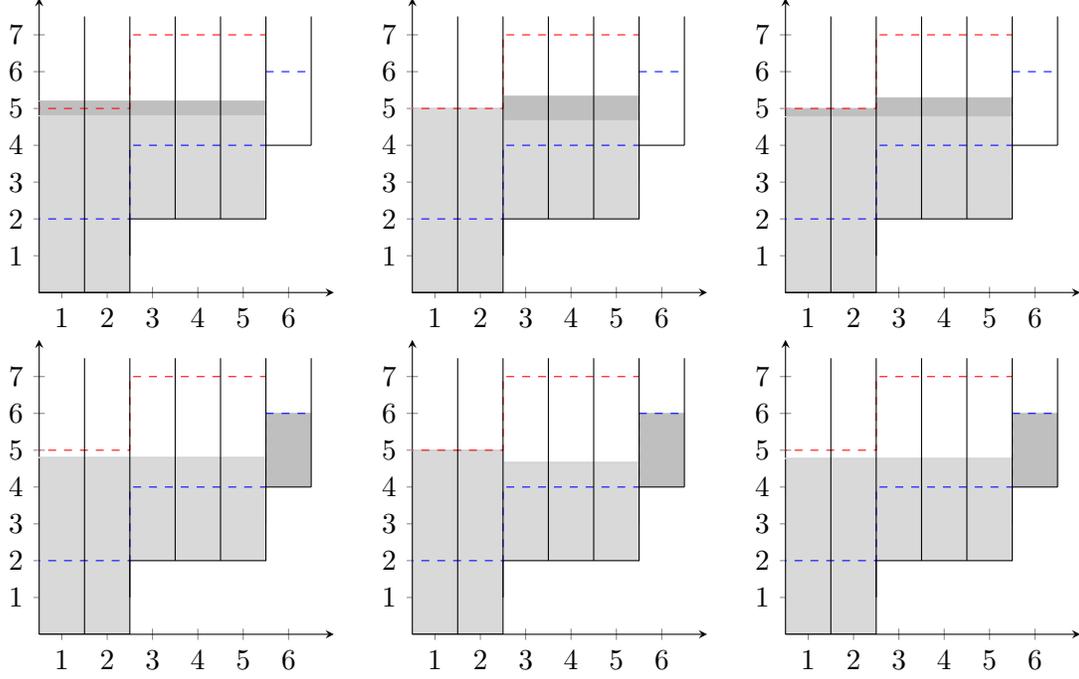
$$x_t' = (\alpha_{n+1} - \alpha_t) + \frac{(n-l'-1)X_{\min}}{2(n-l')}.$$

Because $\alpha_t \neq \alpha_{l'}$, we have $(\alpha_t - \alpha_{l'}) > X_{\min}$ for all $l' < t$. Also, because $l' = l'_x$, for a standard solution x , we know $X_{\max} > x_{l'} = (\alpha_{n+1} - \alpha_{l'})$. So we have $x_t' < x_t^* = (\alpha_{n+1} - \alpha_t) + \frac{(n-l'+1)X_{\min}}{2(n-l')} < X_{\max}$, meaning that both x' and x^* are feasible (partial) solutions. And because of the assumption that $(\alpha_{n+1} - \alpha_{l'}) \geq X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$, we get for all time intervals $l' < t \leq n$:

$$\begin{aligned} x_t^* > x_t' &= (\alpha_{n+1} - \alpha_t) + \frac{(n-l'-1)X_{\min}}{2(n-l')} \\ &\geq (\alpha_{n+1} - \alpha_t) + X_{\max} - (\alpha_{n+1} - \alpha_{l'}) \\ &= X_{\max} - (\alpha_t - \alpha_{l'}). \end{aligned}$$

So both (partial) solutions x^* and x' satisfy equation (6). Again, because all time intervals $l'+1, \dots, n$ are equally lowered, the (partial) solution x' also satisfies the balance equation, implying that x' is optimal. \square

The only open case that remains is, when $X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)} < (\alpha_{n+1} - \alpha_{l'}) < X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$. The examples in Figure 15a and 15b show that in this case we cannot find a (partial) solution $x = x^{n+1}$, where we can lower the amount charged in all time intervals l_x, \dots, n equally.



(a) Lowering all time intervals $1, \dots, 5$ equally implies that before activating we had $x_1 = 5.2 > X_{\max}$
(b) Lowering all time intervals $3, \dots, 5$ equally implies that after activating we get $x_3 + \alpha_3 < x_2 + \alpha_2$.
(c) Activating the next time interval using δ as in equation (26)

Figure 15: Three different ways of activating a new time interval in a situation where $X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)} < (\alpha_{n+1} - \alpha_{l'}) < X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$, with $n = 5$, $l = 0$ and $l' = 2$.
1st row: before activating time interval 6. 2nd row: after activating time interval 6.

Because of Lemma 4.2, we know it is rewardable to activate time interval $n + 1$ for a (partial) solution x^* with

$$l \cdot X_{\max} + \sum_{t=l+1}^n (\alpha_{n+1} - \alpha_t) + \frac{(n-l+1)X_{\min}}{2} < C(x^*) < l' \cdot X_{\max} + \sum_{t=l'+1}^n (\alpha_{n+1} - \alpha_t) + \frac{(n-l'+1)X_{\min}}{2}.$$

Let $x = x^n$ be an optimal partial solution with $C(x)$ between these borders. Because $(\alpha_{n+1} - \alpha_{l'}) > X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)}$, we have $x_t = X_{\max}$ for all $1 \leq t \leq l'$. Set $c = c_x := x_{l'} + (\alpha_{l'} - \alpha_l) - X_{\max}$ and suppose, for the sake of contradiction, that $c_x \geq \frac{X_{\min}}{n-l'}$. Then because

$(\alpha_{n+1} - \alpha_{l'}) < X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$ we get

$$\begin{aligned}
C(x) &= l'X_{\max} + \sum_{t=l'+1}^n X_{\max} + c - (\alpha_t - \alpha_{l'}) \\
&= l'X_{\max} + \sum_{t=l'+1}^n X_{\max} + c - (\alpha_{n+1} - \alpha_{l'}) + (\alpha_{n+1} - \alpha_t) \\
&> l'X_{\max} + \sum_{t=l'+1}^n X_{\max} - X_{\max} + \frac{(n-l'-1)X_{\min}}{2(n-l')} + \frac{X_{\min}}{(n-l')} + (\alpha_{n+1} - \alpha_t) \\
&= l' \cdot X_{\max} + \sum_{t=l'+1}^n (\alpha_{n+1} - \alpha_t) + \frac{(n-l'+1)X_{\min}}{2}.
\end{aligned}$$

This implies $c \leq \frac{X_{\min}}{n-l'}$. Furthermore, we have $c \geq 0$, since otherwise the (partial) solution x with $c_x = c$ does not satisfy equation (6). We use activating δ with:

$$\delta_t = \begin{cases} -\frac{X_{\min} - (n-l')c}{n-l} & \text{if } l < t \leq l', \\ -\frac{X_{\min} - (n-l')c}{n-l} - c & \text{if } l' < t \leq n, \\ X_{\min} & \text{if } t = n+1, \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

to activate time interval $n+1$, see Figure 15c. Because $(\alpha_l - \alpha_{l'}) \geq X_{\min}$, the (partial) solution after activating time interval $n+1$ still satisfies the balance equation and equation (6).

It remains to find an optimal partial solution x^* with $0 < c_{x^*} < \frac{X_{\min}}{n-l'}$, such that $\Delta(x^*, \delta) = 0$.

Lemma 4.3. *Let $1 \leq n \leq T$ be a time interval such that $\alpha_{n+1} \neq \alpha_n$, let $x = x^n$ be the standard solution and let $l = l_x$ and $l' = l'_x$. Furthermore, assume*

$$X_{\max} - \frac{(n-l+1)X_{\min}}{2(n-l)} \leq (\alpha_{n+1} - \alpha_{l'}) \leq X_{\max} - \frac{(n-l'-1)X_{\min}}{2(n-l')}$$

For an optimal partial solution x^* with $l_{x^*} = l'$ and the activating vector δ as in equation (26), we have $\Delta(x^*, \delta) = 0$ if and only if

$$c_{x^*} = \frac{-X_{\min}}{(l'-l)} + \sqrt{X_{\min}^2 \frac{(n-l)(l'-l+1)}{(l'-l)^2(n-l')} + X_{\min} \frac{2(n-l)((\alpha_{n+1} - \alpha_{l'}) - X_{\max})}{(n-l')(l'-l)}}. \quad (27)$$

Proof. Let $c = c_{x^*}$. Because $x_t^* = X_{\max} + c - (\alpha_t - \alpha_{l'})$ for all $l'+1 \leq t \leq n$ and $x_t = X_{\max}$ for all $1 \leq t \leq l'$ the cost difference to activate time interval $n+1$ is:

$$\begin{aligned}
\Delta(x^*, \delta) &= \sum_{t=l'+1}^{l'} d_t \left(X_{\max}, -\frac{X_{\min} - (n-l')c}{n-l} \right) \\
&+ \sum_{t=l'+1}^n d_t \left(X_{\max} + c - (\alpha_t - \alpha_{l'}), -c - \frac{X_{\min} - (n-l')c}{n-l} \right) \\
&+ d_{n+1} \left(0, X_{\min} \right).
\end{aligned}$$

This implies:

$$\begin{aligned}
\frac{\Delta(x^*, \delta)}{2\beta} &= -\frac{(l' - l)(X_{\min} - (n - l')c)}{n - l} \left(X_{\max} + \alpha_{l'} - \frac{X_{\min} - (n - l')c}{2(n - l)} \right) \\
&\quad - \frac{(n - l')(X_{\min} - (n - l')c)}{n - l} \left(X_{\max} + \alpha_{l'} - \frac{X_{\min} - (n - l')c}{2(n - l)} + \frac{c}{2} \right) \\
&\quad - (n - l')c \left(X_{\max} + \alpha_{l'} - \frac{X_{\min} - (n - l')c}{2(n - l)} + \frac{c}{2} \right) + X_{\min} \left(\alpha_{n+1} + \frac{X_{\min}}{2} \right) \\
&= -\left(((l' - l) + (n - l')) \frac{X_{\min} - (n - l')c}{n - l} + (n - l')c \right) \left(X_{\max} + \alpha_{l'} - \frac{X_{\min} - (n - l')c}{2(n - l)} \right) \\
&\quad - (n - l') \left(\frac{X_{\min} - (n - l')c}{n - l} + c \right) \frac{c}{2} + X_{\min} \left(\alpha_{n+1} + \frac{X_{\min}}{2} \right) \\
&= X_{\min} \left((\alpha_{n+1} - \alpha_{l'}) + \frac{(n - l + 1)X_{\min}}{2(n - l)} - X_{\max} - \frac{(n - l')c}{n - l} \right) - \frac{(n - l')(l' - l)c^2}{2(n - l)}.
\end{aligned}$$

This leads to the following second degree polynomial in c :

$$f(c) = c^2 + \frac{2X_{\min}}{(l' - l)}c + X_{\min} \left(\frac{2(n - l)((\alpha_{n+1} - \alpha_{l'}) - X_{\max}) + (n - l + 1)X_{\min}}{(n - l')(l - l')} \right),$$

for which we have $\Delta(x^*, \delta) = 0$ if and only if $f(c) = 0$. Using the *abc*-formula we get $f(c) = 0$ if and only if:

$$c = \frac{-X_{\min}}{(l' - l)} \pm \sqrt{X_{\min}^2 \frac{(n - l)(l' - l + 1)}{(l' - l)^2(n - l')} + X_{\min} \frac{2(n - l)((\alpha_{n+1} - \alpha_{l'}) - X_{\max})}{(n - l')(l' - l)}}$$

Because $\frac{-X_{\min}}{(l' - l)} < 0$ and we want to have $0 \leq c \leq \frac{X_{\min}}{n - l'}$, we only can take the zero

$$c = \frac{-X_{\min}}{(l' - l)} + \sqrt{X_{\min}^2 \frac{(n - l)(l' - l + 1)}{(l' - l)^2(n - l')} + X_{\min} \frac{2(n - l)((\alpha_{n+1} - \alpha_{l'}) - X_{\max})}{(n - l')(l' - l)}}$$

into account. First, using $X_{\max} \leq \frac{(n - l + 1)X_{\min}}{2(n - l)} + (\alpha_{n+1} - \alpha_{l'})$, we get that:

$$\begin{aligned}
c &\geq \frac{-X_{\min}}{(l' - l)} + \sqrt{X_{\min}^2 \frac{(n - l)(l' - l + 1)}{(l' - l)^2(n - l')} - X_{\min}^2 \frac{(n - l + 1)}{(n - l')(l' - l)}} \\
&= \frac{-X_{\min}}{(l' - l)} + \sqrt{\frac{X_{\min}^2}{(l' - l)^2}} \\
&= 0.
\end{aligned}$$

Next, using $X_{\max} \geq (\alpha_{n+1} - \alpha_{l'}) + \frac{(n - l' - 1)X_{\min}}{2(n - l')}$, we get that:

$$\begin{aligned}
c &\leq \frac{-X_{\min}}{(l' - l)} + \sqrt{X_{\min}^2 \frac{(n - l)(l' - l + 1)}{(l' - l)^2(n - l')} - X_{\min}^2 \frac{(n - l)(n - l' - 1)}{(n - l')^2(l' - l)}} \\
&= \frac{-X_{\min}}{(l' - l)} + \sqrt{\frac{(n - l)^2 X_{\min}^2}{(l' - l)^2(n - l')^2}} \\
&= \frac{X_{\min}}{(n - l')}
\end{aligned}$$

□

For x^* with c_{x^*} as in equation (27) we have

$$C(x^*) = l'_{x^*} X_{\max} + \sum_{t=l'+1}^n X_{\max} + c_{x^*} - (\alpha_t - \alpha_{l'}).$$

Summarizing, we activate the next time interval if and only if $C \geq C(x^*)$.

Up to now we have considered only optimal (partial) solutions for time intervals $1 \leq n \leq T$ with $\alpha_{n+1} > \alpha_n$. For these solutions we have calculated the borders for $C(x^n)$, from which it is rewardable to activate the next time interval. However, we also need to specify when it is rewardable to add a new time interval $n + 1$ if we have an optimal partial solution x^n with $1 \leq n < T$ where $\alpha_{n+1} = \alpha_n$.

Lemma 4.4. *Let $1 \leq n < T$ be a time interval such that $\alpha_{n+1} = \alpha_n$ and set $n' := \min\{1 \leq t \leq n \mid \alpha_t = \alpha_n\}$. Let x^* be the non-increasing optimal (partial) solution we get right after activating time interval $n' + 1$ using Lemma 4.2 or 4.3. Then it is optimal to activate time interval $n + 1$ if and only if $C > C(x^*) + (n - n')X_{\min}$.*

Proof. Let \bar{x} be the standard solution on n' intervals and set $l' = l'_{\bar{x}}$. We can use Lemma 4.2 or Lemma 4.3 (it depends on the size of $(\alpha_{n'+1} - \alpha_{l'})$ which of the two we need to use) to find a (partial) solution x^* and an activating vector δ such that $\Delta(x^*, \delta) = 0$ and the (partial) solution after activating time interval $n' + 1$ is still optimal. In both cases we know $x_t^* - (\alpha_{n'+1} - \alpha_t) < X_{\min}$ for all $1 \leq t \leq n'$. So let x^n be the (partial) solution with:

$$x_t^n = \begin{cases} x_t^* & \text{if } 1 < t \leq n' \\ X_{\min} & \text{if } n' < t \leq n \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

Then $\Delta(x^n, \delta) = \Delta(x^*, \delta) = 0$ because $\delta_t = 0$ for all $t \geq n'$. Since x^n also satisfies equation (6) and is non-increasing, it is an optimal (partial) solution and $C(x^n) = C(x^*) + (n - n')X_{\min}$. \square

4.2 Algorithm for the CPLBS

In the previous subsection we shown that for every time interval $n\mathcal{T}$, we can find an optimal partial solution x^n and an activating vector δ such that $\Delta(x^n, \delta) = 0$ and the (partial) solution after activating time interval $n + 1$ using δ is still optimal. The next step is to develop an algorithm to find an optimal solution for a given CPLBS.

To get an optimal non-increasing solution we initialize the vector $x \in \mathbb{R}^T$ with the zero vector. Then we iteratively create partial solutions that charge a bit more in every step until $C(x) = C$. These steps are explained in more detail in several different algorithms below. At the beginning of every iteration we assume we have an optimal partial solution x^n for some $1 \leq n \leq T$ and $C(x)$. After each iteration we make sure the partial solution obtained is non-increasing and optimal.

Definition 4.3. *Let x be a partial solution for a CPLBS and let C be the amount that has to be charged for this CPLBS. We define $r(x, C)$ to be the amount that still has to be charged, i.e. $r(x, C) := C - C(x)$*

4.2.1 Activating time intervals t with $\alpha_t = \alpha_n$.

The iterations of the algorithm are grouped in three different phases. In iterations of the first phase called Algorithm 1 we activate, starting with an optimal partial solution x^n achieved

after activating interval n , all the time intervals $t > n$ with $\alpha_t = \alpha_n$. We start with a partial solution $x = x^n$, we get right after activating time interval n . By Lemma 4.4 we know that it is rewardable to activate a time interval t with $\alpha_t = \alpha_n$ if $C > C(x) + X_{\min}$. Let k be the last time interval k such that $\alpha_t = \alpha_n$ for all $n \leq t \leq k$. We want to set $x_t = X_{\min}$, for all $n + 1 \leq t \leq k$. However, we do not want the amount to be charged on the partial solution to exceed C , so we only do this if $r(x, C) \geq (k - n)X_{\min}$.

More precisely, in Algorithm 1, we activate the time intervals $n + 1, \dots, k$ one at a time, thereby updating $r(x, C)$, until $r(x, C) < X_{\min}$ (Figure 16b) or until all time intervals $n + 1, \dots, k$ are activated (Figure 16a).

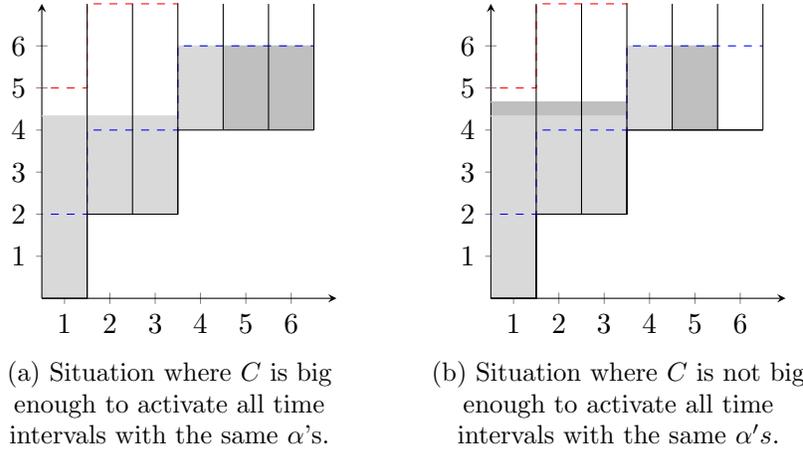


Figure 16: Activating all time intervals t , with $\alpha_t = \alpha_4 = 4$, if possible, using Algorithm 1. The dark gray area represents the charging added in Algorithm 1.

Suppose we are in the situation that $r(x, C) < X_{\min}$ and only $m < k - n$ of the time intervals $n + 1, \dots, k$ are activated. We distinguish between two different scenarios:

1. If it is possible to add $r(x, C)$ to the time intervals that are already activated (i.e. if $\sum_{t=1}^m X_{\max} - x_t \geq r(x, C)$), we do this. Because we started with an optimal partial solution, we achieved right after activating a time interval and because $r(x, C) \leq X_{\min}$, we know there is an optimal solution for the EV-planning problem with m activated time intervals. This case is detailed later on in Algorithm 5.
2. If $\sum_{t=1}^m (X_{\max} - x_t) < r(x, C)$, we have to activate a new time interval and lower some of the already activated time intervals. This case is detailed later on in Algorithm 4.

Algorithm 1 Activate time intervals with the same α 's.

Input: A partial solution x with $r(x, C) > 0$ and a time interval n such that x is the partial solution we achieved, right after activating time interval n .

Output: x is an optimal partial solution with $x_t = X_{\min}$, for all t with $\alpha_t = \alpha_n$, or $r(x, C) = 0$.

```

1: Set  $t = n + 1$ .
2: while  $\alpha_t = \alpha_n$  do
3:   if  $r(x, C) \geq X_{\min}$  then
4:     set  $x_t = X_{\min}$ , update  $r(x, C)$  and  $n$  and set  $t = t + 1$ .
5:   else if  $r(x, C) < X_{\min}$  and  $\sum_{s=1}^{t-1} = X_{\max} - x_s < r(x, C)$  then
6:     use Algorithm 4, to charge the remaining  $r(x, C)$  on the already activated time intervals.
7:   else
8:     use Algorithm 5, to activate the next time interval, while lowering the already activated time intervals.
9:   end if
10: end while
11: return  $x, r(x, C)$  and  $n$ .
```

4.2.2 Making the standard solution

Consider the case that Algorithm 1 returns a remaining capacity $r(x, C) > 0$ and let $x = x^n$ be the partial solution it returns. Because $r(x, C) > 0$, we know that all time intervals t with $\alpha_t = \alpha_n$ are activated in Algorithm 1, meaning that $\alpha_{n+1} > \alpha_n$. In the next iteration we can then try to increase the total charging done to make this the standard solution on n time intervals.

Hereby, it can be that C is not big enough to obtain this standard solution, i.e. when $\sum_{t=1}^n \min\{X_{\max}, (\alpha_{n+1} - \alpha_t)\} > C$. Therefore, in Algorithm 2, we ensure that we add the remaining $r(x, C)$ on the time intervals $1, \dots, n$ in such a way, that when $r(x, C) = 0$, the partial solution we have at that moment is still optimal. Furthermore, we ensure that $r(x, C) \geq 0$ at all time.

To this end, let $l = l_x$ and $l' = l'_x$ as in Definition 4.2. Also let $n' = n'_x$. Because in Algorithm 1 we started with the partial solution we obtained right after activating time interval n' and because Algorithm 1 returned $r(x, C) > 0$, we know there is no charging added to the time intervals $1, \dots, n'$ in Algorithm 1, so $x_{n'} - (\alpha_n - \alpha_{n'}) < X_{\min}$ and $\alpha_{n'+1} = \dots = \alpha_n$. In every iteration of Algorithm 2 we add some extra charging δ to the time intervals $l+1, \dots, n'$, to maintain partial solutions that satisfy the balance equation. After each step we update $r(x, C)$ and the time intervals l, l' and n' . As long as $x'_n < X_{\min} + (\alpha_n - \alpha_{n'})$, we have to ensure:

1. $\delta \leq X_{\max} - x_{l+1}$, since otherwise $x_{l+1} + \delta > X_{\max}$ (see Figure 17a);
2. $\delta \leq \frac{r(x, C)}{n' - l}$, since otherwise after adding δ to the time intervals $l+1, \dots, n'$ and updating $r(x, C)$, we get $r(x, C) < 0$;
3. $\delta \leq (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1}$, since otherwise $x_t + \delta > (\alpha_{n+1} - \alpha_t)$ for all $l+1 \leq t \leq n'$, which is bigger than x_t should be in the standard solution (see Figure 17c);

4. $\delta \leq X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'}$, since otherwise $x_t + \delta + \alpha_t > x_n + \alpha_n = X_{\min} + \alpha_n$ for all $l+1 \leq t \leq n'$, so the partial solution does not satisfy equation (6) anymore and is therefore not optimal (see Figure 17b).

Summarizing, we set:

$$\delta = \min \left\{ X_{\max} - x_{l+1}, \frac{r(x, C)}{n' - l}, (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1}, X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'} \right\}.$$

If, at some point, $\delta = X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'}$, we get $x_t + \alpha_t = X_{\min} + \alpha_n = x_n + \alpha_n$, after adding δ to x_t for all $l+1 \leq t \leq n'$. So from this point, to maintain optimal partial solutions, we have to add

$$\delta = \min \left\{ X_{\max} - x_{l+1}, \frac{r(x, C)}{n' - l}, (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1} \right\},$$

to all time intervals $l+1, \dots, n$. We stop when we have $\delta = \frac{r(x, C)}{n' - l}$ or $\delta = (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1}$, because in this case after adding δ to all time intervals $l+1 \leq t \leq n'$, we have either $r(x, C) = 0$ or x is the standard solution on n intervals.

Algorithm 2 Making the standard solution

Input: An optimal partial solution x , time intervals $0 \leq l \leq l' \leq n' \leq n$ and $r(x, C) > 0$, such that $x_1 = \dots x_l = X_{\max}$, $\alpha_{l+1} = \dots = \alpha_{l'} \neq \alpha_{l'+1}$, $x_{n'} - (\alpha_n - \alpha_{n'}) < x_{n'+1} = \dots = x_n = X_{\min}$ and $\alpha_{n+1} > \alpha_n$.

Output: x is an optimal partial solution and either $r(x, C) = 0$ or x is the standard solution on n time intervals.

- 1: set $\delta = \min\{X_{\max} - x_{l+1}, \frac{r(x, C)}{n' - l}, (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1}, X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'}\}$
 - 2: **while** $\delta < X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'}$ **do**
 - 3: **for** $t = l+1$ to n' **do**
 - 4: set $x_t = x_t + \delta$
 - 5: **end for**
 - 6: update $r(x, C)$, l, l' and n' .
 - 7: **if** $\delta = (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1}$ or $r(C, x) = 0$ **then**
 - 8: **return** $x, r(x, C)$ and n .
 - 9: **end if**
 - 10: update δ .
 - 11: **end while**
 - 12: **while** $r(x, C) > 0$ and $x_t < \min\{X_{\max}, (\alpha_{n+1} - \alpha_t)\}$ for some $t \leq n$ **do**
 - 13: do the same as in the first while-loop, but use $\delta = \min\{X_{\max} - x_{l+1}, \frac{r(x, C)}{n' - l}, (\alpha_{n+1} - \alpha_{l+1}) - x_{l+1}\}$ when updating δ . Also set $n' = n$ after the first iteration.
 - 14: **end while**
-

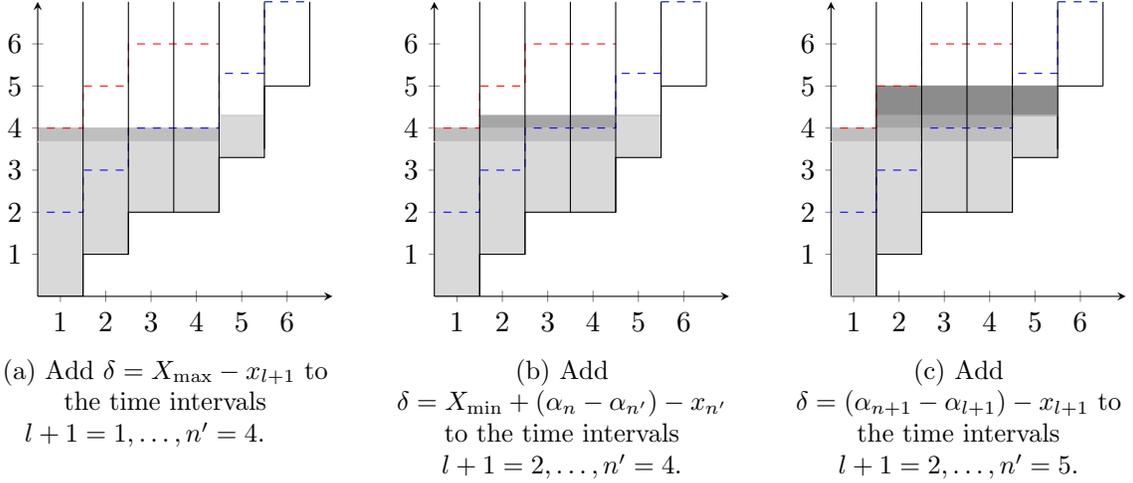


Figure 17: Making the standardsolution on 5 time intervals, in three different steps. The light gray area represents the amount charged before using Algorithm 2

4.2.3 Activating a time interval $n+1$ with $\alpha_{n+1} > \alpha_n$

We now consider the case that x is the standard solution on n time intervals, $r(x, C) > 0$ and $\alpha_{n+1} > \alpha_n$. In Algorithm 3 we activate the next time interval, if possible. Therefore, we set the time intervals $l = l_x$ and $l' = l'_x$ as in Definition 4.2. Furthermore, we define an upper bound $UB := \frac{(n-l+1)X_{\min}}{2(n-l)} + (\alpha_{n+1} - \alpha_{l'})$ and a lower bound $LB := \frac{(n-l'-1)X_{\min}}{2(n-l')} + (\alpha_{n+1} - \alpha_{l'})$ for a case distinction based on the value of X_{\max} . We now distinguish between three different cases:

Case A Suppose $UB \leq X_{\max}$. By Lemma 4.2, we know that it is rewardable to activate time interval $n+1$ when we have an optimal partial solution x with $C(x) = l \cdot X_{\max} + \frac{(n-l)X_{\min}}{2} + \sum_{t=l+1}^n (\alpha_{n+1} - \alpha_t)$, see Figure 18a. To activate time interval $n+1$ we lower all the time intervals $l+1, \dots, n$ equally with $\frac{X_{\min}}{n-l}$ to get the situation shown in Figure 19a. So in Algorithm 3 we add $\min\left\{\frac{(n-l-1)X_{\min}}{2(n-l)}, \frac{r(x, C)}{n-l}\right\}$ extra charging to all time intervals $l+1, \dots, n$, to get in the situation where we can set $x_{n+1} = X_{\min}$ and still have an optimal partial solution, or we get $r(x, C) = 0$.

Case B Suppose $LB \geq X_{\max}$. Again, by Lemma 4.2, we know that it is rewardable to activate time interval $n+1$ when we have a partial solution x with $C(x) = l' \cdot X_{\max} + \frac{(n-l')X_{\min}}{2} + \sum_{t=l'+1}^n (\alpha_{n+1} - \alpha_t)$, see Figure 18b. To activate the time interval $n+1$ we lower all time intervals $l'+1, \dots, n$ equally to get the situation shown in Figure 19b. So in Algorithm 3 we first add $\min\left\{X_{\max} - (\alpha_{n+1} - \alpha_{l'}), \frac{r(x, C)}{n-l'}\right\}$ to all time intervals $l'+1, \dots, n$ implying that $x_t = X_{\max} - (\alpha_t - \alpha_{l'})$ for all $l' < t \leq n$, or $r(x, C) = 0$. If after updating all the parameters we still have $r(x, C) > 0$, we add $\min\left\{\frac{(n-l'-1)X_{\min}}{2(n-l')} - (X_{\max} - (\alpha_{n+1} - \alpha_{l'})), \frac{r(x, C)}{n-l'}\right\}$ to all time intervals $l'+1, \dots, n$ to get in the situation where we can set $x_{n+1} = X_{\min}$ and still have an optimal partial solution, or we get $r(x, C) = 0$.

Case C Suppose $LB < X_{\max} < UB$. Now, by Lemma 4.3, we know that it is rewardable to activate time interval $n+1$ when we have a partial solution x with $C(x) = l' \cdot X_{\max} + \sum_{t=l'+1}^n X_{\max} - (\alpha_t - \alpha_{l'}) + c$, with c as in equation (27), see Figure 18c. To activate time interval $n+1$, we can not lower all time intervals $l'+1, \dots, n$ equally. So we first lower all time intervals $l'+1, \dots, n$ with c , and then we lower all time intervals $l+1, \dots, n$

equally with $\frac{X_{\min} - (n-l')c}{n-l}$ to get the situation shown in Figure 19c. So in Algorithm 3, we add $\min\{X_{\max} - (\alpha_{n+1} - \alpha_{l'}) - \frac{X_{\min} - (n-l')c}{n-l}, \frac{r(x,C)}{n-l}\}$ to all time intervals $l+1, \dots, n$, to get in the situation where we can set $x_{n+1} = X_{\min}$ and still have an optimal partial solution, or we get $r(x, C) = 0$.

In Figure 18 we show these three cases at the moment before we activate the next time interval. For every case we only use a different value for X_{\max} but leave everything else the same.

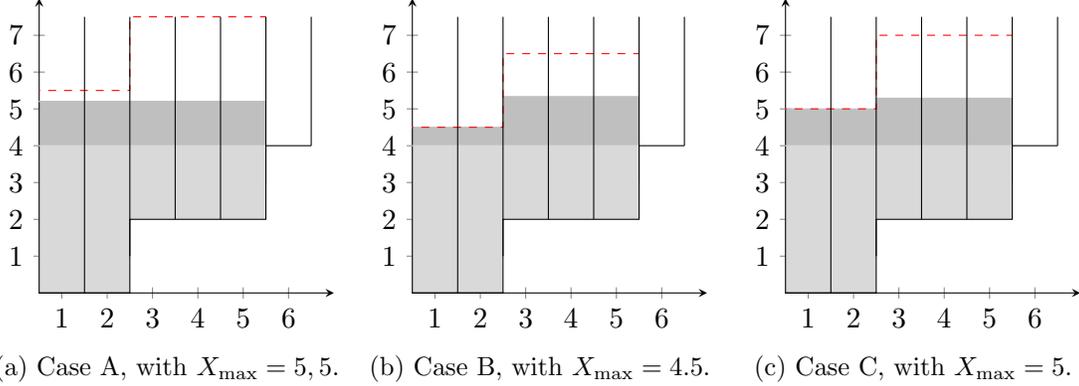


Figure 18: Situation before activating time interval $n+1$.

As we want to achieve a situation where we can set $x_{n+1} = X_{\min}$ and still have an optimal partial solution, the darker gray areas in Figure 19 are added in Algorithm 3 on top of the amount charged in the standard solution.

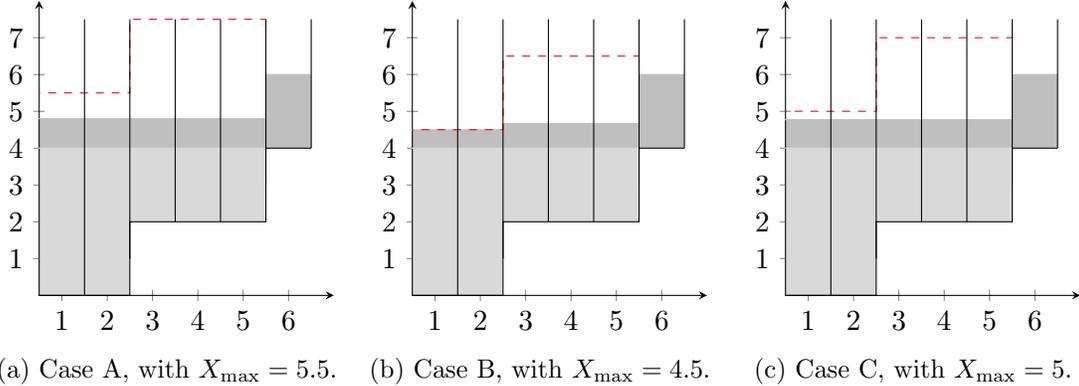


Figure 19: Situation after activating time interval $n+1$.

There is only one problem left that can occur when we are in case C, this is when we have $l' = n$. In this situation we only have the time intervals $l+1, \dots, l' = n$ on which we can charge more. However, X_{\max} may be too low, to get in the situation where we can lower all these time intervals equally to activate the next time interval, i.e. in the situation in case A. If this is the case, we add the next time interval when $x_t = X_{\max}$ for all $1 \leq t \leq n$. So in Algorithm 3 we add $\min\{X_{\max} - (\alpha_{n+1} - \alpha_{l'}), \frac{r(x,C)}{n-l}\}$ to all time intervals $l+1, \dots, l$, to either arrive at the situation where we can set $x_{n+1} = X_{\min}$ while remaining an optimal partial solution, or we get $r(x, C) = 0$, see Figure 20. This is actually the same situation as when $l = l' = n$, only now $\min\{X_{\max} - (\alpha_{n+1} - \alpha_{l'}), \frac{r(x,C)}{n-l}\} = 0$. Thus, nothing is added before we get in the situation where we can set $x_{n+1} = X_{\min}$.

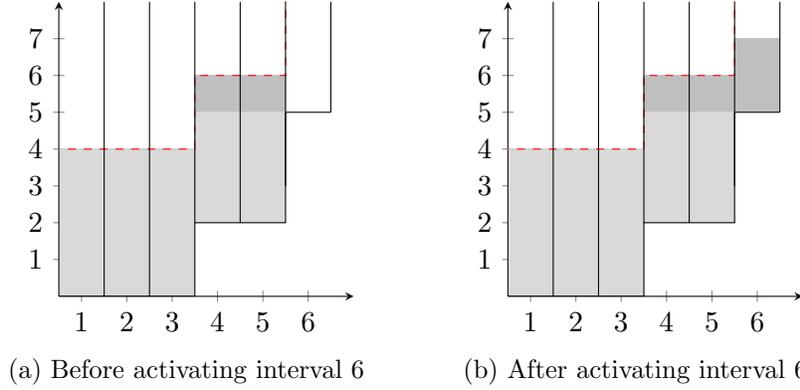


Figure 20: Situation where $l' = n = 5$ and $LB < X_{\max} = 4 < UB$. The dark gray area is added in Algorithm 3.

Consider now the situation where $r(x, C) > 0$ and we can set $x_{n+1} = X_{\min}$. We can do this if and only if $r(x, C) \geq X_{\min}$. However, if this is not the case we have to proceed in a different way. We use Algorithm 4 (if there is not enough space left on the time intervals $1, \dots, n$ to charge the remaining $r(x, C)$) or Algorithm 5 (if there is enough space left on the time intervals $1, \dots, n$ to charge the remaining $r(x, C)$).

Algorithm 3 Activate a new time interval

Input: A standard solution x on n time intervals, time intervals $0 \leq l \leq l' \leq n$ and $r(x, C) > 0$, such that $x_t = X_{\max}$ if and only if $1 \leq t \leq l$, $\alpha_t = \alpha_{l'}$ if and only if $l + 1 \leq t \leq l'$ and $\alpha_{n+1} > \alpha_n$.

Output: x is an optimal partial solution and either $r(x, C) = 0$ or the time interval $n + 1$ is activated.

- 1: Set $UB = \frac{(n-l+1)X_{\min}}{2(n-l)} + (\alpha_{n+1} - \alpha_{l'})$ and $LB = \frac{(n-l'-1)X_{\min}}{2(n-l')} + (\alpha_{n+1} - \alpha_{l'})$
 - 2: **if** $l = l' = n$, or $LB < X_{\max} < UB$ and $l' = n$ **then**
 - 3: set $x_t = x_t + \min \{ X_{\max} - (\alpha_{n+1} - \alpha_{l'}), \frac{r(x, C)}{n-l} \}$ for all $l < t \leq n$ and update $r(x, C)$.
 - 4: **else if** $UB \leq X_{\max}$ (situation A) **then**
 - 5: set $x_t = x_t + \delta$ for all $l < t \leq n$ with $\delta = \min \{ \frac{(n-l-1)X_{\min}}{2(n-l)}, \frac{r(x, C)}{n-l} \}$ and update $r(x, C)$.
 - 6: **else if** $X_{\max} \leq LB$ (situation B) **then**
 - 7: set $x_t = x_t + \delta$ for all $l < t \leq n$ with $\delta = \min \{ X_{\max} - (\alpha_{n+1} - \alpha_{l'}), \frac{r(x, C)}{n-l} \}$ and update $r(x, C)$, l and l' .
 - 8: **if** $r(x, C) > 0$ **then**
 - 9: set $x_t = x_t + \delta$ for all $l < t \leq n$ with $\delta = \min \{ \frac{(n-l-1)X_{\min}}{2(n-l)} - (X_{\max} - (\alpha_{n+1} - \alpha_l)), \frac{r(x, C)}{n-l} \}$ and update $r(x, C)$.
 - 10: **end if**
 - 11: **else if** $LB < X_{\max} < UB$ (situation C) **then**
 - 12: set $x_t = x_t + \delta$ for all $l < t \leq n$ with $\delta = \min \{ X_{\max} - (\alpha_{n+1} - \alpha_{l'}) - \frac{X_{\min} - (n-l')c}{n-l}, \frac{r(x, C)}{n-l} \}$ and c as in equation (27).
 - 13: **end if**
 - 14: **if** $r(x, C) \geq X_{\min}$ **then**
 - 15: Set $x_{n+1} = X_{\min}$, update $r(x, C)$.
 - 16: **return** x and $r(x, C)$.
 - 17: **else**
 - 18: Use Algorithm 4 if $\sum_{t=1}^n X_{\max} - x_t < r(x, C)$ and Algorithm 5 otherwise.
 - 19: **end if**
-

4.2.4 Activating the last time interval

As we have seen in Algorithm 1 and 3 it can be, that we are in the situation where we want to activate the next time interval, but $r(x, C) < X_{\min}$. Let $x = x^n$ be an optimal partial solution such that if we set $x_{n+1} = X_{\min}$ the partial solution is still optimal. Also suppose $r(x, C) < X_{\min}$ and $\sum_{t=1}^n X_{\max} - x_t < r(x, C)$.

In this case we can not charge the remaining $r(x, C)$ on the already activated time intervals, so we have to activate time interval $n + 1$ and therefore we need to lower (some of) the intervals $1, \dots, n$ with a total of $X_{\min} - r(x, C)$. Let $l = l_x$ $l' = l'_x$ and $n' = n'_x$ and also let $b := x_{n'} - X_{\min}$ and $c := x_{l'} + X_{\max} - (\alpha_{l'} - \alpha_l)$. The easiest way to activate the next time interval, would be to lower the time intervals $l + 1, \dots, n'$ all equally with $\delta := \frac{X_{\min} - r(x, C)}{n - l}$. But the precense of b and c causes that we don not always obtain an optimal partial solution if we do that.

When we take b into account we see that we can lower time interval n' with a maximum of b . And when we also take c into account we see that we can lower time intervals $l + 1, \dots, l'$ with a maximum of c before we also have to lower the time intervals $t \leq l$ with $\alpha_t = \alpha_l$. For this, we determine the time interval $0 < m \leq l$ such that $\alpha_t = \alpha_l$ if and only if $m \leq t \leq l$. We have $(\alpha_l - \alpha_m) \geq X_{\min}$, so by the time we also want to lower some of the intervals $t < m$, the time intervals m, \dots, l already have to be lowered with $(\alpha_l - \alpha_m) \geq X_{\min}$. However, since we only have to lower the time intervals $1, \dots, n$ with a total of $X_{\min} - r(x, C) < X_{\min}$, this never happens.

A summary of the sketched is given in Algorithm 4, where in each step we lower $l + 1, \dots, n'$ with $\min\{b, c, \delta\}$ and update all the parameters afterwards. We repeat this step until we have $r(x, C) = X_{\min}$, and we can activate time interval $n + 1$, see Figure 21.

Algorithm 4 Activate the last time interval

Input: An optimal partial solution x , time intervals $0 \leq m \leq l \leq l' \leq n' \leq n$ such that $\sum_{t=1}^n X_{\max} - x_t < r(x, C) < X_{\min}$, $x_t = X_{\max}$ if and only if $1 \leq t \leq l$, $\alpha_t = \alpha_l$ if and only if $m < t \leq l$, $\alpha_t = \alpha_{l'}$ if and only if $l < t \leq l'$ and $x_t = X_{\min}$ if and only if $n' < t \leq n$.

Output: x is an optimal partial solution and $x_{n+1} = X_{\min}$.

- 1: Set $b := x_{n'} - X_{\min}$, $c = x_{l'} + X_{\max} - (\alpha_{l'} - \alpha_l)$ and $\delta = \frac{X_{\min} - r(x, C)}{n - l}$.
 - 2: **while** $r(x, C) < X_{\min}$ **do**
 - 3: **for** $t = l + 1$ to n' **do**
 - 4: Set $x_t = x_t - \min\{b, c, \delta\}$.
 - 5: **end for**
 - 6: Update $m, l, l', n', r(x, C), b, c$ and δ .
 - 7: **end while**
 - 8: Set $x_{n+1} = X_{\min}$ and update $r(x, C)$.
 - 9: **return** x and $r(x, C)$.
-

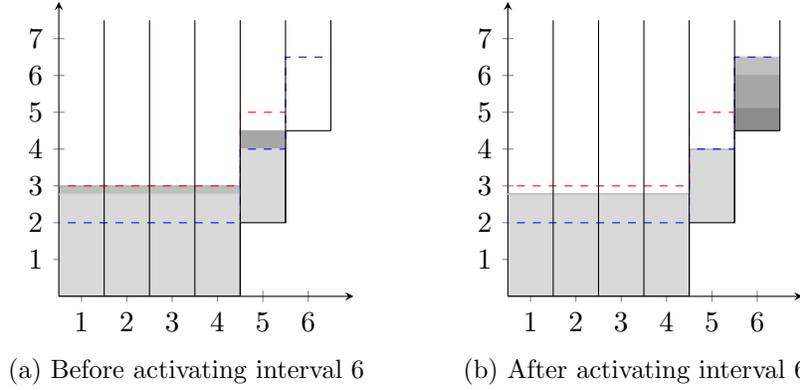


Figure 21: Lower the already activated time intervals with a total of $X_{\min} - r(x, C) = 2 - 0,6 = 1,4$, to activate the last time interval. The darkest gray area represents $r(x, C)$.

4.2.5 Charge the remaining $r(x, C)$ on the already activated time intervals

We consider, again, the situation that $x = x^n$ is an optimal partial solution such that, if we set $x_{n+1} = X_{\min}$, the partial solution is still optimal. Also suppose that $r(x, C) < X_{\min}$ and that there is enough space on the time intervals $1, \dots, n$ to charge the remaining $r(x, C)$ on these time intervals, i.e. $\sum_{t=1}^n X_{\max} - x_t > r(x, C)$. We know that if Algorithm 5 is called in Algorithm 1 or Algorithm 3 that we are always in the situation where it is cheaper to charge the remaining $r(x, C) < X_{\min}$ on the already activated time intervals. So if we add the remaining $r(x, C)$ in the right way, we get an optimal solution.

We also will use Algorithm 5 in Algorithm 6, which is detailed in the next subsection. However, in that case we only use it when all time intervals $1, \dots, T$ are already activated and $r(x, C) > 0$. Note that, in this case, it can also be that $r(x, C) \geq X_{\min}$. But, for the execution of Algorithm 5, it is not relevant if $r(x, C) < X_{\min}$ or if $r(x, C) \geq X_{\min}$, because in both cases the algorithm works the same.

Again, we define the time intervals $l = l_x$, $l' = l'_x$ and $n' = n'_x$ and we set $b := X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'}$, meaning that b is the maximum amount we can charge extra on the time intervals $l + 1, \dots, n'$, before we also need to charge extra on the time intervals $n' + 1, \dots, n$. We have to make sure that we do not charge more than $X_{\max} - x_{l'}$ extra on the time intervals $l + 1, \dots, l'$ and we do not charge more than $r(x, C)$ extra on all time intervals $l + 1, \dots, n'$ together. So we define $\delta := \min \{ X_{\max} - x_{l'}, \frac{r(x, C)}{n' - l} \}$ and we distinguish between two different loops in the algorithm:

1. We add δ extra charging to the time intervals $l + 1, \dots, n'$ as long as $\delta \leq b$ and $r(x, C) > 0$, while we update all the parameters after each time we added extra charging. One execution of this loop is shown in Figure 22a.
2. If $r(x, C) > 0$ after we stop the first loop, we add δ extra charging to all time intervals $l + 1, \dots, n$ until $r(x, C) = 0$, while we update all the parameters after each time we added extra charging. One execution of this loop is shown in Figure 22b.

Algorithm 5 Charge the remaining $r(x, C)$

Input: An optimal partial solution x , time intervals $0 \leq l \leq l' \leq n' \leq n$ such that $\sum_{t=1}^n X_{\max} - x_t \geq r(x, C)$, $x_t = X_{\max}$ if and only if $1 \leq t \leq l$, $\alpha_t = \alpha_{l'}$ if and only if $l < t \leq l'$ and $x_t = X_{\min}$ if and only if $n' < t \leq n$.

Output: x is an optimal solution with n active time intervals and $r(x, C) = 0$.

- 1: Set $\delta = \min \left\{ X_{\max} - x_{l'}, \frac{r(x, C)}{n' - l} \right\}$ and $b := X_{\min} + (\alpha_n - \alpha_{n'}) - x_{n'}$.
 - 2: **while** $\delta \leq b$ and $r(x, C) > 0$ **do**
 - 3: Set $x_t = x_t + \delta$ for all $l < t \leq n'$ and update $l, l', r(x, C), b$ and δ .
 - 4: **end while**
 - 5: **if** $r(x, C) = 0$ **then**
 - 6: **return** x
 - 7: **else**
 - 8: Set $x_t = x_t + \delta$ for all $l < t \leq n'$ and update $l, l', r(x, C)$ and δ .
 - 9: **while** $r(x, C) > 0$ **do**
 - 10: Set $x_t = x_t + \delta$ for all $l < t \leq n$ and update $l, l', r(x, C)$ and δ .
 - 11: **end while**
 - 12: **return** x
 - 13: **end if**
-

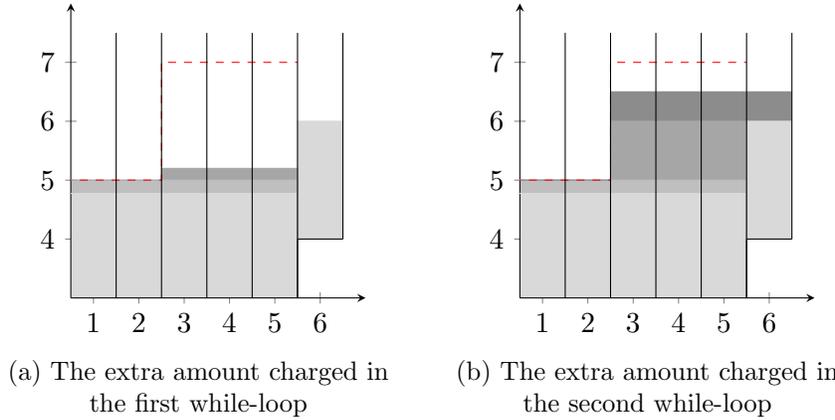


Figure 22: Add the remaining $r(x, C) =$ to the time intervals $1, \dots, 6$

4.2.6 Finding the optimal solution

In the previous subsections we have specified all the different iterations and steps needed to solve the considered CPLBS. We now describe in Algorithm 6 how we combine all these algorithms to find the optimal solution for the considered CPLBS. We start with the optimal partial solution x , with $x_t = 0$ for all $1 \leq t \leq T$. In the first step we set $r(x, C) = C$ and $x_1 = X_{\min}$ if and only if $r(x, C) \geq X_{\min}$. If we have set $x_1 = X_{\min}$, we set $n = 1$ and repeat the following three steps until we have either activated all time intervals $1 \dots T$, or $r(x, C) = 0$.

Step 1: Use Algorithm 1 to get a partial solution with n activated time intervals such that $\alpha_n < \alpha_{n+1}$, or $r(x, C) = 0$.

Step 2: If $r(x, C) > 0$, use Algorithm 2, to get the standard solution on n time intervals, or $r(x, C) = 0$.

Step 3: If $r(x, C) > 0$, use Algorithm 3, to get an optimal partial solution with $n+1$ activated time intervals, or $r(x, C) = 0$.

If we have activated all time intervals $1, \dots, T$, using these three steps repeatedly and we still have $r(x, C) > 0$, we use Algorithm 5 to charge the remaining $r(x, C)$ on the time intervals $1, \dots, T$. To ensure we can actually do all these steps, we first make sure there exists a solution for the CPLBS. There are two situations when there is no solution for a CPLBS for given values of C, T, X_{\min} and X_{\max} :

1. The minimum number of time intervals needed is larger than the number of time intervals, i.e., $\lceil \frac{C}{X_{\max}} \rceil > T$;
2. The minimum number of time intervals needed is larger than the number of time intervals we can maximal use given the capacity C , i.e., $\lceil \frac{C}{X_{\max}} \rceil > \lfloor \frac{C}{X_{\min}} \rfloor$.

If we exclude these two cases, we can find the optimal solution of the considered CPLBS using Algorithm 6.

Algorithm 6 Create an optimal solution

Input: $C, X_{\min}, X_{\max}, T \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}^T$ such that $(\alpha_s - \alpha_t) = 0$ or $(\alpha_s - \alpha_t) \geq X_{\min}$ for all $1 \leq t \leq s \leq T$ and $\lceil \frac{C}{X_{\max}} \rceil \leq \min\{T, \lfloor \frac{C}{X_{\min}} \rfloor\}$.

Output: x is an optimal solution with $n \leq T$ active time intervals and $C(x) = C$.

- 1: Set $x_t = 0$ for all $1 \leq t \leq T$, $r(x, C) = C$ and $n = 0$.
 - 2: **if** $r(x, C) \geq X_{\min}$ **then**
 - 3: Set $x_1 = X_{\min}$ and update $r(x, C)$ and n .
 - 4: **else**
 - 5: **return** x .
 - 6: **end if**
 - 7: **while** $r(x, C) > 0$ and $n < T$ **do**
 - 8: Use Algorithm 1 to activate all time intervals t with $\alpha_t = \alpha_n$, update n and $r(x, C)$.
 - 9: **if** $r(x, C) = 0$ **then**
 - 10: **return** x
 - 11: **end if**
 - 12: Use Algorithm 2 to get the standard solution on n time intervals and update $r(x, C)$.
 - 13: **if** $r(x, C) = 0$ **then**
 - 14: **return** x
 - 15: **end if**
 - 16: Use algorithm 3 to activate time interval $n + 1$, update n and $r(x, C)$.
 - 17: **end while**
 - 18: **if** $r(x, C) = 0$ **then**
 - 19: **return** x
 - 20: **else**
 - 21: Use Algorithm 5 to charge the remaining $r(x, C)$ on the time intervals $1, \dots, T$.
 - 22: **return** x
 - 23: **end if**
-

5 Conclusion

The idea of solving CPLBs like we did in this thesis, i.e. by add charge a bit more in every step, initially seemed less complex than it turned out to be. When we added the lower bound, we had to distinguish between a lot of different corner cases. Thus, determining the borders from which we have to activate the next time interval, in order to keep the solution optimal, became more difficult. Also, determining the time intervals which have to be lowered to activate the next time interval was not as easy as expected. We showed it is possible to solve this problem to optimality. However, because of the large expressions we obtained for the borders for $C(X)$, which denotes the total charging required before activating a new interval in the solution, we only showed that these borders must exist without determining all the borders exactly.

In practice, we can assume that the steering signals differ sufficiently. Hence, we can consider the case where the α 's are either equal or differ by a minimum amount of X_{\min} for each time interval. That is why, in Section 4, we considered a special case of CPLB, called CPLBS. This special case appeared to be more easy to solve; a lot of corner cases disappeared. Also, in this variant it was a lot easier to determine the time intervals which have to be lowered to activate the next time interval. For the CPLBS we determined the borders from which we have to activate the next time interval in all cases exactly. Using these borders we constructed an algorithm that solves the CPLBS to optimality and in polynomial time.

5.1 Future work

The original idea, when I started working towards this thesis, was to look at a variant of the CPLB, in which there are always at least two consecutive time intervals on which the EV is charging. So when we want to charge a positive amount on a certain time interval, we have to make sure a positive amount is also charged on the time interval before or after that time interval. This way the battery is not switched on and off too much, because that can have negative side-effects for the battery.

Because the original CPLB already was more difficult to solve than expected, we did not get the chance to look into this problem. In this problem we only get useful solutions if we have a minimum amount X_{\min} that has to be charged on each time interval, otherwise we can charge an arbitrary low amount on time intervals t with a large value for α_t . However, the way we solved the CPLB and CPLBS, maybe can help to solve this problem in the future. That is also why we used this method in the first place. There are some problems that arise immediately, when considering this variant.

1. When we activate a new time interval t and neither the time interval $t - 1$ nor the time interval $t + 1$ is charging a positive amount, we have to activate one of these time intervals too. So, sometimes we have to activate two time intervals at a time.
2. In this thesis we ordered the time intervals in a way that $\alpha_t \geq \alpha_s$ for time intervals $t > s$. However in this ordering, two consecutive time intervals do not necessarily have to be consecutive in time. So, in this variant there does not always exist an optimal non-increasing solution, with respect to a non-decreasing α .

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