Hoffman colorings

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Abstract

We study equality in the Hoffman bound for the chromatic number and Hoffman colorings in regular and irregular graphs. We investigate the connection between Hoffman colorability and several graph operations, of which the tensor product is especially interesting in this context. We then introduce the Decomposition Theorem revealing structural properties that Hoffman colorings must obey. Using the Decomposition Theorem we are able to completely classify Hoffman colorability of cone graphs and line graphs. We also prove a partial converse, the Composition Theorem, allowing us to find various new infinite families of Hoffman colorable graphs, many of which are irregular. Lastly we introduce a new parameterization and type system for strongly regular graphs, that show connections between Hoffman colorability, spreadability, pseudo-geometricity and unique vector colorability.

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Chapter 1

Introduction

Algebraic graph theory lies in the intersection of linear algebra and graph theory. The point of this area of mathematics is to apply the powerful tools of linear algebra to the mysterious world of graphs. One way of doing this is by capturing the information of a graph in a matrix, after which we can compute the eigenvalues of this matrix. The big question is, how do the eigenvalues correspond with combinatorial properties of a graph? These relations are important, for one because the eigenvalues of a matrix can be computed in polynomial time, whereas various combinatorial invariant might not be easily computable, but there are also some theoretical applications to other fields, like the Erdős-Ko-Rado Theorem in extremal set theory (see [16, Theorem 7.8.1]).

Before we explore the intersection of linear algebra and graph theory, let's first settle some notation in graph theory. In this thesis, a graph G consists of a set of vertices $V(G)$ together with a symmetric, irreflexive relation $\sim_G$ on the vertices, called the *adjacency relation*. If it is clear from the context, we will omit the G. We will also need the set of edges $E(G)$ of a graph, which is a set consisting of the pairs of adjacent vertices of the graph. A graph without any edges is called *empty*.

Let's return to the algebraic setting. As said before, we would like to capture the information of a graph in a matrix. One way of doing this is the following. Let G be a graph. The *adjacency matrix* $A(G)$ of G is defined as

$$A(G)_{u,v} = \begin{cases} 1 & \text{if } u \sim_G v, \\ 0 & \text{otherwise.} \end{cases}$$

So the entry of $A(G)$ corresponding to the pair of vertices (u, v) is 1 if u and v are adjacent in the graph and 0 otherwise. Since A is a real, symmetric matrix, it has an orthogonal eigenbasis over $\mathbb{R}$. The *spectrum* $\mathrm{Spec}(G)$ of a graph G is the multiset of eigenvalues of its adjacency matrix. We write $\lambda_{\min}(G)$ for the least eigenvalue of the adjacency matrix of G, and $\lambda_{\max}(G)$ for the greatest eigenvalue.

There are other matrices of interest in algebraic graph theory, most notably the different Laplacians (standard, signless, normalized), but the adjacency matrix is the most elementary way and the one that has attracted the most attention in the literature, and for these reasons we will focus on the adjacency matrix only in this thesis.

We will now look at three spectral bounds using the adjacency matrix, of which two are with respect to the independence number. The third bound is the Hoffman bound on the chromatic number, which is the main point of interest of this thesis.

Independence number: ratio bound and inertia bound

The following spectral bound on the independence number of a graph was first stated and proved by Delsarte in [10] for strongly regular graphs. Later, although this was not published, Hoffman proved it for all regular graphs. In 2021, Haemers wrote a note [20] clarifying the history of the bound. The inequality is called the *ratio bound.*

Theorem 1.1 (Ratio bound, [20])**.** *Let G be a regular graph. Then*

$$\alpha(G) \leq \frac{n\lambda_{\min}(G)}{\lambda_{\min}(G) - k},$$

where n is the number of vertices of G and k is the valency of G. If a coclique C meets this bound, then every vertex not in C is adjacent to precisely $-\lambda_{\min}(G)$ vertices in C.

Cocliques meeting the ratio bound are called *Delsarte-Hoffman cocliques.* A clique that turns into a Delsart-Hoffman coclique in the complement is called a *Delsarte-Hoffman clique.*

Another bound on the independence/clique number is the inertia bound, introduced by Cvetković.

Theorem 1.2 (Inertia bound, [9])**.** *The independence number of a graph is bounded above by the minimum of the number of non-negative eigenvalues of the adjacency matrix and the number of non-positive eigenvalues of the adjacency matrix (counted with multiplicity).*

Either of these bounds can be used for a concise algebraic proof of the Erdős-Ko-Rado Theorem (see [16, Theorem 7.8.1]), which gives an upper bound for the independence number of a Kneser graph. This is one of the most fundamental results in extremal set theory.

For a comparison of the two spectral bounds on the independence number (also extended to weighted adjacency matrices), we refer to [23]. In this paper, Kwan and Wigderson find a familiy of graphs (C_4-free graphs with independence number asymptotically less than the number of vertices) where the inertia bound dramatically undeeperforms the ratio bound. In general however, it is not so clear which bound performs better than the other, since there are also many cases in which the inertia bound is better.

Chromatic number: Hoffman's bound

In 1970, Alan J. Hoffman gave the following spectral bound on the chromatic number of a graph.

Theorem 1.3 (Hoffman bound, [21]). *If G is a non-empty graph, then*

$$\chi(G) \geq 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}.$$

For regular graphs, this follows from the ratio bound, using the inequality

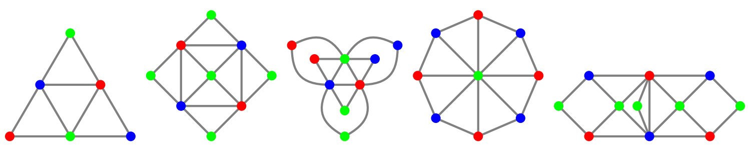
$$|V(G)| \leq \chi(G)\alpha(G),$$

which holds for every graph since $V(G)$ can be partitioned into $\chi(G)$ independent sets. However, for irregular graphs this is a new result, as the ratio bound does not hold necessarily for irregular graphs.

The Hoffman bound has seen various applications. In [17], Haemers classifies which strongly regular graphs have chromatic number equal to 4, using the Hoffman bound as a key ingredient. In [11], this is extended to 5 colors, but the classification is not complete. In [5], this idea is taken into a different direction, namely to classify all distance regular graphs with chromatic number 3.

For any inequality, one might ask in which situations the bound is tight. In general, this is an important question, because investigating which situations give tightness indirectly also investigates in which situations the bound might be improved. More specifically, studying equality of the Hoffman bound also has another point of interest. As we will see, various graph parameters are sandwiched in between the Hoffman bound and the chromatic number, for example the vector chromatic number and the quantum chromatic number. The quantum chromatic number is not known to be computable in general, but if the Hoffman bound is tight then it is.

With this motivation in mind, in this thesis we study the question of tightness of the Hoffman bound. If the Hoffman bound of a graph is equal to its chromatic number, then we call the graph *Hoffman colorable* and every optimal coloring a *Hoffman coloring*. A complete characterization of Hoffman colorability seems to be far out of reach at this point of time.



Since the smallest irregular Hoffman colorable graphs are very structured, as shown by the above examples, the original goal of this thesis was to investigate the structure of Hoffman colorings in irregular graphs. However, while writing the thesis it turned out to be natural to also include regular graphs.

Although completely classifying Hoffman colorability is not feasible, in this thesis we take some significant steps in this direction and we will also see a detailed description of the structure of Hoffman colorings, by way of the Decomposition and Composition Theorems that can be found in Chapters 6 and 7 (Theorem 6.1 and Theorem 7.1).

Outline and summary of new results of this thesis

This thesis is organized as follows.

In Chapter 2, we will set out all the preliminary definitions.

In Chapter 3 we will explore the literature on what has been done previously on the problem of classifying Hoffman colorability.

In Chapter 4, we explore the power of the Hoffman bound when seen as just another graph parameter, like the independence number and the chromatic number. We will classify Hoffman colorability of various graph operations. Especially the tensor product of graphs is suited to Hoffman colorability very well (Proposition 4.6). In fact, we find infinitely many irregular Hoffman colorable graphs here, which is a new result. Furthermore, in Theorem 4.12 we prove a new necessary and sufficient condition for strong regularity, only depending on the Hoffman bound of a regular graph and its complement.

In Chapter 5 we study strongly regular graphs using a new parameterization of strongly regular graphs, which we call the "geometric" parameterization. This parameterization originates from partial geometries and it is distinct from the standard combinatorial parameterization and the spectral parameterization of strongly regular graphs. Using this new parameterization we prove that pseudo-geometricity of strongly regular graphs is expressible solely using the Hoffman bound (Proposition 5.5). Also, using the geometric parameters, we introduce a type system for strongly regular graphs using eight different types and classify the smallest strongly regular graphs into these types. The idea of these types is to capture Hoffman colorability, the existence of Delsarte-Hoffman cliques and pseudo-geometricity all at once, since these three concepts interact a lot.

Chapter 6 contains the first of the central new results of this thesis, the Decomposition Theorem 6.1. Very briefly, the Decomposition Theorem describes the structure of Hoffman colorings by decomposing a Hoffman colorable graph into a collection of compatible bipartite parts. The Decomposition Theorem has various consequences. For instance, we were able to completely classify Hoffman colorability of cone graphs (Corollary 6.14) and line graphs (Theorem 6.22), and also Hoffman colorability of graphs where the number of vertices is less than three times the chromatic number (Corollary 6.16).

In Chapter 7, we will meet the other central new result, the Composition Theorem 7.1. The Composition Theorem is a partial converse to the Decomposition Theorem, and it shows that if you compose a graph out of a collection of compatible bipartite parts, there is only one requirement needed for Hoffman colorability: that the least eigenvalue of the graph is what you want it to be. This theorem is very useful for constructing various infinite families Hoffman colorable graphs, and that is exactly what we will do in the remaining of the chapter.

Furthermore, the Decomposition and Composition Theorems form the basis of an algorithm that we wrote for computing every connected Hoffman colorable graph given a number of vertices and a chromatic number. The algorithm and its results are discussed in Chapter 8.

Chapter 2

Preliminaries

In this chapter we will set out all the necessary definitions for the content of this thesis. We will discuss the eigenvalues and chromatic number of various standard graph families and graph operations first, and here we will encounter two classes of graphs that are trivially Hoffman colorable, namely regular complete multipartite graphs and bipartite graphs. Then we will investigate the Interlacing Theorem, which provides many spectral inequalities for graph theory. After that, we will study strongly regular graphs, which is an algebraically interesting class of graphs, and finite geometry, a great source for strongly regular graphs. Lastly, we will consider some interesting variations on the chromatic number.

Recall that the overall goal is to study the relation between the algebraic properties of a graph and its combinatorial properties. Of main interest in this thesis will be the independence number, the clique number and the chromatic number of a graph. The *independence number* $\alpha(G)$ of a graph G is the size of the largest independent set in the graph, where an *independent set* is a subset I of the vertices of the graph such that no pair of vertices from I is adjacent. Similarly, a *clique* is a subset of the vertices such that every pair of vertices from it is adjacent. The *clique number* $\omega(G)$ of a graph is the size of the largest clique in G. If we denote $\overline{G}$ for the complement of G (that is, the graph on the same vertex set, such that a pair of vertices is adjacent in the complement $\overline{G}$ if and only if they are not adjacent in G), then we see that $\alpha(G) = \omega(\overline{G})$, since independent sets turn into cliques in the complement, and vice versa. In this context, independent sets are also called *co-cliques*. Another graph theoretic concept can be expressed with independent sets. A *c-coloring* of a graph G is a partition of the vertex set $V(G)$ into c independent sets (each independent set belonging to a color). Equivalently, a c-coloring is a function $f : V(G) \to C$ (where C is a set of c colors), assigning a color to every vertex such that adjacent vertices get different colors. The set of vertices of a specific color is an independent set, hence the two expressions are equivalent.

We also include here some basic facts on the adjacency matrix. Note that the adjacency relation of a graph is assumed to be irreflexive, so that the diagonal of the adjacency matrix only contains zeroes. As a consequence, the trace is zero, and so the eigenvalues of any graph sum to 0.

Further, the eigenvectors of the adjacency matrix have a particularly nice combinatorial interpretation. If we consider the vector space of $\mathbb{R}$-valued function with the vertices $V(G)$ as domain, then A can be seen as a linear operator on this space. If $x \in \mathbb{R}^{V(G)}$, then

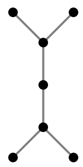
$$(Ax)(u) = \sum_{v \in V(G)} A_{uv} x(v),$$

but A_{uv} only contributes to the sum if u and v are adjacent. Write $N(u)$ for the *neighborhood* of u; that is, the set of vertices that are adjacent to u (the *neighbors* of u). Then

$$(Ax)(u) = \sum_{v \in N(u)} x(v).$$

So the linear operator A sends a function x to the function where the u-value is determined by the sum over all the neighbors of the values of x. If x is an eigenvector, then this must result in a scalar multiple of x. We can therefore interpret an eigenvector of A with eigenvalue λ as an assignment of a real number to each vertex, such that for every vertex u, the sum of the numbers assigned to the neighbors of u is exactly λ times the number assigned to u.

Example 2.1. As an example, consider the following graph.



Consider the function assigning 1 to the four leaves (a *leaf* is a vertex adjacent to only one other vertex), and 2 to the other three vertices. One can check that this function defines an eigenvector. The eigenvalue corresponding to this eigenvector is 2.

2.1 Chromatic number and eigenvalues of standard graph families and operations

The simplest graphs (from the spectral perspective) are the empty graphs (graphs with no edges). Note that empty graphs have the null matrix as adjacency matrix, and therefore have a trivial spectrum: every eigenvalue is 0. Conversely, if a graph has only 0 as eigenvalue, then their adjacency matrix must be the null matrix, so the graph cannot have any edges. The empty graphs are precisely the graphs with chromatic number 1.

In this section we will investigate the chromatic number and the spectra of various graph families, and various graph operations. We will start by stating the Perron-Frobenius Theorem and its applications to regular graphs. We will then consider several families of regular graphs, where we establish Hoffman colorability of one of the two classes of trivially Hoffman colorable graphs, namely the regular complete multipartite graphs. After that we will study bipartite graphs, the second class of trivially Hoffman colorable graphs; in general and for specific subclasses.

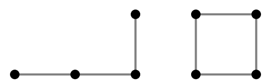
We conclude this section by investigating the effects of various graph operations on the spectrum and the chromatic number. Most of the content of this section is from [7].

2.1.1 The Perron-Frobenius Theorem and regular graphs

A very important result in algebraic graph theory is the Perron-Frobenius Theorem applied to the adjacency matrix, because of its relevance to regular graphs and to positive eigenvectors of irregular graphs. The Perron-Frobenius Theorem is stated for connected graphs. However, after spectrally studying the disjoint union of graphs we can also use the theorem for disconnected graphs.

A graph is *connected* if from any vertex one can reach any other vertex by moving between adjacent vertices. A graph is *disconnected* if this is not the case; that is, if there exists a pair of vertices u, v for which it is impossible to walk from u to v only by traversing edges of the graph.

A *subgraph* H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$. A connected subgraph that is maximal (that is, there exists no connected graph K with $H \subsetneq K \subseteq G$) is called a *component*. Equivalently, a component consists of every vertex that is reachable by a walk from a given vertex. A graph is connected if and only if it has just one component.



a disconnected graph with two components

The *disjoint union* of two graphs G and H, is the graph K with $V(K) = V(G) \sqcup V(H)$ and $E(K) = E(G) \sqcup E(H)$. We write $K = G \sqcup H$. Note that G can be written as the disjoint union of all of its components.

Connected graphs

If a graph is non-empty and connected, the Perron-Frobenius Theorem (see [7, Theorem 2.2.1]) applies to the adjacency matrix. We obtain the following.

Theorem 2.2 (Perron-Frobenius Theorem)**.** *Let G be a non-empty connected graph. Then the following hold.*

(i) The largest eigenvalue $\lambda_{\max}(G)$ is simple (called the Perron eigenvalue*), and it has a positive eigenvector (called the* Perron eigenvector*);*

(ii) If G is bipartite, then $\lambda_{\min}(G) = -\lambda_{\max}(G)$;

(iii) If G is not bipartite, then every eigenvalue of G other than the Perron eigenvalue has absolute value strictly bounded above by the Perron eigenvalue.

A vector is called *positive* if every entry of the vector is positive. Note that if we have a positive eigenvector x distinct from the Perron eigenvector p, then since their inner product is positive, they are not orthogonal and hence have the same eigenvalue. This eigenvalue must be the Perron eigenvalue, but this was a simple eigenvalue, so x is actually just a scalar multiple of p. This means that (up to rescaling) the Perron eigenvector is the unique positive eigenvector.

Disconnected graphs and disjoint unions

If we write $K = G \sqcup H$ for the disjoint union of G and H, then we get the following adjacency matrix for K.

$$A(K) = \begin{pmatrix} A(G) & 0 \\ 0 & A(H) \end{pmatrix}$$

Therefore, the multiset of eigenvalues of a disjoint union $K = G \sqcup H$ is the multiset-theoretic union of the eigenvalues of G and H. As a consequence, $\lambda_{\max}(K) = \max(\lambda_{\max}(G), \lambda_{\max}(H))$ and $\lambda_{\min}(K) = \min(\lambda_{\min}(G), \lambda_{\min}(H))$.

It is easy to verify that $\chi(K) = \max(\chi(G), \chi(H))$, since any coloring of the disjoint union K reduces to a coloring of both G and H. Actually, something more general is true; the disjoint union is the coproduct in the category of graphs. We will meet the categorical product in Section 2.1.3.

When we apply the Perron-Frobenius Theorem to the components of a graph, we get some useful information for disconnected graphs as well. We can for example classify this way which graphs have a positive eigenvector and what this positive eigenvector should look like. This is important, as for the Decomposition Theorem we require the graph to have a positive eigenvector; we need to know about the scope of the theorem.

Corollary 2.3. *Let G be a graph. Then G has a positive eigenvector x if and only if the largest eigenvalues of the components of G are all equal. In this case, the eigenvalue for x will be the largest eigenvalue of G, and furthermore, if G is non-empty, for every component C the restriction $x|_C$ is the Perron eigenvector of C.*

Proof. If G has no edges, then any vector is an eigenvector of G so G certainly has a positive eigenvector. Note further that the maximum eigenvalue of G is equal to the maximum eigenvalue of every component, namely 0. So, from now on assume that G is non-empty.

Suppose that the largest eigenvalue of every component is equal to, say, λ. The concatenation of the Perron eigenvectors of the components will give an eigenvector of G for λ, and this vector will be positive.

Conversely, assume that x is a positive eigenvector of G of eigenvalue $\lambda(x)$. Since G is non-empty, we must have $\lambda(x) > 0$. For every component C, the restriction $x|_C$ will then also be an eigenvector for $\lambda(x)$, but now for the graph C. Since x is positive, also $x|_C$ is positive, and since $\lambda(x) > 0$, C cannot be empty. Hence $x|_C$ is the Perron eigenvector of C and therefore $\lambda(x) = \lambda_{\max}(C)$. Since

$$\lambda_{\max}(G) = \max_{C \text{ component}} (\lambda_{\max}(C)),$$

we must have $\lambda(x) = \lambda_{\max}(G)$ as well. □

Note that whereas positive eigenvectors of connected graphs were unique, this is not the case anymore for disconnected graphs as we can rescale the Perron eigenvectors of all the components as we wish. In fact, if a graph G has a positive eigenvector, then the dimension of the eigenspace of the largest eigenvalue of G is equal to the number of components of G, as every component provides a unique eigenvector (namely the Perron eigenvector) for this eigenspace.

Regular graphs

The Perron eigenvector/positive eigenvector of a regular graph behaves particularly nicely. Let $v \in V(G)$ be a vertex of a graph G. Recall that the neighborhood $N(v)$ of v is the set of vertices adjacent to v. The *degree* $\deg(v)$ of v is the size of its neighborhood, or equivalently it is the number of neighbors of v. We write Δ for the maximum degree. A graph is *regular* if and only if every vertex has degree Δ. In this case, the *valency* is equal to Δ, but in this context it is usually denoted k.

Given a regular graph, consider the constant function $\mathbb{1}$, assigning 1 to every vertex. Then

$$(A\mathbb{1})(u) = \sum_{v \in V(G)} A_{u,v}\mathbb{1}(v) = |N(u)| = k,$$

so $\mathbb{1}$ is a positive eigenvector for G with eigenvalue k. So k is the largest eigenvalue of G.

If G is regular, and has eigenvalues

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 < k,$$

then the complement $\overline{G}$ has eigenvalues

$$-1 - \lambda_2 \leq -1 - \lambda_3 \leq \cdots \leq -1 - \lambda_n \leq n - k - 1.$$

This is because

$$A(\overline{G}) + A(G) + I = J,$$

where J is the all-ones matrix, (see [7, Section 1.3.2]). For irregular graphs we do not have this particularly nice relation between the eigenvalues of a graph and the eigenvalues of its complement, as we do not have a constant eigenvector. This constant eigenvector is needed, because of the all-ones matrix J.

Regular graphs are a very special class of graphs in algebraic graph theory, as many of the different matrices that capture the structure of graphs are equivalent for regular graphs. For this thesis, regular graphs are of interest because the ratio bound applies to regular graphs. Next we study various regular families of graphs.

Complete graphs

A *complete* graph is a graph where every pair of vertices is adjacent. The complete graph on c vertices is denoted K_c. Note that the complete graph K_c is $c-1$-regular. The complement is the empty graph. By the previous section, the spectrum is given by:

$$\mathrm{Spec}(K_c) = \{c - 1, (-1)^{c-1}\}.$$

We can also explicitly write down the spectrum by finding eigenvectors. Since complete graphs are regular, the Perron eigenvector will be the constant vector. The eigenspace for the eigenvalue -1 consists of all the vectors orthogonal to the constant vector, in other words all the vectors $x : V \to \mathbb{R}$ whose entries sum to 0. Indeed, in this case

$$(Ax)(u) = \sum_{v \in V} A_{uv} x(v) = \sum_{v \in V, u \neq v} x(v) = -x(u).$$

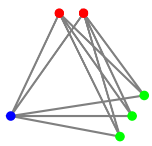
We can now compute the Hoffman bound (which we denote with a small h) of a complete graph, and it is equal to its number of vertices:

$$h(K_c) = 1 - \frac{c-1}{-1} = c.$$

The chromatic number of a complete graph is its number of vertices, as independent sets of complete graphs cannot contain two distinct vertices. We immediately see that complete graphs are Hoffman colorable.

Regular complete multipartite graphs

In a *complete multipartite graph*, the vertex set is partitioned into classes, such that two vertices are adjacent if and only if they belong to different classes. If the classes are of size $m_1, \dots, m_c$ respectively, then we write $K_{m_1,\dots,m_c}$.



the complete tripartite graph $K_{1,2,3}$

The complement of a complete multipartite graph is a disjoint union of complete graphs. A complete multipartite graph is regular if and only if the classes are of the same size, and in this case the complement will consist of complete graphs of the same size. If we write c for the number of classes of the regular complete multipartite graph, and m for the class size, we get

$$\overline{G} = \bigsqcup_{i=1}^{c} K_m.$$

Now the spectra of the complement is given by:

$$\mathrm{Spec}(\overline{G}) = \{(m-1)^c, (-1)^{n-c}\},$$

which is just c times the spectrum of K_m. Therefore, the spectrum of the regular complete multipartite graph is given by

$$\mathrm{Spec}(G) = \{(c-1)m, 0^{c(m-1)}, (-m)^{c-1}\}.$$

We can now easily calculate the Hoffman bound to be c, equal to the chromatic number. Therefore regular complete multipartite graphs are Hoffman colorable.

2.1.2 Bipartite graphs and cycles

A graph is *bipartite* if its chromatic number is at most 2. A non-empty bipartite graph has chromatic number 2. If G is bipartite, we can partition the vertices $V(G) = V_1 \sqcup V_2$ into two independent sets. All edges now have one endpoint V_1 and the other endpoint in V_2.

If x is an eigenvector for a bipartite graph G with eigenvalue λ, then the vector y defined by

$$y(u) = \begin{cases} x(u) & \text{if } u \in V_1, \\ -x(u) & \text{if } u \in V_2. \end{cases}$$

gives an eigenvector of eigenvalue $-\lambda$. This implies that the spectrum of bipartite graphs is symmetric around 0:

$$\{-\lambda : \lambda \in \mathrm{Spec}(G)\} = \mathrm{Spec}(G).$$

Therefore, the greatest eigenvalue and the least eigenvalue are additively inverse to each other, as we have already seen for connected bipartite graphs from the Perron-Frobenius Theorem. As a consequence, the Hoffman bound of bipartite graphs is 2. Since the chromatic number is also 2, all non-empty bipartite graphs are Hoffman colorable.

Furthermore, if x is an eigenvector as above and $\lambda \neq 0$, then y is an eigenvector to a different eigenvalue, hence x and y are orthogonal. Writing out the inner product, we get

$$\sum_{u \in V_1} x(u)^2 = \sum_{u \in V_2} x(u)^2.$$

In other words, the two projections of an eigenvector to the two bipartite classes have equal norm. This fact will be important to Lemma 6.8 and to the Composition Theorem.

Biregular graphs

We will need biregular graphs for Construction 7.17 and also for Proposition 7.12. A graph is *biregular* if it is bipartite, and there exist positive integers k_1, k_2 such that the degree of any vertex in color class V_i is equal to k_i. A simple counting argument shows that $k_1|V_1| = k_2|V_2|$.

For this class of graphs, we can guess an eigenvector for the largest eigenvalue. Since regular graphs have a constant eigenvector for their largest eigenvalue, we might guess a vector x by

$$x(v) := \begin{cases} \frac{1}{\sqrt{|V_1|}} & \text{if } v \in V_1, \\ \frac{1}{\sqrt{|V_2|}} & \text{if } v \in V_2, \end{cases}$$

since the norms of the projections of the vectors on the two color classes must be equal (we just set it to 1). Now let $v \in V_1$, then the sum of the entries of x of the neighbors of v is precisely $\frac{k_1}{\sqrt{|V_2|}}$, which is equal to $\sqrt{k_1 k_2} \cdot x_v$. Similarly if $v \in V_2$, so

x is an eigenvector for the graph with eigenvalue $\sqrt{k_1 k_2}$. Since we have a positive eigenvector, this is the maximal eigenvalue.

The *complete bipartite graphs* are a special class of biregular graphs. In complete bipartite graphs, there are two vertex classes and two vertices are connected if and only if they belong to different classes. In this case, k_1 is equal to $|V_2|$ and k_2 is equal to $|V_1|$. The Perron eigenvalue will be $\sqrt{|V_1||V_2|}$ in this case.

Cycle graphs

Let $n \geq 3$. The *cycle graph* on n vertices C_n is the graph with vertices indexed by $\mathbb{Z}/n\mathbb{Z}$, where every a is adjacent to $a+1$ and $a-1$. If n is even, then by having all even-indexed vertices in one class and all odd-indexed vertices in the other class, C_n is bipartite. If n is odd, this does not work anymore and $\chi(C_n) = 3$. For the eigenvalues we have the following proposition.

Proposition 2.4 ([7, Section 1.4.3]). *The cycle on n vertices has adjacency eigenvalues*

$$\lambda = 2\cos\left(\frac{2k\pi}{n}\right),$$

for $k = 1, \ldots, n$.

It is now clear that $\lambda_{\max}(C_n) = 2$ and $\lambda_{\min}(C_n) = 2\cos\left(\frac{2k\pi}{n}\right)$ for k an integer as close to $\frac{1}{2}n$ as possible. For even cycles, we can take $k = \frac{1}{2}n$, and we get $\lambda_{\min}(C_n) = -2$, which checks out with the bipartiteness of C_n. If n is odd however, then we get that $-2 < \lambda_{\min}(C_n) \leq -\frac{1}{2}$, with equality on the right only for $n = 3$. The Hoffman bound for odd cycles is now $2 < h(C_n) \leq 3$, with equality on the right only for $n = 3$. So C_3 is Hoffman colorable, but all other odd cycles are not.

Path graphs

The *path graph* on n vertices P_n is the graph with vertices n vertices $v_1, \ldots, v_n$ such that

$$v_1 \sim_G v_2 \sim_G \cdots \sim_G v_n,$$

in a chain. By having all even-indexed vertices in one class and all odd-indexed vertices in the other class, it is clear that path graphs are bipartite. For the spectrum, we have the following proposition.

Proposition 2.5 ([7, Section 1.4.4]). *The path graph on n vertices P_n has n simple adjacency eigenvalues, given by*

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right),$$

for $k = 1, \ldots, n$.

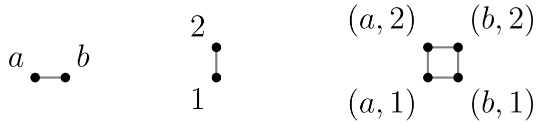
In particular, the largest eigenvalue of P_n is $2\cos\left(\frac{\pi}{n+1}\right)$. As a consequence, we can read off the length of a path from its largest eigenvalue. This fact will be an essential part of the classification of Hoffman colorability of line graphs. Namely, it is the main ingredient of Corollary 6.17.

2.1.3 Graph operations

We study various graph operations and the effects on the chromatic number and the eigenvalues. We already covered the disjoint union above, which is a binary graph operation, as it was needed for the discussion of disconnected graph. Here we will study the Cartesian product, the tensor product and the line graph. We will revisit these in Chapter 4, when we study the interaction of these operations and the Hoffman bound.

Cartesian product

The *Cartesian product* of two graphs G and H is the graph K with vertices $V(K) = V(G) \times V(H)$ such that $(u, v) \sim_K (w, x)$ whenever $u = w$ and $v \sim_H x$ or $u \sim_G w$ and $v = x$. We write $K = G \square H$, inspired by $K_2 \square K_2$, which looks like $\square$:



Cartesian product of K_2 with itself

The multiset of eigenvalues of K is the multiset of sums of eigenvalues of G and H (see [7]):

$$\mathrm{Spec}(K) = \{\lambda + \mu : \lambda \in \mathrm{Spec}(G), \mu \in \mathrm{Spec}(H)\}.$$

This implies that

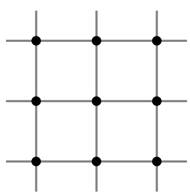
$$\lambda_{\max}(K) = \lambda_{\max}(G) + \lambda_{\max}(H)$$

and

$$\lambda_{\min}(K) = \lambda_{\min}(G) + \lambda_{\min}(H).$$

In [30, Lemma 2.6] it was shown that $\chi(G \square H) = \max(\chi(G), \chi(H))$. We give the short proof here. Note that G and H are subgraphs of $G \square H$, so certainly $\chi(G \square H) \geq \max(\chi(G), \chi(H))$. In fact this is an equality; take a pair of colorings $f_G : V(G) \to \mathbb{Z}/c\mathbb{Z}$ and $f_H : V(H) \to \mathbb{Z}/c\mathbb{Z}$, where $c = \max(\chi(G), \chi(H))$, then we can define $f_K : V(K) \to \mathbb{Z}/c\mathbb{Z}$ by $f_K(u, v) = f_G(u) + f_H(v)$, and this defines a valid c-coloring.

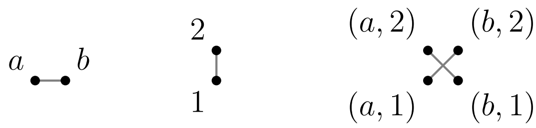
A nice example of graphs obtained from a Cartesian product are the *square lattice graphs*. The square lattice graph of order m can be obtained as $K_m \square K_m$, or alternatively as a line graph, which we will see later.



the square lattice graph of order 3

Tensor product

The *tensor product*, *direct product*, *categorical product* or *Kronecker product* of two graphs G and H is the graph K with vertices $V(K) = V(G) \times V(H)$ such that $(u, v) \sim_K (w, x)$ whenever $u \sim_G w$ and $v \sim_H x$ simultaneously. We write $K = G \times H$, inspired by $K_2 \times K_2$, which looks like $\times$:



tensor product of K_2 with itself

The multiset of eigenvalues of K is the multiset of products of eigenvalues of G and H:

$$\mathrm{Spec}(K) = \{\lambda\mu : \lambda \in \mathrm{Spec}(G), \mu \in \mathrm{Spec}(H)\},$$

because the adjacency matrix of $G \times H$ is the matrix Kronecker product of the adjacency matrices of G and H (see [7, Section 1.4.7]). This implies

$$\lambda_{\max}(K) = \lambda_{\max}(G)\lambda_{\max}(H),$$

and

$$\lambda_{\min}(K) = \min(\lambda_{\min}(G)\lambda_{\max}(H), \lambda_{\max}(G)\lambda_{\min}(H)).$$

The tensor product is the category-theoretic product of the graphs. As a consequence, $\chi(K) \leq \min(\chi(G), \chi(H))$ (alternatively: the pullback through the projection of a c-coloring of G gives a c-coloring of K, and similarly for H). The famous Hedetniemi's conjecture from 1966 states that this is actually an equality. In 2019, Shitov refuted this conjecture in [32], finding a tensor product with a chromatic number lower than the chromatic number of the components. However, the inequality still holds.

The tensor product $K_3 \times K_3$ turns out to be isomorphic to the Cartesian product $K_3 \square K_3$.

Line graph

If G is a graph, then we define the *line graph* $L(G)$ such that $V(L(G)) = E(G)$, and two edges are adjacent in the line graph if they share a vertex. Write N for the incidence matrix of G, so that $N_{v,e} = 1$ if vertex v lies on edge e and 0 otherwise. If B is the adjacency matrix of $L(G)$, A the adjacency matrix of G, then we get $N^T N = B + 2I$, and $NN^T = D + A$, where D is the diagonal matrix of the degrees of the vertices of G. The non-zero eigenvalues of $N^T N$ are the same (and of the same multiplicity) as the non-zero eigenvalues of NN^T. We have the following proposition.

Proposition 2.6 ([7, Proposition 1.4.1]). *Suppose G has m edges, and let $\rho_1 \geq \cdots \geq \rho_r$ be the positive eigenvalues of NN^T. Then the eigenvalues of $L(G)$ are $\theta_i = \rho_i - 2$ for $i = 1, \ldots, r$ and $\theta_i = -2$ for $i = r + 1, \ldots, m$.*

In particular, every eigenvalue of a line graph is at least -2.

Corollary 2.7 ([7, Corollary 1.4.2])**.** *If G is k-regular with $k \geq 2$ on n vertices and $m = kn/2$ edges and eigenvalues $\lambda_1 \geq \ldots \lambda_n$, then $L(G)$ is $(2k-2)$-regular and has eigenvalues $\lambda_i + k - 2$ and then -2 of multiplicity $m - n$.*

The chromatic number of the line graph of G is called the *edge chromatic number* or the *chromatic index* of G. By Vizing's famous result in [34], the chromatic index of a graph is either Δ or $\Delta + 1$, where Δ is the maximum degree. A graph where the chromatic index is equal to Δ is called *class 1* and a graph with chromatic index equal to $\Delta + 1$ is *class 2*.

Examples of class 1 graphs are K_{2m} and $K_{m,m}$. The line graph of $K_{m,m}$ is the square lattice graph, which can also be described as $K_m \square K_m$. Examples of class 2 graphs include the complete graphs of odd order K_{2m+1}.

2.2 Eigenvalue interlacing

Eigenvalue interlacing has a central role in algebraic graph theory, and it is the source of many interesting results. Hoffman's bound is an example of this, as it can be proved using this technique. Also the proof of the Decomposition Theorem, which we will see in Section 6.1, is based on interlacing.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \mu_m$ be two sequences of real numbers such that $m < n$. The latter sequence *interlaces* the former whenever

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \text{ for } i = 1, \ldots, m.$$

So the i'th largest member of the shorter sequence is at most the i'th largest member of the longer sequence, and also the i'th least member of the shorter sequence is at least the i'th least member of the longer sequence. The interlacing is called *tight* if there exists an integer $k \in \{0, \ldots, m\}$ such that $\mu_i = \lambda_i$ for $i \leq k$ and $\mu_i = \lambda_{n-m+i}$ for $i < k$. So that is if the k largest members of the shorter sequence are equal to the k largest members of the longer sequence, and also the $m-k$ least members of the shorter sequence are equal to the $m-k$ least members of the longer sequence. If $m = n-1$, then interlacing becomes

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_m.$$

We will study interlacing with respect to eigenvalues of matrices, hence we speak of *eigenvalue interlacing.*

2.2.1 The Interlacing Theorem

The following theorem, the Interlacing Theorem, explains why this previous definition is interesting.

Theorem 2.8 (Interlacing Theorem, [7, Theorem 2.5.1: (i),(iv)])**.** *Let S be an $n \times m$-matrix such that $S^T S = I$. Let A be a symmetric $n \times n$ matrix. Define $B = S^T AS$. Then the eigenvalues of B interlace those of A. If this interlacing is tight, then $SB = AS$.*

A particular case of this is called Cauchy interlacing. In this case, we take S to be the $m \times m$-identity matrix with a bunch of zeroes added to make it an $n \times m$-matrix.

Corollary 2.9 (Cauchy interlacing, [7, Corollary 2.5.2])**.** *If B is a principal submatrix of A, then the eigenvalues of B interlace those of A.*

This applies to graphs. Let X be a subset of the vertices of a graph G. Then the *induced subgraph of G on X* is the graph that has vertices X and all the edges between vertices of X that were in G. We write $G[X]$ for the induced subgraph of G on X. The adjacency matrix of an induced subgraph is a principal submatrix of the adjacency matrix of the original graph, so Cauchy interlacing applies. We obtain that the eigenvalues of $G[X]$ interlace those of G. This is a very powerful tool, which is exemplified in the following corollary.

Corollary 2.10 ([3, Proposition 6.1])**.** *If G is a non-empty graph, then $\lambda_{\min}(G) \leq -1$. If this is an equality, then G is a disjoint union of complete graphs.*

This result will ensure the positivity of the geometric parameters of strongly regular graphs that we will introduce in Section 5.1.

2.2.2 Partitions and interlacing

One particularly interesting application of interlacing, is the application to partitions and (weight-)quotient matrices. Weight-quotient interlacing and weight-regularity are the main ingredients of Proposition 3.2 ([1, Proposition 5.3(i)]), the interlacing proof of the Hoffman bound, and of Lemma 6.8, leading to the Decomposition Theorem. Before we go to the weighted definitions, let's first go over the standard quotient matrix.

Let G be a graph, and $\mathcal{P}$ a partition of $V(G)$ into $V_1, \ldots, V_m$. The *quotient matrix* given this partition is the matrix with entries

$$B_{ij} = \frac{1}{|V_i|} \mathbb{1}^T A_{ij} \mathbb{1},$$

which is the average row sum of A_{ij}, the submatrix of A indexed by the vertices in V_i and V_j. In other words, B_{ij} is the average over the vertices in V_i of the number of neighbors in V_j. A partition is *regular* or *equitable* if for each row of A_{ij}, the sum is equal to B_{ij}, or equivalently, if $A_{ij}\mathbb{1}$ is a constant vector. This is also equivalent to the number of neighbors in V_j of a vertex in V_i not depending on the specific vertex one chooses from V_i. In this case, the value of the constant vector $A_{ij}\mathbb{1}$ or the number of neighbors in V_j for a vertex in V_i is called the *intersection number* b_{ij}.

The class partition of complete multipartite graphs is an example of an equitable partition. Here the intersection number b_{ij} is equal to the size of class j.

Taking for S the characteristic matrix of this partition, the Interlacing Theorem implies the following.

Corollary 2.11 ([7, Corollary 2.5.4])**.** *Let G be a graph, with A its adjacency matrix. Let $\mathcal{P}$ a partition of $V(G)$, inducing the quotient matrix B. Then*

(i) The eigenvalues of B interlace those of A;

(ii) If the interlacing is tight, then the partition is regular.

In [12], Fiol uses weights, coming from a positive eigenvector of the graph, to "regularize" graphs. One interesting consequence is that we can consider weight-quotient matrices. Again, let $\mathcal{P}$ be a partition. Suppose x is a positive vector (we will mostly use an eigenvector of the graph). Write y_i for the projection of x onto color class V_i. So y_i is a vector indexed by the vertices in V_i, and $y_i(u) = x(u)$ for $u \in V_i$. Then the *weight-quotient matrix* given the partition is the matrix with entries

$$B^*_{ij} = \frac{1}{||y_i||^2} y_i^T A_{ij} y_j,$$

which is a weighted average of the weighted row sums. A partition is *weight-regular* or *weight-equitable* if the vector $A_{ij}y_j$ is a scalar multiple of y_i for every i and j. In this case, this scalar is called the *weight-intersection number* b^*_{ij} and it is equal to the ij-entry in B^*. For every pair of distinct colors i, j we must have in this case

$$x(u)b^*_{ij} = \sum_{v \in N_j(u)} x(v),$$

where u is a vertex of V_i, and $N_j(u)$ is the set of neighbors of u in V_j. If x is a constant vector, then all of the above is equivalent to a regular partition.

We can again use interlacing for weight-regularity, to get the following statement, which is analogous to Corollary 2.11.

Corollary 2.12 ([12, Lemma 2.3])**.** *Let G be a graph, with A its adjacency matrix. Let $\mathcal{P}$ be a partition of $V(G)$, inducing the weight-quotient matrix B given a positive eigenvector. Then*

(i) The eigenvalues of B interlace those of A;

(ii) If the interlacing is tight, then the partition is weight-regular.

2.3 Strongly regular graphs

In order to solve a difficult problem involving graphs and their eigenvalues, one should first try to solve the simplest cases. The simplest case of connected regular graphs is complete graphs, with just two eigenvalues, and the next case is strongly regular graphs, with just three eigenvalues. We will give a short introduction to strongly regular graphs. For a more thorough coverage, see Chapter 8 of [3], Chapter 9 of [7] or Section 9 of Chapter II of [4]. A graph G is *strongly regular* if there exist parameters (n, k, a, c) such that:

- The number of vertices $|V(G)|$ is equal to n;
- Every vertex in G is of degree k;
- Every pair of adjacent vertices in G have exactly a neighbors in common;
- Every pair of distinct non-adjacent vertices in G have exactly c neighbors in common.

Examples of strongly regular graphs are regular complete multipartite graphs, square lattice graphs ($L(K_{m,m})$), and the triangular graphs ($L(K_m)$). Other examples include the pentagon and the Petersen graph.

If G is strongly regular, then also $\overline{G}$ is strongly regular, with parameters

$$(n, n-k-1, n-2k+c-2, n-2k+a).$$

A strongly regular graph G is *primitive* if G and $\overline{G}$ are connected. The only imprimitive strongly regular graphs are regular complete multipartite graphs and their complements, namely disjoint unions of complete graphs of the same size. We state the following result on strongly regular graphs, which consists of various propositions of [7] and [3].

Proposition 2.13 ([7], [3]). *Let G be a strongly regular graph with parameters (n, k, a, c). Then the following hold.*

(i) $c(n-k-1) = k(k-a-1)$;

(ii) If G is primitive, then G has exactly three eigenvalues $\tau < \theta < k$ with $\tau < -1$ and $0 < \theta$, and

$$(x-\tau)(x-\theta) = x^2 + (c-a)x + (c-k);$$

(iii) Every regular connected graph that has at most three distinct eigenvalues is strongly regular;

(iv) The multiplicity of k is 1, and the multiplicities of τ and θ satisfy $m_\tau + m_\theta = n-1$ and $\tau m_\tau + \theta m_\theta = -k$.

2.4 Finite Geometry

In this thesis, we will touch on some finite geometry as well. The first reason is that finite geometry is a great source of strongly regular graphs in the shape of partial geometries, which we will study further in Section 3.2. In addition to partial geometries, we will also need block designs, as they capture precisely the requirements for Construction 7.17.

Both of these concepts are written in the language of incidence structures. An *incidence structure* is an ordered triple (P, B, I) such that $I \subseteq P \times B$. The elements of P are called *points* and the elements of B are called *blocks* or *lines*. If $(p, b) \in I$, then we say that p and b are *incident* to each other, and we write pIb.

Given an incidence structure, we can construct the bipartite *incidence graph* G with vertices $P \sqcup B$ and an edge $\{p, b\}$ whenever pIb. Here the incidence relation of the incidence structure acts as the adjacency relation of the graph.

2.4.1 Block designs

As said before, we will need block designs for the Construction 7.17 later, so let's briefly go over the definition of block designs. The interested reader should consult [4, Chapter I, Definition 2.9] for more details.

A *block design with parameters* v, k, λ is an incidence structure (P, B, I) such that the following three conditions are satisfied.

- $v = |P|$;
- Every block is incident to exactly k points;
- Any two distinct points are incident to exactly λ common blocks.

In a block design, the two parameters b, r can also be defined (see [4, Theorem 2.10]):

- Every point is incident to exactly r blocks, where $r = \lambda(v-1)/(k-1)$;
- There are precisely b blocks, where $b = \lambda v(v-1)/k(k-1)$.

For practical reasons, we usually speak of a (v, b, r, k, λ)-block design, rather than just a (v, k, λ)-block design, even though b and r are completely determined by the other parameters. For any block design, we must have $bk = vr$.

There is a duality in block designs: if (P, B, I) is a block design with parameters (v, b, r, k, λ), then we can construct a corresponding dual block design $(P, B, \overline{I})$, where $p\overline{I}b$ if and only if not pIb. So if a point and a block were not incident in the block design, they will be in the dual and vice versa. The parameters of the dual block design will be $(v, b, b-r, v-k, b-2r+\lambda)$.

For Construction 7.17, we will mostly only investigate trivial designs. An incidence structure where the incidence relation pairs up the points and blocks (and no other incidences occur) is a $(v, v, 1, 1, 0)$-block design. We will also consider the duals of these designs, which will have parameters $(v, v, v-1, v-1, v-2)$.

2.4.2 Partial geometries

A great source of strongly regular graphs is the theory of partial geometries. Also they will be the source of the geometric parameterization that we will introduce for strongly regular graphs in Section 5.1. Partial geometries and especially special cases of partial geometries can be defined in various contexts. The way we will do it is mostly based on [3]. However, it will contain results and definitions from [4], which in general covers these structures in a broader context. We will exclude partial geometries that are disconnected. Considering other combinatorial structures, like transversal designs, we will only consider those that are also partial geometries, and exclude those that are not a partial geometry. Interested readers should see [4] for the broader notion.

In the context of partial geometries, we do not speak of blocks but of *lines*. Two points are *collinear* if they lie on a common line. A *partial geometry* is defined as an incidence structure $(V, \mathcal{L}, I)$, where $I \subseteq V \times \mathcal{L}$ is an incidence relation, together with positive integral parameters (s, t, α), such that the following conditions are met.

- Every line $\ell \in \mathcal{L}$ is incident to $s+1$ points:

$$|\{v \in V : vI\ell\}| = s + 1;$$

- Every point $v \in V$ is incident to $t+1$ lines:

$$|\{\ell \in \mathcal{L} : vI\ell\}| = t+1;$$

- Every pair of distinct lines $\ell_1, \ell_2 \in \mathcal{L}$ intersects in at most one point:

$$|\{v \in V : vI\ell_1 \text{ and } vI\ell_2\}| \leq 1;$$

- Every pair of distinct points $v_1, v_2 \in V$ lies on at most one common line:

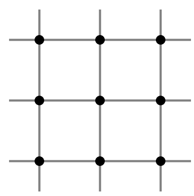
$$|\{\ell \in \mathcal{L} : v_1 I\ell \text{ and } v_2 I\ell\}| \leq 1;$$

- If point $v \in V$ is not incident to line $\ell \in \mathcal{L}$, then v is collinear with exactly α points on ℓ, or equivalently, ℓ intersects exactly α lines going through v:

$$|\{(\ell', v') : vI\ell',\ v'I\ell',\ v'I\ell\}| = \alpha.$$

It is clear that for any partial geometry it holds that $\alpha \leq 1 + \max(s,t)$. The dual of a partial geometry is formed by exchanging the roles of V and $\mathcal{L}$, and it is again a partial geometry but now with parameters (t,s,α). The number of points in an (s,t,α)-partial geometry is $(s+1)\frac{st+\alpha}{\alpha}$. The number of lines is $(t+1)\frac{st+\alpha}{\alpha}$. These need to be integers, so this excludes the feasibility of some parameter sets, such as $(s,t,\alpha) = (4,4,3)$.

The *point graph* or *collinearity graph* of a partial geometry is the graph with the points as vertices, and collinearity as the adjacency relation. For example, if we consider the partial geometry where the points are arranged in a $n \times n$-grid, and the lines are given by the rows and the columns, then the collinearity graph will be a square lattice graph. For example, for $n = 3$:



The *line graph* of a partial geometry is the graph with the lines as vertices, and edges whenever lines intersect. The line graph of a partial geometry G is the collinearity graph of the dual of G. Note that the graph-theoretic line graph of the collinearity graph is often very different from the geometric line graph. We have the following proposition.

Proposition 2.14 ([3, Theorem 4.1]). *The point graph of an (s,t,α)-partial geometry is strongly regular with parameters*

$$\left((s+1)\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

This result will be the basis of the geometric parameterization for strongly regular graphs in Section 5.1. A graph is called *geometric* if it is the collinearity graph of some partial geometry. If, for a strongly regular graph G, positive integers s, t, and α exist such that the parameters of G are given by the expressions in the above proposition, then G is called *pseudo-geometric*. Note that all geometric graphs are pseudo-geometric, but the converse is not true as we will see in Section 5.1.

Several geometric objects are of great interest to Hoffman colorings, which we will investigate further in Section 3.2. A *spread* or *parallel class* in a partial geometry is a collection of lines that covers every point exactly once. Dually, an *ovoid* is a collection of points such that every line is incident to exactly one point of the collection. Spreads correspond to ovoids in the dual partial geometry. A *resolution*, *parallelism* or *packing* is a partition of the lines of a partial geometry into spreads. A *fan* is a partition of the points of a partial geometry into ovoids. Unsurprisingly, resolutions correspond to fans in the dual. If a partial geometry admits a spread, it is called *spreadable*, and if it additionally admits a resolution, it is called *resolvable*.

If a spread exists, then the set of $(s+1)\frac{st+\alpha}{\alpha}$ points can be partitioned into sets of $s+1$ points using the lines in the spread. This means that $\frac{st+\alpha}{\alpha}$ needs to be an integer, and so α has to divide st. Since st is constant under duality, if an ovoid exists then also α divides st. This division relation is not a sufficient condition, as we will see examples of partial geometries admitting ovoids but not spreads.

Generally, three subclasses of partial geometries are distinguished, to which (up to duality) many of the known partial geometries belong. A partial geometry with $\alpha = s+1$ has various names, like *Steiner system* or *2-design*, but we will go with *linear space*. The duals of linear spaces are the partial geometries with $\alpha = t+1$ and are called *dual linear spaces*. A partial geometry with $\alpha = s$ is called a *transversal design*, and their duals are *dual transversal designs* or *nets*. A partial geometry with $\alpha = 1$ is called a *generalized quadrangle*. All other partial geometries are called *proper partial geometries*, which have much less theory devoted to them. Note that the collinearity graphs of linear spaces or transversal designs are imprimitive, so they are not of interest to us. However, the line graphs of those partial geometries are still very interesting. We will go a little bit deeper into each of the three subclasses.

Linear spaces

A linear space is a partial geometry with $\alpha = s+1$. This means that every point v not on a line ℓ is collinear with all of the points on ℓ. A consequence of this is that every pair of points is collinear. If an ovoid exists, it must be a singleton (as every pair of points is collinear). In this case, the lines going through this single point are all the lines, so there are just $t+1$ lines. This implies that $st = 0$, which is not possible. So no ovoids exist in linear spaces, let alone fans.

See [4, Chapter I, Corollary 2.11] and [4, Chapter I, Proposition 5.10] for necessary conditions for the existence and resolvability of ilnear spaces respectively.

An important example of a linear space is the complete linear space, with $s = 1$. Now the lines contain exactly 2 points, and there are exactly $t+2$ points. The only possibility is that $\mathcal{L}$ is in bijection with $\binom{V}{2}$, or the set of all 2-subsets of V. If t is odd, then a spread cannot exist, because there is an odd number of points. Hence, this is not resolvable. If t is even, then there exists a resolution, which in this case just means that the chromatic index of the complete graph on an even number of vertices is equal to the valency. This is also part of [4, Chapter I, Theorem 5.11] (the case $k = 2$).

If $s = 2$, then the linear space is called a *Steiner triple system*, as the lines will be triples in this case.

Translating the statements made about spreads and ovoids for linear spaces, we see that dual linear spaces are not spreadable or resolvable, and only in some cases an ovoid or fan exists.

Transversal designs and nets

The concept of transversal designs is broader than just within partial geometries. For simplicity, we will only consider the transversal designs that are partial geometries. For this notion in a broader context, see [4].

A transversal design is a partial geometry with $s = \alpha$. So if point v is not on line ℓ, then there exists a unique point on ℓ that is not collinear with v. One can prove that non-collinearity is now an equivalence relation, and that there are exactly $s + 1$ equivalence classes (called *point classes*). All point classes must have $t + 1$ points, and every line must have exactly one point from every point class (hence "transversal").

The collinearity graph of a transversal design will therefore always by a regular complete multipartite graph.

The point classes in the transversal design form ovoids, and the partition into the point classes is a fan. Resolvability is related to the existence of transversal designs, as is shown by the following proposition, which we state without proof.

Proposition 2.15 ([4, Chapter I, Proposition 7.15])**.** *Let s and t be positive integers. Consider a transversal design D with parameters (s, t). Then the following hold.*

(i) If $s \geq 2$, then the incidence structure D_0 obtained by removing one of the point classes is a resolvable transversal design with parameters $(s - 1, t)$;

(ii) Suppose D is resolvable. Note that every point is on exactly $t + 1$ lines, so the resolution has $t + 1$ spreads $S_0, \ldots, S_t$. Construct an incidence structure D_1 by adding $t + 1$ points $p_0, \ldots, p_t$, adjoining p_i to every line in spread S_i. Then D_1 is a transversal design with parameters $(s + 1, t)$.

If $s = 1$, then lines contain just two points, and there are just two point classes. It is easy to see that the transversal design now is related to the complete bipartite graph $K_{t+1,t+1}$, and this is resolvable. For $s = 2$, we can therefore construct the following transversal design (from [4, Chapter I, Example 6.4]). The point set V is $\{0, \ldots, t\} \times \{0, 1, 2\}$, where the second coordinate will refer to the point class. We will catch every triple $\{(x, 0), (y, 1), (z, 2)\}$ with $x + y + z \equiv 0 \pmod{t + 1}$ into a line. It is easy to verify that this gives a transversal design.

Another important standard construction is the dual of the affine plane of order q (from [4, Section I.2]). Here q needs to be a prime power, and it will give a transversal design with parameters $(q, q - 1)$. We will not give the construction here.

The dual of a transversal design is a net. The point classes of a transversal design turn into spreads, and the fan consisting of the point classes of a transversal design will give a resolution of a net.

The smallest example of a net is for $t = \alpha = 1$, and then it is related to the line graph of $K_{s+1,s+1}$ (a square lattice graph). Nets with $t = \alpha = 2$ are related to Latin squares (see [3]). An $n \times n$ *Latin square* is an arrangement of n symbols in an $n \times n$-grid such that every row and every column contains every symbol. We can construct a partial geometry in the following way. As the points take the n^2 cells of the grid. For the lines, we make a line for every row and column and make it incident to every cell that is in that row/column. Additionally, make for every symbol a line and make it incident to every cell containing that symbol. One can check that this gives a partial geometry with parameters $(n-1, 2, 2)$.

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

A	B	C	D
B	D	A	C
C	A	D	B
D	C	B	A

the two non-isomorphic Latin squares of order 4

Generalized quadrangles

A generalized quadrangle is a partial geometry with $\alpha = 1$. So every point and line that are not incident are joined by a unique other line and point. The number of points has to be $(s+1)(st+1)$, and the number of lines is $(t+1)(st+1)$. Triangles (that is, three points that are joined by three different lines each incident to two of the points) cannot exist.

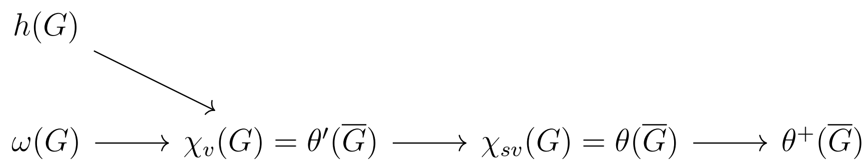
There are six classes of *classical* generalized quadrangles. They are denoted $Q(3,q)$, $Q(4,q)$, $Q(5,q)$, $H(3,q^2)$, $H(4,q^2)$, $W(q)$, with parameters (s,t) equal to $(q,1)$, (q,q), (q,q^2), (q^2,q), (q^2,q^3), (q,q) respectively, with q a prime power. The generalized quadrangles denoted with a Q can be obtained from quadrics in projective space, where $Q(3,q)$ arises from a hyperbolic quadric, $Q(4,q)$ from a parabolic quadric, and $Q(5,q)$ from an elliptic quadric. Hermitian varieties $H(n,q^2)$ are another source of generalized quandrangles, for $n = 3$ and $n = 4$.

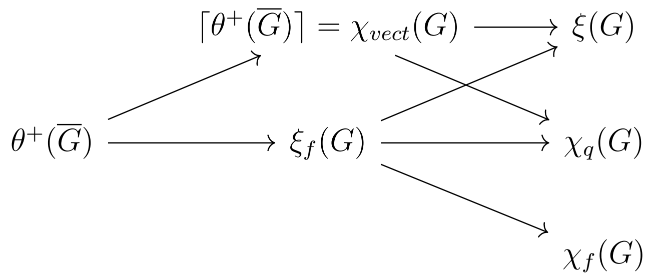
The $Q(3,q)$ are also nets so not of interest now. The $Q(4,q)$ and $W(q)$ are dual to each other, and also the $H(3,q^2)$ and $Q(5,q)$ are dual. There are also non-classical generalized quadrangles.

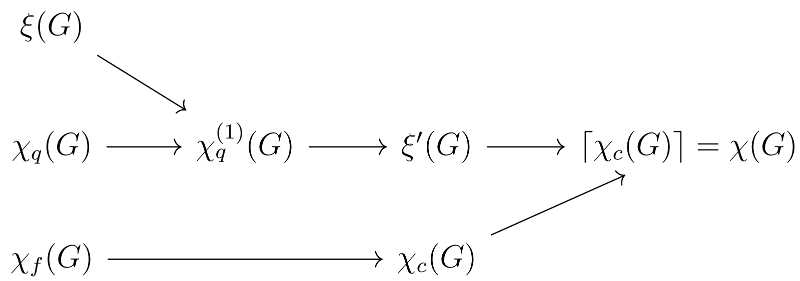
Spreads and fans in generalized quadrangles (both classical and non-classical) have been surveyed and further studied by Thas and Payne in [33]. We will cover their survey/work in Section 3.2.

2.5 Variations on the chromatic number

There are numerous variations of the chromatic number introduced in the literature. These variations are interesting in the context of the Hoffman bound, since many of these parameters are sandwiched in between the Hoffman bound and the chromatic number. As a starting point, we show the following diagram, which comes from [35, Section 2]. However, we added Hoffman's bound and also the Schrijver variation of the Lovász number, which we will justify below. As the number of parameters is too large, the diagram is split into three sections. The second diagram starts on the left where the first diagram ended, and similarly for the transition of the second diagram into the third.







The following parameters are included in the diagram:

- $h(G)$, the Hoffman bound;
- $\omega(G)$, the clique number;
- $\chi_v(G)$, the vector chromatic number;
- $\theta'(G)$, the Schrijver variant of the Lovász number;
- $\chi_{sv}(G)$, the strict vector chromatic number;
- $\theta(G)$, the Lovász number;
- $\theta^+(G)$, the Szegedy variant of the Lovász number;
- $\chi_{\mathrm{vect}}(G)$, the vectorial chromatic number (different from the vector chromatic number!);
- $\xi_f(G)$, the projective orthogonal rank;

- $\xi(G)$, the orthogonal rank;
- $\chi_q(G)$, the quantum chromatic number;
- $\chi_f(G)$, the fractional chromatic number;
- $\chi_q^{(1)}$, the rank-1 quantum chromatic number;
- $\xi'(G)$, the normalized orthogonal rank;
- $\chi_c(G)$, the circular chromatic number;
- $\chi(G)$, the standard chromatic number.

The diagram from [35] was composed from various sources that are listed there. In this paper also the definitions of all these variants are included.

For now we pick out three interesting variations, namely the quantum chromatic number χ_q, the vector chromatic number χ_v and the strict vector chromatic number χ_{sv}. For each of these, we will discuss their definition and relevance to Hoffman colorings in the respective sections.

2.5.1 Quantum chromatic number

The quantum chromatic number is interesting in this context, because it is not known to be computable in general (see [25]). However, for Hoffman colorable graphs it is, as the quantum chromatic number is sandwiched by the Hoffman bound and the standard chromatic number (by [25, Corollary 4.1]).

We define the *quantum chromatic number* $\chi_q(G)$ as the least integer k such that a quantum-k-coloring exists. A *quantum-k-coloring* of a graph G is a matrix consisting of projections

$$(E_{v,i})_{v\in V(G),\ 1\le i\le k},$$

such that:

- For all $v \in V(G)$ we have

  $$\sum_{i=1}^{k} E_{v,i} = I,$$

  the identity matrix.

- For all $v \in V(G)$ and $1 \le i < j \le k$ we have

  $$E_{v,i}E_{v,j} = E_{v,j}E_{v,i} = 0.$$

  That is, the projections belonging to v are orthogonal.

- For all connected vertices $u \sim_G v$ and for all $1 \le i \le k$ we have

  $$E_{u,i}E_{v,i} = 0.$$

The size of the projection matrices $E_{v,i}$ here is free. So for every vertex in the graph we take a decomposition of the space into k orthogonal subspaces, where we denote the projection onto the subspace with $E_{v,i}$. Adjacent vertices $u \sim v$ should give orthogonal subspaces $E_{u,i}$ and $E_{v,i}$ for every i.

If a graph admits a k-coloring $c : V(G) \to \{1, \ldots, k\}$ in the regular sense, then we can make a quantum k-coloring by taking

$$E_{v,i} = \begin{cases} I & \text{if } c(v) = i, \\ 0 & \text{if } c(v) \neq i. \end{cases}$$

This shows that

$$\chi_q(G) \leq \chi(G).$$

For example, for the 5-cycle we have the quantum-3-coloring with the matrix E given by

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

2.5.2 Vector colorings and Lovász numbers

In this section, we will investigate two versions of vector colorings and two versions of Lovász numbers, which turn out to coincide, and which furthermore have many equivalent pairs of characterizations. Galtman surveyed spectral characterizations of these two closely related graph invariants in [13], but only in the context of the Lovász numbers.

These variations are of interest to the Hoffman bound because it turns out that a special class of graphs, the 1-walk regular graphs, has vector chromatic number equal to the Hoffman bound. We will cover this in more detail in Section 3.3. Also of interest is the application of the inequalities proved by Lovász to the Hoffman bound. Namely, we will be able to conclude that for regular graphs G the product of the Hoffman bound of G and the Hoffman bound of the complement of G is bounded above by the number of vertices of G. We will investigate and elaborate on this fact in Section 4.2.

Vector colorings

Vector colorings were first introduced by Karger et al. in [22] (see Definition 2.1 and Definition 8.1). Given a graph G and a real number $t \geq 2$, a *vector t-coloring* is an assignment of unit vectors from a finite-dimensional real vector space to the vertices of G such that for any pair of adjacent vertices, the assigned vectors have inner product bounded above by $-1/(t-1)$. A vector coloring is *strict* if for any pair of adjacent vertices, the inner product of the assigned vectors is equal to $-1/(t-1)$.

The *vector chromatic number* χ_v of a graph is the least t such that a vector t-coloring of the graph exists. The *strict vector chromatic number* χ_{sv} of a graph is the least t such that a strict vector t-coloring exists. A strict vector coloring is in particular a vector coloring, so it immediately follows that $\chi_v(G) \leq \chi_{sv}(G)$ for every graph G.

Consider a regular simplex on $t+1$ vertices in $\mathbb{R}^t$ centered at the origin and inscribed in the unit sphere. For example, the regular triangle in two-dimensional space or the tetrahedron in three-dimensional space. It holds that the inner product of the vectors indicating the positions of two distinct vertices is exactly $-1/(t-1)$. For an explicit construction, see [22, Lemma 4.1]. A consequence of this, is that any standard t-coloring can be extended to a strict vector t-coloring, by sending every vertex in color class V_i to the i'th vertex of the regular simplex. This shows that in fact

$$\chi_v(G) \leq \chi_{sv}(G) \leq \chi(G).$$

Lovász number and variant

In [24], Lovász introduced a graph invariant $\theta(G)$, the Lovász number, as an upper bound for the Shannon capacity. Independently, Schrijver [31] and McEliece, Rodemich and Rumsey [26] introduced a variation $\theta'(G)$ on this. We will follow the notation of Schrijver (in [26] $\alpha_L(G)$ is written).

The *Lovász number* $\theta(G)$ has many equivalent definitions, but one is that it is defined as the maximum of the sum of all entries of a matrix B, where we require B to be a real symmetric, positive semi-definite $V(G) \times V(G)$-matrix that has trace 1 and has zeroes if u and v are adjacent. For $\theta'(G)$, we also require B to be non-negative. It is now clear that $\theta'(G) \leq \theta(G)$.

Furthermore (see [31, Theorem 1]), if C is a coclique in G, then the matrix B given by $B_{uv} = 1/|C|$ if u and v are in the coclique and 0 elsewhere gives a valid matrix for the optimization, and we conclude that

$$\alpha(G) \leq \theta'(G) \leq \theta(G).$$

In [13], Galtman surveyed all of the known spectral characterizations of the Lovász number and its variation. Of interest to us is characterization 5. The Lovász number of the complement $\theta(\overline{G})$ is equal to the maximum of $1 - \frac{\lambda_{\max}(B)}{\lambda_{\min}(B)}$ over all real non-zero symmetric $V \times V$-matrices B that have zeroes if u and v are not adjacent (this was already proven by Lovász [24, Theorem 6]). Additionally, $\theta'(\overline{G})$ is the same, but we again require B to be non-negative as well. Since the adjacency matrix of a graph satisfies all of these requirements, we can conclude that

$$h(G) \leq \theta'(\overline{G}) \leq \theta(\overline{G}).$$

In [24], Lovász also proved that $\theta(G)\theta(\overline{G}) \geq |V(G)|$ for all graphs, and equality is attained if G is vertex-transitive (Corollary 2 and Theorem 8). Furthemore, if G is regular, then the Lovász number can be sandwiched in between the independence number and the ratio bound:

$$\alpha(G) \leq \theta(G) \leq \frac{|V(G)|}{h(G)}.$$

If the graph is furthermore edge-transitive, then $\theta(G)$ is actually equal to $\frac{|V(G)|}{h(G)}$ ([24, Theorem 9]).

Relating

To relate the vector colorings and the Lovász numbers, we refer back to Karger et al., namely, [22] and [14]. In particular, [22, Theorem 8.1] states that $\theta(\overline{G}) = \chi_{sv}(G)$ for all graphs, and furthermore in [14] and in [15] is stated that $\theta'(\overline{G}) = \chi_v(G)$. To avoid the need of complements in our notation, we will from this point on mostly use the (strict) vector chromatic numbers instead of the Lovász number and its Schrijver variant.

As stated in [25, Corollary 4.1], for all graphs, the quantum chromatic number upper bounds the strict vector chromatic number. We collect all of these results in a theorem. The results are from [24], [22], [14], [15], [25].

Theorem 2.16. *Let G be a graph. Then the following hold.*

(i) We have the following chain of inequalities:

$$\omega(G) \leq \chi_v(G) \leq \chi_{sv}(G) \leq \chi_q(G) \leq \chi(G);$$

(ii) Furthermore, the Hoffman bound is a lower bound for the vector chromatic number:

$$h(G) \leq \chi_v(G);$$

(iii) We have the following relation:

$$\chi_{sv}(G)\chi_{sv}(\overline{G}) \geq n,$$

with equality if G is vertex-transitive;

(iv) If G is regular, then

$$\chi_{sv}(\overline{G}) \leq \frac{n}{h(G)},$$

with equality if G is edge-transitive.

Statements (i) and (ii) make clear that the (strict) vector chromatic number and the quantum chromatic number are sandwiched in between the Hoffman bound and the standard chromatic number. With statements (iii) and (iv) we will be able to prove Corollary 4.11.

Chapter 3

Literature Review

In the literature, some work has been done on the problem of classifying Hoffman colorability; especially for regular graphs. This has recently been extended to a more general scenario by Abiad [1], and we will progress further with this.

In addition, quite some results are known about Hoffman colorability in strongly regular graphs. Also, the relation between the Hoffman bound and the (strict) vector chromatic number has been studied.

3.1 Color partitions: (weight-)regularity

In [5], Blokhuis, Brouwer and Haemers determine 3-chromaticity of all of the then known distance-regular graphs, using, among other techniques, the Hoffman bound. We include the first part of Proposition 2.3 of [5] here.

Proposition 3.1 ([5, Proposition 2.3]). *Let $V_1, \dots, V_\chi$ be a Hoffman coloring of a regular graph G, then*

(i) The partition $V_1, \dots, V_\chi$ is regular;

(ii) All intersection numbers b_{ij} of this regular partition with $i \neq j$ equal $-\lambda_{\min}(G)$;

(iii) All color classes have equal size.

In particular, this means that a Hoffman coloring of a regular graph is equivalent to a partition of the vertices into Delsarte-Hoffman cocliques. This already gives a lot of information about the structure of Hoffman colorings in regular graphs.

In [1], Abiad uses Corollary 2.12 ([12, Lemma 2.3]) to prove the following necessary condition on Hoffman colorings.

Proposition 3.2 ([1, Proposition 5.3(i)]). *The partition defined by a Hoffman coloring is weight-regular.*

The power of Proposition 3.2 is that it generalizes the (weight-)regularity of the color partition to irregular graphs as well. We will state and prove a generalization to both of these results, namely Lemma 6.8. This will be a key ingredient for the Decomposition Theorem 6.1.

3.2 Strongly regular graphs

It is an easy calculation to verify that for strongly regular graphs, the ratio bound is equal to the Hoffman bound of the complement graph, and vice versa. This gives the following nice result:

Corollary 3.3 ([29, Equation (3)]). *Let G be a strongly regular graph. Then we have*

$$\omega(G) \leq h(G) \leq \chi(G).$$

Roberson [29] uses these inequalities to categorize all strongly regular graphs into four types (A, B, C an X) depending on whether these inequalities are tight or not. In Section 5.2 we will extend this type system.

In [19], Haemers and Tonchev generalise the notion of spreads in partial geometries to strongly regular graphs. A *spread* in a strongly regular graph is defined as a partition of the vertices into Delsarte-Hoffman cliques. Spreads correspond to Hoffman colorings of the complement.

Note that the Hoffman bound of the collinearity graph of an (s, t, α)-partial geometry is $s + 1$, the size of the lines (unless $\alpha = s + 1$, then the collinearity graph is a disjoint union of equal-sized complete graphs and the Hoffman bound is equal to the size of these complete graphs). This means that collinearity graphs of partial geometries always have Delsarte-Hoffman cliques. Furthermore, spreads in partial geometries will lead to spreads in their collinearity graphs, and thus to Hoffman colorings in the complements.

An ovoid in a partial geometry gives a Delsarte-Hoffman coclique and vice versa. Since a fan is a partition of the points in a partial geometry into ovoids, a fan corresponds with a Hoffman coloring of the point graph ([19, Proposition 3.1]).

For generalized quadrangles, note that all Delsarte-Hoffman cliques in their point graphs come from the lines in the partial geometry. This is because generalized quadrangles do not contain triangles. This means that spreads in the point graphs give spreads of the partial geometry. In general, this is not true. For example, a transversal design (a partial geometry with $\alpha = s$) has as collinearity graph the regular complete multipartite graph with $s + 1$ vertex classes of size $t + 1$. Any set of vertices having one vertex of each class will give a Delsarte-Hoffman clique, so there are $(t + 1)^{s+1}$ Delsarte-Hoffman cliques in the collinearity graph. However, there are only $(t + 1)^2$ lines in a transversal design. Hence, if $s \geq 2$, there must be Delsarte-Hoffman cliques in the collinearity graph that do not come from lines.

We can now state the following proposition, which collects various results about fans and spreads in partial geometries.

Proposition 3.4 (From [19], [4] and [33]).

(i) The line graph of any linear space is Hoffman colorable if and only if the linear space is resolvable, and in this case the parameters of the linear space satisfy $s + 1|t$. In particular, $L(K_{2m})$ is Hoffman colorable, and $L(K_{2m+1})$ is not.

(ii) The line graph of a transversal design is Hoffman colorable if and only if the transversal design is resolvable. In particular, $L(K_{m,m})$ is Hoffman colorable.

(iii) The complement of the line graph of any transversal design is Hoffman colorable. In particular, $\overline{L(K_{m,m})}$ *is Hoffman colorable.*

(iv) If we write $G(GQ)$ *for the point graph of generalized quadrangle* GQ*, then the following are Hoffman colorable:*

- $\overline{G(W(q))}$*;*
- $\overline{G(Q(4,q))}$ *with* q *even;*
- $\overline{G(Q(5,q))}$*;*
- $G(H(3,q^2))$*.*

In particular, the Schläfli graph is Hoffman colorable, as its complement is the point graph of $Q(5,2)$*.*

(v) The following are not Hoffman colorable, but do still contain Delsarte-Hoffman cocliques:

- $G(W(q))$ *with* q *even;*
- $G(Q(4,q))$ *with* q *even;*
- $\overline{G(Q(4,q))}$ *with* q *odd;*
- $\overline{G(H(3,q^2))}$*;*
- *Complement of line graph of* $H(4,q^2)$*;*
- $\overline{G(H(4,4))}$*.*

(vi) The following do not contain Delsarte-Hoffman cocliques and are therefore not Hoffman colorable:

- $G(W(q))$ *with* q *odd;*
- $G(Q(5,q))$*;*
- $G(H(4,q^2))$*.*

In particular, the complement of the Schläfli graph is not Hoffman colorable.

3.3 Vector colorings

A graph is called *k-walk regular* if for every $m \in \mathbb{N}$, for every $0 \le i \le k$ all pairs of vertices u and v at distance i have the same number of walks of length m from u to v (which is recorded in A^m_{uv}). So A^m_{uv} only depends on the distance $d(u,v)$, provided that this distance is at most k. Any k-walk regular graph is regular in the standard sense. Every strongly regular graph is k-walk regular for every k (as it is 2-walk regular and has diameter 2). Every graph that is 1-walk regular and has a 1-walk regular complement is strongly regular.

The reason 1- and 2-walk regularity is interesting is, among other because of the following propositions regarding the vector chromatic number.

Proposition 3.5 ([15, Lemma 5.2]). *Let G be a 1-walk regular graph, then*

$$h(G) = \chi_v(G) = \chi_{sv}(G).$$

This means that 1-walk regular graphs are vector Hoffman colorable. The authors define a so-called "canonical vector coloring" for these graphs, which are optimal vector colorings that attain the Hoffman bound. Strongly regular graphs are 1-walk regular, so in particular this result gives the vector chromatic number for strongly regular graphs. In a following paper, the canonical vector coloring is proven to be locally injective.

Proposition 3.6 ([14, Lemma 3.11]). *Let G be a 2-walk regular graph that is not bipartite or complete multipartite. The canonical vector coloring of G is locally injective.*

A coloring is *locally injective* if any two vertices that share a common neighbor get different colors. Note that primitive strongly regular graphs are 2-walk regular, and not bipartite or complete multipartite, so they satisfy the conditions of this proposition. We will use this result in Section 5.4 to prove that Hoffman colorable imprimitive strongly regular graphs are not uniquely vector colorable.

3.4 Other results

In this thesis, we will not elaborate on the results in this section. However, as they are still relevant to the topic, we have included them as well.

In [5], the authors also find a condition on the multiplicity of the least eigenvalue of Hoffman colorable graphs, as the second part of Proposition 2.3. Furthermore, the authors find an infinite family of Hoffman colorable graphs, namely all tripartite edge-regular graphs. A graph is *edge-regular* with parameters (v, k, λ), if it has v vertices, valency k, such that for every edge e there are exactly λ vertices that are adjacent to both endpoints of e. Note that 1-walk regular graphs must be edge regular.

Proposition 3.7 ([5, Proposition 2.3]). *If G is a Hoffman colorable graph, then the multiplicity of $\lambda_{\min}(G)$ is at least $\chi(G) - 1$. In case of equality, then the graph is uniquely colorable.*

Proposition 3.8 ([5, Theorem 2.6]). *Every 3-coloring of a (v, k, t)-edge-regular graph G with $t > 0$ is a Hoffman coloring.*

In [8], Cioabă, Guo and Haemers examine the chromatic index (that is, the chromatic number of the line graph) of strongly regular graphs. Interesting to us is the following theorem, which uses Proposition 3.1.

Theorem 3.9 ([8, Theorem 2.3]). *Suppose G is a regular graph with a Hoffman coloring and an even chromatic number, then G and $\overline{G}$ are class 1 (meaning their chromatic indices are equal to the valency), or $\overline{G}$ is a disjoint union of cliques of odd order.*

Chapter 4

Hoffman's bound as a graph parameter

As we have seen in the preliminaries, many variations of the chromatic number exist, and many of them are sandwiched in between the Hoffman bound (denoted $h(G)$) and the chromatic number:

$$h(G) \leq \chi_v(G) \leq \chi_{sv}(G) \leq \chi_q(G) \leq \chi(G).$$

The relevance of this to Hoffman colorings, is that this chain of inequalities allows us to find (for free) the (strict) vector chromatic number and the quantum chromatic number for Hoffman colorable graphs, by sandwiching. This means in particular that for the class of Hoffman colorable graphs, the quantum chromatic number is computable: it is equal to the standard chromatic number and to the Hoffman bound.

This is just one consequence of seeing the Hoffman bound as just another graph parameter, instead of as a spectral bound on other graph parameters. We will explore the potential of this shift of perspective in this chapter. First, we will study the Hoffman bound of several graph operations that we met in the preliminaries, namely disjoint unions, Cartesian products, tensor products and line graphs. The tensor product turns out to be a great source of Hoffman colorable graphs. Further, we will study the information that the Hoffman bound contains for regular graphs. In fact, we will prove a new characterization of a graph being strongly regular. In particular, we will prove that strong regularity can be read off from the Hoffman bound of a regular graph and its complement.

4.1 Hoffman bound and graph operations

In this section, we will express the Hoffman bound of several graph operations in terms of the Hoffman bounds of the graphs that are the inputs of the operations. We also aim to express Hoffman colorability in terms of Hoffman colorability of the input graphs.

We will use the results on the spectra and chromatic numbers of these operations set out in Section 2.1.3.

4.1.1 Disjoint unions

Let G and H be graphs. We say that G *dominates* H whenever all eigenvalues of H are in between the least and greatest eigenvalues of G (non-strictly). Note that domination is reflexive and transitive, but not symmetric or antisymmetric.

Lemma 4.1 (Hoffman bound of disjoint unions). *Let G and H be non-empty graphs. Then*

$$\min(h(G), h(H)) \leq h(G \sqcup H) \leq \max(h(G), h(H)).$$

Furthermore, we have that

$$h(G \sqcup H) = h(G)$$

if and only if G dominates H or $h(G) = h(H)$.

Proof. Since

$$\lambda_{\max}(G \sqcup H) = \max(\lambda_{\max}(G), \lambda_{\max}(H))$$

and

$$\lambda_{\min}(G \sqcup H) = \min(\lambda_{\min}(G), \lambda_{\min}(H)),$$

we have the four cases

$$h(G \sqcup H) \in \left\{ h(G), h(H), 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)}, 1 - \frac{\lambda_{\max}(H)}{\lambda_{\min}(G)} \right\},$$

depending on either G or H having the larger largest eigenvalue or the smaller smallest eigenvalue.

If G supplies both the greatest and least eigenvalue (G dominates H), then $h(G \sqcup H) = h(G)$ so the inequality holds.

If G supplies the greatest eigenvalue and H supplies the least eigenvalue (so if $\lambda_{\max}(H) < \lambda_{\max}(G)$ and $\lambda_{\min}(H) < \lambda_{\min}(G)$), then it is clear that $h(H) < h(G \sqcup H) < h(G)$, so now by possibly interchanging the roles of G and H the inequality follows.

For the equivalence, if G dominates H then we have already seen that $h(G \sqcup H) = h(G)$. Furthermore, if $h(G) = h(H)$, then one dominates the other, and the above applies, so that $h(G) = h(H) = h(G \sqcup H)$.

Conversely, suppose $h(G \sqcup H) = h(G)$. If $\lambda_{\max}(G) = \lambda_{\max}(G \sqcup H)$, then by equality of their Hoffman bounds also $\lambda_{\min}(G) = \lambda_{\min}(G \sqcup H)$ follows, and so G dominates H. If $\lambda_{\max}(G) \neq \lambda_{\max}(G \sqcup H)$, then by equality of their Hoffman bounds it follows that $\lambda_{\min}(G) \neq \lambda_{\min}(G \sqcup H)$, and so $h(G \sqcup H) = h(H)$, which implies $h(G) = h(H)$. So $h(G \sqcup H) = h(G)$ is equivalent to G dominating H or G and H sharing their Hoffman bound. □

Now we know about the Hoffman bound of the disjoint union of two graphs, we can characterize Hoffman colorability of the disjoint union.

Proposition 4.2 (Hoffman colorings of disjoint unions). *Let G and H be non-empty graphs. Then $G \sqcup H$ is Hoffman colorable if and only if all of the following statements hold (up to interchanging the roles of G and H).*

(i) G is Hoffman colorable;

(ii) $\chi(H) \leq \chi(G)$*;*

(iii) G *dominates* H *or* G *and* H *have the same Hoffman bound.*

Furthermore, if $G \sqcup H$ *is Hoffman colorable, then* G *and* $G \sqcup H$ *are Hoffman colorable with the same number of colors.*

Proof. We have the chain of inequalities

$$h(G \sqcup H) \leq \max(h(G), h(H)) \leq \max(\chi(G), \chi(H)) = \chi(G \sqcup H).$$

If $G \sqcup H$ is Hoffman colorable, then the outermost values are equal, hence all inequalities are equalities. Then at least one of the equalities $h(G) = \chi(G)$ or $h(H) = \chi(H)$ has to hold. If both hold and are furthermore equal, then clearly both are Hoffman colorable and have the same chromatic number and Hoffman bound, so the three conditions hold. Alternatively, if both hold but are unequal, say $h(H) < h(G)$ or if, say, only $h(G) = \chi(G)$ holds but $h(H) < \chi(H)$, then for the equalities to work in the chain of inequalities, we must have that $h(H) < h(G)$ and $\chi(H) \leq \chi(G)$. Therefore $h(G \sqcup H) = h(G)$ and by Lemma 4.1 this implies that G dominates H.

Conversely, suppose that the three conditions from the statement hold for G and H. Then by the previous proposition we have $h(G \sqcup H) = h(G)$, and also we know $\chi(G \sqcup H) = \chi(G)$. By Hoffman colorability of G the result now follows easily. □

This motivates us to only look at connected Hoffman colorable graphs, as disconnected ones can be easily reduced to the connected ones with Proposition 4.2. This is very convenient for the Decomposition Theorem, as every connected graph has a positive eigenvector (which is required for the Decomposition Theorem) by the Perron-Frobenius Theorem.

4.1.2 Cartesian products

For the Cartesian product, again we will first investigate the Hoffman bound, after which we can classify Hoffman colorability of the Cartesian product of two graphs.

Lemma 4.3 (Hoffman bound of Cartesian products)**.** *Let* G *and* H *be non-empty graphs. Then*

$$\min(h(G), h(H)) \leq h(G \square H) \leq \max(h(G), h(H)),$$

and $h(G \square H) = h(G)$ *if and only if* $h(G) = h(H)$*.*

Proof. We have $\lambda_{\max}(G \square H) = \lambda_{\max}(G) + \lambda_{\max}(H)$ and $\lambda_{\min}(G \square H) = \lambda_{\min}(G) + \lambda_{\min}(H)$. Therefore

$$h(G \square H) = 1 - \frac{\lambda_{\max}(G) + \lambda_{\max}(H)}{\lambda_{\min}(G) + \lambda_{\min}(H)},$$

which is clearly in between $h(G)$ and $h(H)$, and is equal to $h(G)$ if and only if $h(G) = h(H)$. □

Proposition 4.4 (Hoffman colorings of Cartesian products)**.** *Let G and H be non-empty graphs. Then $G \square H$ is Hoffman colorable if and only if G and H are Hoffman colorable with the same number of colors. In this case, also $G \square H$ is Hoffman colorable with that number of colors.*

Proof. We have the chain of inequalities

$$h(G \square H) \leq \max(h(G), h(H)) \leq \max(\chi(G), \chi(H)) = \chi(G \square H).$$

If the outermost values are equal (so if $G \square H$ is Hoffman colorable), then all inequalities are equalities. From $h(G \square H) = \max(h(G), h(H))$ and Lemma 4.3, we conclude that $h(G) = h(H) = h(G \square H)$. Now from $\max(h(G), h(H)) = \max(\chi(G), \chi(H))$ we conclude that $h(G) = \chi(G)$ and $h(H) = \chi(H)$, so G and H are Hoffman colorable, and all of the Hoffman colorings of G, H and $G \square H$ use the same number of colors.

Conversely, suppose that G and H are Hoffman colorable with the same number of colors. Then $\chi(G) = h(G) = \chi(H) = h(H)$. By the previous proposition we obtain $h(G \square H) = h(G)$, and since $\chi(G \square H) = \max(\chi(G), \chi(H))$, we are done. □

As an easy consequence of this, the square lattice graphs $L(K_{m,m})$ are Hoffman colorable with m colors, as they are the Cartesian product of two copies of K_m.

4.1.3 Tensor products and self loops

In order to express Hoffman colorability of tensor products in the most powerful and meaningful way, we have to first investigate allowing graphs to have self loops.

Self loops

Where we see (simple) graphs as a pair $(V, \sim)$ of a set of vertices and a symmetric, irreflexive relation, for a graph with possible self loops, we leave out the irreflexivity. A vertex now has a *self loop* if it is adjacent to itself. We would like to extend the chromatic number and Hoffman bound to graphs with self loops. If a graph has a vertex with a self loop, then assigning a color to this vertex immediately creates a forbidden coloring. Therefore there is no way to color a graph with a self loop, so we define the chromatic number to be infinite.

The adjacency matrix of a graph with self loops is made in the same way as for loopless graphs:

$$A_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{if } u \not\sim v. \end{cases}$$

Note that we put a 1 in the matrix, so we have to go with the convention that a self loop contributes 1 to the degree (instead of 2, which might make sense in other settings).

Further note that the trace of the adjacency matrix is now equal to the number of self loops in the graph. As a consequence, we do not have anymore automatically that a negative eigenvalue exists. We can classify when this happens, because if $\tilde{G}$ denotes the underlying loopless graph, we have

$$A(G) = A(\tilde{G}) + B,$$

where B is a diagonal 0-1-matrix. Consequently, if $A(G)$ has no negative eigenvalues, then $A(\tilde{G})$ has least eigenvalue at least -1, and so $\tilde{G}$ is a disjoint union of complete graphs by Corollary 2.10. A straight-forward calculation leads to the conclusion that G is a disjoint union of fully complete graphs FK_n (that is the complete graph K_n with n self loops) and isolated vertices.

If G has a negative eigenvalue, we define the Hoffman bound in the usual way, as

$$h(G) = 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}.$$

If G has no negative eigenvalues, then we define

$$h(G) = \begin{cases} 1 & \text{if } \lambda_{\max}(G) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Note that the statements in the Preliminaries on the tensor product and connected graphs still hold if we allow self loops, as we still have the property that the adjacency matrix of $G \times H$ is the Kronecker product of the adjacency matrices of G and H. Also the statements on the chromatic number still hold, since $G \times H$ has a self loop if and only if both G and H have a self loop, and only in this case we get $\chi(G \times H) = \infty$.

Also note that if either G of H contains a self loop, then the pair (G, H) does satisfy Hedetniemi's conjecture: if, without loss of generality, H has a self loop, then G is a subgraph of $G \times H$, so we must have $\chi(G) \leq \chi(G \times H)$. Since $\chi(H) = \infty$, we get $\chi(G \times H) = \chi(G) = \min(\chi(G), \chi(H))$.

Tensor products

We now investigate the Hoffman bound of tensor products.

Lemma 4.5 (Hoffman bound of tensor products)**.** *Let G and H be non-empty graphs where we allow H to have self loops. Then*

$$h(G \times H) = \min(h(G), h(H)).$$

Proof. Since $G \times H$ now is a non-empty simple graph, by definition

$$h(G \times H) = 1 - \frac{\lambda_{\max}(G \times H)}{\lambda_{\min}(G \times H)}.$$

The largest eigenvalue of $G \times H$ is the product of the largest eigenvalues of the two factors. The least eigenvalue is

$$\lambda_{\min}(G \times H) = \min(\lambda_{\min}(G)\lambda_{\max}(H), \lambda_{\max}(G)\lambda_{\min}(H)).$$

If H has no negative eigenvalues, then $\lambda_{\min}(H)$ will be non-negative, so the least eigenvalue of $G \times H$ is $\lambda_{\min}(G)\lambda_{\max}(H)$, so that

$$h(G \times H) = 1 - \frac{\lambda_{\max}(G)\lambda_{\max}(H)}{\lambda_{\min}(G)\lambda_{\max}(H)} = 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} = h(G),$$

which is certainly less than $h(H) = \infty$, so equal to $\min(h(G), h(H))$.

Now suppose that H has a negative eigenvalue. First suppose that $h(G) \leq h(H)$, then

$$1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} \leq 1 - \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)},$$

which implies that

$$\lambda_{\min}(G)\lambda_{\max}(H) \leq \lambda_{\max}(G)\lambda_{\min}(H),$$

so that the least eigenvalue of $G \times H$ is $\lambda_{\min}(G)\lambda_{\max}(H)$ and $h(G \times H) = h(G)$ as before. If $h(H) \leq h(G)$, analogously $h(G \times H) = h(H)$ follows. □

Recall that Hedetniemi's conjecture says that the chromatic number of the tensor product of two graphs is the minimum of the chromatic numbers of the two input graphs. If this is the case (which is at least if one of the factors has a self loop), then we will be able to classify Hoffman colorability.

Proposition 4.6 (Hoffman colorings of tensor products)**.** *Let G and H be non-empty graphs, where we allow H to have self loops. Suppose that $\chi(G) \leq \chi(H)$. Then the following are equivalent.*

- *The pair (G, H) satisfies Hedetniemi's conjecture and $G \times H$ is Hoffman colorable;*
- *G is Hoffman colorable and $h(G) \leq h(H)$.*

In this case, $G \times H$ is Hoffman colorable with the same number of colors as G.

Proof. Suppose that (G, H) satisfies Hedetniemi's conjecture and that $G \times H$ is Hoffman colorable. Since $\chi(G) \leq \chi(H)$ we then have

$$\chi(G) = \chi(G \times H) = h(G \times H) = \min(h(G), h(H)),$$

so that $\chi(G) \leq h(G)$ and $\chi(G) \leq h(H)$. It follows that G is Hoffman colorable and $h(G) \leq h(H)$.

Now suppose that G is Hoffman colorable and that $h(G) \leq h(H)$, then

$$\chi(G \times H) \leq \chi(G) = h(G) = h(G \times H),$$

which implies that $G \times H$ is Hoffman colorable and that $\chi(G \times H) = \chi(G)$. □

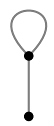
In the case of counterexamples to Hedetniemi's conjecture, it is hard to give an explicit statement about the components. However, the graphs (G, H) satisfying the requirements of Proposition 4.6 automatically satisfy Hedetniemi's conjecture.

Proposition 4.6 gives a productive way of making Hoffman colorings. In particular, we can create an infinite family of irregular Hoffman colorable graphs, as follows. Start with G_0, an irregular Hoffman colorable graph, say with c colors. Then for any step do $G_{n+1} = G_n \times FK_2$, where FK_2 is the fully complete graph on 2 vertices. Note that the eigenvalues of FK_2 are 0 and 2, so the above result applies, and G_{n+1} is irregular and Hoffman colorable with c colors.

Let's look at some specific examples of tensor products.

Example 4.7. If we take G to be K_c the complete graph on c vertices, and H to be FK_m the fully complete graph on m vertices, then G is Hoffman colorable, while $h(G) = c \leq \infty = h(H)$, so the requirements of Proposition 4.6 are satisfied. The tensor product of G and H will be the regular complete multipartite graph with c classes of size m.

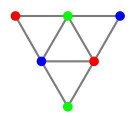
Example 4.8. As a different example, consider the lollipop graph L, which can be obtained from adding a self loop to one of the vertices of K_2.



The adjacency matrix is $A(X) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which has characteristic polynomial $x^2 - x - 1$. Therefore

$$\mathrm{Spec}(L) = \{\phi, 1 - \phi\},$$

with ϕ the golden ratio. Hoffman's bound of the lollipop graph now is equal to $2+\phi$, which is in between 3 and 4. For $G \times L$ to be Hoffman colorable, we therefore need that G is Hoffman colorable graph with $h(G) \leq 3$. If we restrict G to the complete graphs, then we see we must have that G is either K_2 or K_3. For $K_2 \times L$ we get the path graph on four vertices, and for $K_3 \times L$ we get the following Hoffman colorable graph:



This graph will pop up two times as a special case, namely in Theorem 6.22 and Corollary 6.15.

4.1.4 Line graphs

A *1-factor* or *perfect matching* of a graph is a set of edges such that every vertex is on exactly one of those edges. A *1-factorization* of a graph is a partition of the edges into 1-factors. A graph is *1-factorable* if it admits a 1-factorization. Those graphs are necessarily regular, and have an even number of vertices. A 1-factorization is necessarily a coloring of the edges with just Δ colors. Therefore 1-factorable graphs are of class 1. Conversely, every regular graph of class 1 must be 1-factorable.

Recall that the eigenvalues of a line graph are bounded below by -2 (see Proposition 2.6). For the Hoffman bound, we can only say something if the least eigenvalue is equal to -2.

Proposition 4.9 (Hoffman colorability of line graphs with least eigenvalue -2)**.** *Let G be a connected graph with at least two edges, such that $\lambda_{\min}(L(G)) = -2$. Then $L(G)$ is Hoffman colorable if and only if G is 1-factorable.*

Proof. Write Q for the signless Laplace matrix of G, so $Q = NN^T$ where N is the incidence matrix given by $N_{v,e} = 1$ if and only if edge e is incident to vertex v and 0 otherwise. We have $Q = D+A$, where D is the diagonal matrix recording the degrees of the vertices. We get $\lambda_{\max}(D) = \Delta$, the maximum degree, and by [7, Proposition 3.1.2] we have $\lambda_{\max}(A) \leq \Delta$ with equality if and only if G is regular. By linearity of the Rayleigh quotient ([7, Section 2.4]), we have $\lambda_{\max}(Q) \leq \lambda_{\max}(D) + \lambda_{\max}(A)$. By [7, Proposition 1.4.1]

$$\lambda_{\max}(L(G)) = \lambda_{\max}(Q) - 2 \leq \lambda_{\max}(D) + \lambda_{\max}(A) - 2 \leq 2\Delta - 2,$$

and so $h(L(G)) \leq \Delta$. By Vizing's Theorem, $\chi(L(G)) \in \{\Delta, \Delta + 1\}$. So we have

$$h(L(G)) \leq \Delta \leq \chi(L(G)).$$

Now it is evident that $L(G)$ is Hoffman colorable if and only if both inequalities are equalities. The first equality is equivalent to $\lambda_{\max}(A) = \Delta$ and to G being Δ-regular. The second equality is equivalent to G being of class 1. Together, regularity and class 1 are equivalent to 1-factorability, concluding the proof. □

Now we have tackled Hoffman colorablility of line graphs with least eigenvalue equal to -2, we should investigate which line graphs have this property.

Lemma 4.10 (Line graphs with least eigenvalue -2)**.** *Let G be a connected graph with at least two edges. Then $\lambda_{\min}(L(G)) = -2$ if and only if one of the following holds.*

(i) G has more edges than vertices;

(ii) G has an equal number of edges and vertices and the unique cycle in G is of even length.

Proof. By [7, Proposition 1.4.1], if G has more edges than vertices, then $r < m$ (with r and m taken from the Proposition) and so we must have that $\lambda_{\min}(L(G)) = -2$.

If G has fewer edges than vertices, then G must be a tree. The incidence matrix N now has n rows and $n-1$ columns. Note that the columns of N are linearly independent. If we have a linear combination of the columns equal to 0, then we observe that the coefficient of an edge incident to a leaf must be zero, after which we can remove the edge and argue inductively. So now $\text{rk}(N) = n - 1$, so that $\text{rk}(NN^T) = n - 1$ and NN^T has $n - 1$ positive eigenvalues. By [7, Proposition 1.4.1], $L(G)$ can only have eigenvalues greater than -2.

For the other case, G has an equal number of edges and vertices. We note that N is a square matrix. We note that NN^T is invertible if and only if N is invertible. Moreover, NN^T is invertible if and only if $L(G)$ only has eigenvalues greater than -2. Therefore we calculate $\det(N)$ using the Leibniz formula:

$$\det(N) = \sum_{\sigma : V(G) \to E(G)} \text{sgn}(\sigma) \prod_{v \in V(G)} N_{v,\sigma(v)}.$$

For σ to contribute, v must be incident to $\sigma(v)$ for all $v \in V(G)$. It now follows directly that if v is not on the unique cycle, then $\sigma(v)$ must be the edge that is the first step in the shortest path from v to the cycle. For the vertices on the cycle, there are just two options, going clockwise around the cycle or going counterclockwise around the cycle. If we number the vertices around the cycle $v_1, \ldots, v_k$ and the edges $e_1, \ldots, e_k$, where e_i is incident to v_i and v_{i+1}, then the options are $\sigma(v_i) = e_i$ and $\sigma(v_i) = e_{i-1}$. These are related by a cyclic permutation τ of length k on the edges. If k is even, then $\operatorname{sgn}(\tau) = -1$, so that $\det(N) = 0$. If k is odd, then $\operatorname{sgn}(\tau) = 1$, so that $|\det(N)| = 2$. Therefore in this case $\lambda_{\min}(L(G)) = -2$ if and only if the unique cycle is even. □

One consequence of these two results is that $L(K_{m,m})$ and $L(K_{2m})$ are Hoffman colorable for all m, since $K_{m,m}$ and K_{2m} are known to be 1-factorable. One can also conclude Hoffman colorability of these graphs by Proposition 3.4.

We cannot answer in full yet the question when exactly $L(G)$ is Hoffman colorable, in case G is a tree or has a unique cycle which is odd. However, we can quickly see some Hoffman colorable cases: if G is a path graph, then $L(G)$ is also a path graph hence Hoffman colorable. If G is isomorphic to $K_{1,m}$ for some m, then its line graph will be the complete graph on m vertices, so also Hoffman colorable. Next, if G is isomorphic to K_3, then also $L(G)$ is isomorphic to K_3 hence Hoffman colorable. Lastly, if G is the graph obtained by adding one leaf to every vertex of K_3, then $L(G)$ is isomorphic to the tensor product of K_3 and the lollipop graph from Example 4.8, hence Hoffman colorable.

In Section 6.4, we will see that these two infinite families of trees, the two exceptional cases, and the 1-factorable graph are all of the G with $L(G)$ Hoffman colorable. This is the content of Theorem 6.22.

4.2 A new condition for strong regularity

Considering the Hoffman bound in relation to the strict vector chromatic number, we obtain the following corollary.

Corollary 4.11. *Let G be a non-empty, non-complete regular graph. Then*

$$h(G)h(\overline{G}) \leq n.$$

If G is strongly regular, then this is an equality.

Proof. From Theorem 2.16 we obtain

$$h(G) \leq \chi_v(G) \leq \chi_{sv}(G) \leq \frac{n}{h(\overline{G})},$$

and so $h(G)h(\overline{G}) \leq n$. If G is strongly regular, then we can write its eigenvalues $\tau < \theta < k$, and it follows that

$$h(G) = 1 - \frac{k}{\tau}, \quad h(\overline{G}) = 1 - \frac{n-k-1}{-1-\theta}.$$

As is said in [29], only minor arithmetic operations are needed, together with Proposition 2.13 to find $h(G)h(\overline{G}) = n$. □

We can actually also prove this directly, without using the vector chromatic number. This direct proof also allows us to prove that in the case of equality, the graph is actually strongly regular. This means that in order to determine strong regularity of a regular graph, we only need the Hoffman bound of the graph and the Hoffman bound of its complement. The argument we use here is based on the convexity of the parabola.

Theorem 4.12. *Let G be a non-empty, non-complete regular graph. Then*

$$h(G)h(\overline{G}) \leq n,$$

with equality if and only if G is strongly regular.

Proof. Write k for the valency of G and n for the number of vertices of G, write θ for the second greatest eigenvalue of G, and τ for the least eigenvalue. Note that $-1-\theta$ is now the least eigenvalue of $\overline{G}$.

If G is disconnected, then $\overline{G}$ is connected so assume without loss of generality that G is connected, so that $\theta < k$. If $\theta = \tau$, then G has just two eigenvalues, making G complete. Similarly, if $\tau \geq -1$ then G is complete. Therefore we can assume that $\tau < \theta < k$ and $\tau < -1$. Moreover $-1-\theta < 0$.

The first ingredient is [7, first part of Proposition 1.3.1]. For completeness we include the proof here as well. If A is the adjacency matrix of G, then the spectrum of A^2 consists of the squares of the eigenvalues of A, with the same eigenvectors. If $v \in V(G)$, then $(A^2)_{vv}$ is equal to the number of neighbors of v, which is k. So the trace of A^2 is kn, so that

$$\sum_{\lambda \in \mathrm{Spec}(G)} \lambda^2 = kn.$$

By convexity of the function $f(x) = x^2$, if we have $\tau \leq \lambda \leq \theta$ and $t_\lambda \in (0,1)$ such that $\lambda = t_\lambda \cdot \tau + (1-t_\lambda)\cdot\theta$, then $\lambda^2 \leq t_\lambda \cdot \tau^2 + (1-t_\lambda)\cdot\theta^2$. Since the convexity is strict this is an equality if and only if λ is equal to either τ or θ. Applying this to all $n-1$ eigenvalues in the interval $[\tau,\theta]$, and writing $t = \sum_{\lambda\neq k} t_\lambda$, we obtain

$$kn = \sum_{\lambda \in \mathrm{Spec}(G)} \lambda^2 \leq k^2 + t\cdot\tau^2 + (n-1-t)\cdot\theta^2,$$

with equality if and only if G has all eigenvalues λ with $\tau \leq \lambda \leq \theta$ equal to τ or θ, if and only if G has at most three eigenvalues, if and only if G is strongly regular (by [7, Theorem 9.1.2]). Note that we know as well that

$$t\cdot\tau + (n-1-t)\cdot\theta + k = \sum_{\lambda \in \mathrm{Spec}(A)} \lambda = 0,$$

so that

$$t = \frac{(n-1)\theta + k}{\theta - \tau}, \quad n-1-t = \frac{(n-1)(-\tau) - k}{\theta - \tau}.$$

We can now substitute this in the inequality we found, to get that

$$kn \leq k^2 + \tau^2\frac{(n-1)\theta + k}{\theta-\tau} + \theta^2\frac{(n-1)(-\tau) - k}{\theta-\tau},$$

with equality if and only if G is strongly regular. Combining the two fractions, combining like terms and factorizing, we get

$$kn \leq k^2 + \frac{(n-1)\theta\tau(\tau-\theta) + k(\tau-\theta)(\tau+\theta)}{\theta-\tau},$$

so that the denominator cancels out to

$$kn \leq k^2 - (n-1)\theta\tau - k(\tau+\theta),$$

with equality if and only if G is strongly regular. We distribute, rearrange and refactorize the terms in the inequality further:

$$\begin{aligned} kn &\leq k^2 - n\tau\theta + \theta\tau - k\tau - k\theta, \\ k\tau - n\tau - \theta\tau - k^2 + kn + k\theta &\leq -n\tau - n\tau\theta, \\ (\tau-k)(k-n-\theta) &\leq -n\tau(1+\theta), \end{aligned}$$

with equality if and only if G is strongly regular. Note that $-\tau(1+\theta)$ is positive, so we can divide by it without changing the inequality, to get

$$\begin{aligned} \frac{\tau-k}{\tau} \cdot \frac{k-n-\theta}{-1-\theta} &\leq n, \\ \left(1-\frac{k}{\tau}\right)\left(1-\frac{n-k-1}{-1-\theta}\right) &\leq n. \end{aligned}$$

One recognizes the two Hoffman bounds. After substituting, we get

$$h(G)h(\overline{G}) \leq n,$$

with equality if and only if G is strongly regular. This concludes the proof. □

In addition to the fact that this result is interesting in and of itself, it has the following corollary, excluding Delsarte-Hoffman cliques for Hoffman colorable graphs that are regular but not strongly regular.

Corollary 4.13. *Let G be a regular graph. If G is Hoffman colorable and has a Delsarte-Hoffman clique, then G is strongly regular.*

Proof. Since G is Hoffman colorable, we have $\chi(G) = h(G)$. Since G has a Delsarte-Hoffman clique, we have $\omega(G) = \frac{n}{h(\overline{G})}$. Since $\omega(G) \leq \chi(G)$, we must now have

$$\frac{n}{h(\overline{G})} \leq h(G),$$

which by Theorem 4.12 implies that G is strongly regular. □

This result applies to the class of strictly Neumaier graphs (see [2]). A *Neumaier graph* is an edge-regular graph that contains a Hoffman-Delsarte clique. A graph is *strictly Neumaier* if it is not strongly regular. We get the following.

Corollary 4.14. *No strictly Neumaier graph is Hoffman colorable.*

Chapter 5

Hoffman's bound in strongly regular graphs

In this chapter we will investigate the Hoffman bound in strongly regular graphs. Quite some work already has been done in the literature (see [19], [33], [4]), see Section 3.2 for more details. In Theorem 4.12 we proved that strong regularity of a regular graph can be deduced from the Hoffman bound of the complement and of the graph itself only. In this chapter we will introduce a new parameterization of strongly regular graphs based on the parameters of partial geometries. The geometric parameters show the connection between the Hoffman bound and pseudo-geometricity. Next, we introduce a type system for strongly regular graphs, which is a refinement of the types set out in [29] that also uncovers the connection to pseudo-geometricity in strongly regular graphs. Lastly we will revisit vector colorings and show a connection between Hoffman colorability and unique vector colorability.

5.1 Geometric parameters

Recall from Proposition 2.14 that the collinearity graph of a partial geometry with parameters (s, t, α) is a strongly regular graph with parameters

$$\left((s+1)\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

The three eigenvalues of a strongly regular graph with these parameters are

$$-t-1 < s-\alpha < s(t+1).$$

We can use the geometric parameters s, t, α for geometric graphs (graphs that are the collinearity graph of a partial geometry), instead of the usual "combinatorial" parameters (n, k, a, c) or "spectral" parameters (k, θ, τ). However, in this section we extend this geometric parameterization not only to pseudo-geometric graphs, but to all strongly regular graphs.

In the context of Neumaier graphs, the parameters s and α (which in that context was called e) are introduced in [2], and it turns out that these parameters are very important in the context of strongly regular graphs in conjunction with the parameter t. This is because they uncover the connection of the Hoffman bound to pseudo-geometricity itself. We find the geometric parameters from the three eigenvalues of a primitive strongly regular graph.

Proposition 5.1. *Let G be a primitive strongly regular graph with parameters (n, k, a, c) and eigenvalues $\tau < \theta < k$. Define $s = \frac{-k}{\tau}$, $t = -\tau - 1$, $\alpha = s - \theta$. Then*

$$(n, k, a, c) = \left((s+1)\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

Moreover, the Hoffman bound of G is equal to $s+1$. If m_τ and m_θ are the multiplicities of τ and θ, then

$$m_\tau = s + \frac{(s-\alpha)st(s+1)}{\alpha(1+t+s-\alpha)},$$
$$m_\theta = \frac{(t+1)st(s+1)}{\alpha(1+t+s-\alpha)}.$$

Proof. It is clear by definition that $h(G) = s+1$ and $k = s(t+1)$. By Proposition 2.13, $(x-\tau)(x-\theta) = x^2 + (c-a)x + (c-k)$. Firstly, this implies that $\tau\theta = c - k < 0$, and so $\theta = \frac{-k-(-c)}{\tau} = s - \frac{-c}{\tau}$, so that $\alpha = \frac{-c}{\tau}$ and $c = \alpha(t+1)$. Secondly, this implies $\tau + \theta = a - c$, and so

$$a = c + \tau + \theta = \alpha(t+1) - (t+1) + s - \alpha = s - 1 + t(\alpha - 1).$$

From the same proposition, $c(n-k-1) = k(k-a-1)$. Rearranging this, we get

$$cn = c(k+1) + k(k-a-1) = k^2 + (c-a)k + (c-k) = (k-\tau)(k-\theta).$$

We consider both factors independently.

$$k - \tau = s(t+1) + t + 1 = (s+1)(t+1),$$
$$k - \theta = s(t+1) - (s-\alpha) = st + \alpha.$$

Now

$$n = \frac{(t-\tau)(t-\theta)}{c} = \frac{(s+1)(t+1)(st+\alpha)}{\alpha(t+1)} = (s+1)\frac{st+\alpha}{\alpha},$$

with which we are done for the parameters. For the multiplicities, note that $k + m_\tau\tau + m_\theta\theta = 0$, and $m_\tau + m_\theta = n - 1$. Introducing the new variable $m'_\tau := m_\tau - s$ and substituting $k = -s\tau$, we get $m'_\tau\tau + m_\theta\theta = 0$ and $m'_\tau + m_\theta = n - s - 1$. Now this is easily solvable, with

$$m'_\tau = \frac{\theta(n-1-s)}{\theta - \tau},$$
$$m_\theta = \frac{-\tau(n-1-s)}{\theta - \tau}.$$

Note that

$$n - 1 - s = (s+1)\frac{st+\alpha}{\alpha} - (1+s) = (s+1)\frac{st}{\alpha},$$

which we can substitute to get

$$m_\tau = s + \frac{\theta st(s+1)}{\alpha(\theta - \tau)},$$
$$m_\theta = \frac{-\tau st(s+1)}{\alpha(\theta - \tau)}.$$

Substituting $\theta = s - \alpha$ and $\tau = -t - 1$, we get the posed equations. □

This means that every strongly regular graph is pseudo-geometric in a way. However, we will reserve the term "pseudo-geometric" for the case where the parameters are positive integers. Note that others define geometric and pseudo-geometric also for the imprimitive strongly regular graphs (as is done in [3] and [7]), however we will not consider this case. Let's first see some examples of the geometric parameters of several strongly regular graphs.

Example 5.2. If a strongly regular graph is the collinearity graph of an (s, t, α)-partial geometry, then its geometric parameters will also be (s, t, α). Consequently, the parameters of the square lattice graph $L(K_{m,m})$ are $(m-1, 1, 1)$, and the parameters of the triangular graph $L(K_m)$ are $(m-2, 1, 2)$. The parameters of the pentagon are

$$\left(\sqrt{5} - 1, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right),$$

providing an example with irrational geometric parameters. An example of rational but not integral parameters is given by the Petersen graph, which has geometric parameters $(3/2, 1, 1/2)$.

The values of s, t, and α obtained from Proposition 5.1 are all positive, since $\tau < -1$ by Corollary 2.10. However, they might not be integral, or even rational. Furthermore, for partial geometries, we have $\alpha \leq 1 + \min(s, t)$, but it turns out that for the geometric parameters of strongly regular graphs this does not hold, as we will see in the following example.

Example 5.3. The *Clebsch graph* be obtained as the complement of the folded 5-cube. The *folded 5-cube* is a strongly regular graph obtained by identifying antipodal vertices of the 5-dimensional hypercube graph. The folded 5-cube has combinatorial parameters $(16, 5, 0, 2)$. Now the Clebsch graph has combinatorial parameters $(16, 10, 6, 6)$, and its geometric parameters are therefore given by $(5, 1, 3)$. Note that we have $\alpha > 1 + t$. We can conclude that the Clebsch graph is pseudo-geometric but not geometric.

Recall that in partial geometries α stood for the number of points on a given line that are collinear with a given point not on this line. We see something very similar for the geometric parameters of strongly regular graphs, however now with Delsarte-Hoffman cliques instead of lines. This proposition is just the ratio bound (Theorem 1.1, [10]) translated through a complement to the geometric parameters. Also compare [2, Proposition 2.3], which concerns the ratio bound in the context of Neumaier graphs.

Proposition 5.4. *Let G be a primitive strongly regular graph with geometric parameters (s, t, α). If C is a clique in G, then*

$$|C| \leq 1 + s,$$

and equality implies that every vertex outside of C is adjacent to precisely α vertices in C.

Proof. The ratio bound on cliques is in the case of strongly regular graphs equal to the Hoffman bound, which is $1 + s$.

In case of equality, in $\overline{G}$, every vertex out of the (now coclique) C is connected to $-\lambda_{\min}(\overline{G})$ vertices of C. By [7, Section 1.3.2] we know $-\lambda_{\min}(\overline{G}) = \theta + 1$. In G, this implies that every vertex outside of C is connected to precisely $(1 + s) - (1 + \theta) = s - \theta = \alpha$ vertices of C. □

The rationality or integrality of the parameters and the eigenvalues are related to each other. This is expressed in the following proposition. One part of this is that for strongly regular graphs, being pseudo-geometric is equivalent to having an integral Hoffman bound.

Proposition 5.5. *Let G be a primitive strongly regular graph with parameters (n, k, a, c) and geometric parameters (s, t, α). Then the following are equivalent.*

(i a) The Hoffman bound $h(G)$ is rational;

(i b) s is rational;

(i c) α is rational;

(i d) t is integral;

(i e) τ is integral;

(i f) θ is integral;

(i g) G is integral, meaning that all eigenvalues are integral;

(i h) s and α have equal fractional parts.

Furthermore, the following are equivalent.

(ii a) The Hoffman bound $h(G)$ is integral;

(ii b) s is integral;

(ii c) α is integral;

(ii d) G is pseudo-geometric.

Proof. Ad (i): Since $h(G) = 1 + s$, it is clear that (i a) and (i b) are equivalent. Since $s/\alpha = k/c \in \mathbb{Q}$, also (i b) and (i c) are equivalent. By $t = -\tau - 1$, (i d) and (i e) are equivalent. Since k is integral, and since $(x - \tau)(x - \theta)$ is a polynomial with integer coefficients, (i e), (i f) and (i g) are equivalent. Since $\theta = s - \alpha$, (i f) and (i h) are equivalent. Finally, since $s\tau = -k \in \mathbb{Q}$ and τ is an algebraic integer, also (i b) is equivalent to (i e).

Ad (ii): By $h(G) = 1 + s$ (ii a) and (ii b) are equivalent. By (i h) also (ii b) and (ii c) are equivalent. It is clear that (ii d) implies both (ii b) and (ii c), and conversely (ii b), (ii c) and (i d) imply (ii d). □

The Petersen graph provides an example of a graph satisfying the (i)-statements but not the (ii)-statements of the above proposition (see Example 5.2). For Hoffman colorable graphs we get the following consequence.

Proposition 5.6. *Every primitive strongly regular graph with a Hoffman coloring is pseudo-geometric. Moreover, then α divides st.*

Proof. Let G be primitive strongly regular. Suppose that G is Hoffman colorable. Then in particular $h(G)$ is an integer. By the previous result it is pseudo-geometric. By Proposition 3.1 all color classes have the same size, so n must be a multiple of χ. In terms of the geometric parameters, this means that $\frac{st+\alpha}{\alpha}$ must be an integer, implying that α divides st. □

The complement of a strongly regular graphs is again strongly regular. We know what effect this has on the combinatorial and the spectral parameters, so let's also investigate this for the geometric parameters.

Proposition 5.7. *Let G be primitive strongly regular with parameters (n, k, a, c) and (s, t, α). Then the complement $\overline{G}$ has the following parameters:*

$$(\overline{s}, \overline{t}, \overline{\alpha}) = \left(\frac{st}{\alpha}, s - \alpha, \frac{t(s-\alpha)}{\alpha}\right).$$

Proof. If $\tau < \theta < k$ are the eigenvalues of G, then $-1 - \theta < -1 - \tau < n - k - 1$ are the eigenvalues of $\overline{G}$ (see [7, Section 1.3.2]). We saw in Proposition 5.1 that $\theta = s - \alpha$. So $\overline{t} = -\overline{\tau} - 1 = -(-1 - \theta) - 1 = \theta = s - \alpha$. This also implies $t = \overline{s} - \overline{\alpha}$.

Note that $t = -\tau - 1$ and $\overline{\tau} = -1 - \theta$, so $t\overline{\tau} = (-1 - \tau)(-1 - \theta)$, is the value of $(x-\tau)(x-\theta)$ for the input $x = -1$. We know that $(x-\tau)(x-\theta) = x^2+(c-a)x+c-k$, so $t\overline{\tau} = 1 + a - k$. The equation $c(n - k - 1) = k(k - a - 1)$ now boils down to $c\overline{k} = -kt\overline{\tau}$. Dividing by τ now gives $-\alpha\overline{k} = st\overline{\tau}$, and so $\frac{-\overline{k}}{\overline{\tau}} = \frac{st}{\alpha}$, giving $\overline{s} = \frac{st}{\alpha}$.

By $t = \overline{s} - \overline{\alpha}$ and $\overline{s} = \frac{st}{\alpha}$, we see $\overline{\alpha} = \frac{st-t\alpha}{\alpha} = \frac{t(s-\alpha)}{\alpha}$. □

This means that we can predict when the complement of a pseudo-geometric graph is again pseudo-geometric.

Proposition 5.8. *Let G be a pseudo-geometric graph with parameters (s, t, α). Then $\overline{G}$ is pseudo-geometric if and only if α is a divisor of st.*

Proof. By Proposition 5.5, $\overline{G}$ is pseudo-geometric if and only if $\overline{s}$ is integral. Since $\overline{s} = \frac{st}{\alpha}$ by the previous proposition, this is equivalent to α dividing st. □

Note that we have met the condition that α divides st in Proposition 5.6. Namely, Hoffman colorable graphs satisfy this divisibility condition. We conclude with the following new necessary condition on Hoffman colorability of strongly regular graphs.

Theorem 5.9. *If G is primitive strongly regular and Hoffman colorable, then G and $\overline{G}$ are pseudo-geometric.*

5.2 Types

In this section we will introduce a type system for strongly regular graphs. The goal is to classify Hoffman colorability, pseudo-geometricity and spreadability at once in a type system. The basis of the type system is the types introduced by Roberson [29], but it also has elements of the types from [3, Chapter 8] and new elements.

Recall Corollary 3.3 ([29, Equation (3)]), which states that for strongly regular graphs we have the chain of inequalities $\omega(G) \leq h(G) \leq \chi(G)$. Roberson uses this to divide strongly regular graphs up into four distinct types (see [29, Section 4.1]), according to whether tightness occurs:
Type A: $\omega(G) < h(G) = \chi(G)$,
Type B: $\omega(G) = h(G) = \chi(G)$,
Type C: $\omega(G) = h(G) < \chi(G)$,
Type X: $\omega(G) < h(G) < \chi(G)$.
As said before, we want to capture spreadability as well in these types, and therefore we split up types B and C. We define a graph of type B to be BS if it admits a spread and BN if not. Similarly we define types CS and CN. Also, roughly following Types I and II from [3, Chapter 8] (also compare [4, Chapter II, Remark 9.13]), we subdivide type X into type XZ, where Hoffman's bound is integral, XQ, where Hoffman's bound is rational but not integral and XX, where Hoffman's bound is irrational. Whenever it is convenient, we will use Roberson's types instead of our refinement.

Hoffman colorability is now obtained in classes A, BN or BS. The strongly regular graphs that are spreadable are those of type BS and CS. Lastly, the graphs that are not pseudo-geometric are of type XQ or XX. In this way the concepts of Hoffman colorability, spreadability and pseudo-geometricity are captured all at once.

Geometric graphs have Delsarte-Hoffman cliques, so they are of type B or C, although not every graph of these types is automatically geometric. The graphs that are of type B or C are the strongly regular Neumaier graphs (see [2]).

We have a lot of information regarding taking complements. A spread in a strongly regular graph corresponds to a Hoffman coloring in the complement. Also, by $h(G)h(\overline{G}) = n$ (Corollary 4.11), if the Hoffman bound of a strongly regular graph is rational then also the Hoffman bound of its complement is rational. This gives the following result.

Proposition 5.10. *Let G be a primitive strongly regular graph. Then exactly one of the following holds.*

(i) Both G and $\overline{G}$ are of type BS;

(ii) One of G and $\overline{G}$ is of type CS and the other is of type A or BN;

(iii) Both G and $\overline{G}$ are of type CN, XZ or XQ (not necessarily of the same type);

(iv) Both G and $\overline{G}$ are of type XX.

For geometric graphs, we have the following result, which follows directly from the discussion in Section 3.2.

Corollary 5.11. *Let PG be a partial geometry. Write G for its point graph, then G, $\overline{G}$, respectively have to be of the types in the cell corresponding to the properties in the following table.*

Property	*G has no spread*	*G has a spread*
PG has no ovoid	*CN, XZ / CN, XQ*	*CS, A*
PG has an ovoid, but no fan	*CN, CN*	*CS, BN*
PG has a fan	*BN, CS*	*BS, BS*

If PG has a spread, then G has a spread. The converse is not true in general. If G has no more Delsarte-Hoffman cliques than those coming from the lines in PG or if every edge of G is in a unique Delsarte-Hoffman clique, then the properties in the top row can be stated as "PG has no spread" and "PG has a spread", purely in terms of the partial geometry. This holds for generalized quadrangles, as in those partial geometries no triangles exist. However, it does not hold in general.

5.3 Classifying primitive strongly regular graphs with up to 41 vertices

In this section we will find the type of all primitive strongly regular graphs with up to 41 vertices, as this is how far the database of computer algebra system Magma [6], goes. This database of strongly regular graphs is based on McKay's catalogue [27].

Spreadability (and indirectly Hoffman colorability) has already been studied in [19], and unique vector colorability in [14]. What we add is to consider these concepts simultaneously.

We will use the table of strongly regular graphs in [7, Section 9.9]. Every primitive strongly regular graph with up to 41 vertices, or its complement, is one of the following.

- A conference graph;
- A triangular graph $L(K_m)$ $(m \geq 5)$;
- A square lattice graph $L(K_{m,m})$ $(m \geq 3)$;
- The Schläfli graph;
- The Clebsch graph;
- The Shrikhande graph;
- One of the three Chang graphs;

- A strongly reular graph with one of the following parameter sets:
  - $(26, 10, 3, 4)$;
  - $(35, 16, 6, 8)$;
  - $(36, 14, 4, 6)$;
  - $(36, 15, 6, 6)$;
  - $(40, 12, 2, 4)$.

Some of these graphs are geometric, namely triangular graphs, square lattice graphs and the Schläfli graph. To these graphs we can apply Corollary 5.11. We will cover the cases one by one.

5.3.1 Conference graphs

Conference graphs are those strongly regular graphs with the multiplicities of τ and θ equal. As a consequence, they have the same parameters as their complements (see [3, Chapter 8]). By the discussion after Theorem 3.1 in [3, Chapter 8], every graph of type XX is a conference graph.

So, for conference graphs the parameters of the complements are the same, and so $s = \overline{s}$, which implies $n = (s+1)(\overline{s}+1) = (s+1)^2 = h(G)^2$. If n is not a square, then we can immediately conclude that the graph is of type XX, as Hoffman's bound will be irrational. The only interesting cases are therefore when n is a square.

Paley graphs

An important family of conference graphs are the Paley graphs, and they are constructed as follows. Let q be a prime power that is one more than a multiple of four, then the *Paley graph* Paley(q) is defined as the graph with the field of q elements $\mathbb{F}_q$ as vertices, such that $a \sim b$ if and only if $a - b$ is a non-zero square in $\mathbb{F}_q$ (see [7, Section 9.1.1] or [3, Chapter 8]).

Paley graphs are self-complementary: multiplication by a non-square in $\mathbb{F}_q$ gives an isomorphism of the Paley graph and its complement. Now the Paley graph obviously has the same parameters as its complement, and is therefore a conference graph. By Proposition 5.10, self-complementary strongly regular graphs are of type X, CN or BS. If q is not a square, then we have already seen that Paley(q) is of type XX. For the Paley graphs of square order we will see shortly that they are of type BS. We need the following lemma.

Lemma 5.12. *Let $q = r^2$ be a square of an odd prime power r, and consider the inclusion of fields*

$$\mathbb{F}_r \subseteq \mathbb{F}_q.$$

Then every element of $\mathbb{F}_r$ is a square in $\mathbb{F}_q$.

Proof. The unit groups of $\mathbb{F}_q$ and $\mathbb{F}_r$ are cyclic, and we have

$$\mathbb{F}_r^* \subseteq \mathbb{F}_q^*.$$

Write g for a generator of $\mathbb{F}_q^*$, then $\mathbb{F}_r^*$ consists of those elements of $\mathbb{F}_q^*$ for order dividing $r-1$. As a consequence,

$$\mathbb{F}_r^* = \langle g^{(q-1)/(r-1)} \rangle,$$

and $(q-1)/(r-1) = r+1$ is even, so every element of $\mathbb{F}_r$ is a square in $\mathbb{F}_q$. □

Proposition 5.13. *If q is a square, then Paley(q) has a spread and a Hoffman coloring. So Paley(q) is of type BS.*

Proof. Since the Paley graphs are self-complementary we only have to find a spread. Write r for the square root of q, then r is still an odd prime power. In the previous lemma we showed that every element of $\mathbb{F}_r$ is a square in $\mathbb{F}_q$. As a consequence, for any $w \in \mathbb{F}_q$ the coset $w + \mathbb{F}_r$ will form a clique in the Paley graph. This way the cosets form a spread. □

Remaining small conference graphs

At this point we have classified the Paley graphs and the conference graphs of non-square order. The only remaining parameter set is $(25, 12, 5, 6)$. There are fifteen graphs with these parameters, of which one is Paley(25), of type BS. We revert to a computer search. We find one graph of type BN, with complement of type CS. The twelve remaining graphs are type CN. The one graph of type CS is related to a Latin square graph of order 5.

5.3.2 Geometric graphs

The next couple of graphs, namely the triangular graphs, square lattice graphs and the Schläfli graphs, are geometric, so Proposition 3.4 and Corollary 5.11 apply. We can immediately see that the square lattice graphs are of type BS, and the Schläfli is of type A (so by Proposition 5.10 its complement is of type CS). We also know that the triangular graphs of even order $L(K_{2m})$ are Hoffman colorable, and the ones of odd order $L(K_{2m+1})$ are not, and that $\overline{L(K_m)}$ contains Delsarte-Hoffman cocliques. We also know that the complete linear space (the partial geometry with parameters $(1, t, 2)$, which is equivalent to the complete graph on $t+2$ vertices) does not have ovoids, and so the dual of the complete linear space (of which the point graph is $L(K_{t+2})$) does not have spread. However, we do not know yet if triangular graphs have a spread as strongly regular graphs (which is equivalent to the Hoffman colorability of $\overline{L(K_m)}$).

One can see that $L(K_m)$ does not have a spread by showing that every Delsarte-Hoffman clique in the strongly regular graph $L(K_m)$ comes from a point in the complete linear space. The points in the complete linear space give the cliques in $L(K_m)$ consisting of all edges that are incident to a certain pre-specified vertex: for $v \in V(K_m)$, we have

$$C_v = \{\{u, v\} : u \in V(K_m) \setminus \{v\}\}.$$

Let C be a clique in $L(K_m)$ of size $m-1$. Since $m \geq 5$, the clique must be of size at least 4. Pick two edges e_1, e_2 of K_m in the clique C. They must be adjacent, so they must share a vertex, $v \in V(K_m)$, say $e_i = \{u_i, v\}$ for some u_i. Consider a third edge e_3 of the clique. It must share a vertex with both e_1 and e_2. If $e_3 = \{u_1, u_2\}$, then the fourth edge in the clique cannot be adjacent to e_1, e_2, e_3 simultaneously. So $e_3 = \{u_3, v\}$ for some u_3. Now every other edge in the clique must go from vertex v in order to be adjacent to e_1, e_2, e_3, and so $C = C_v$.

This means that every Delsarte-Hoffman clique of a triangular graph comes from a line in the partial geometry, which does not have a spread, and so the complement of a triangular graph is never Hoffman colorable. We get that $L(K_{2m})$ is of type BN and $\overline{L(K_{2m})}$ of type CS. We also know that $L(K_{2m+1})$ is of type CN, and $\overline{L(K_{2m+1})}$ of type X. In fact, it is of type XQ since its Hoffman bound is $m + \frac{1}{2}$ (see Proposition 5.7).

5.3.3 Remaining graphs

Recall from Example 5.3 that the Clebsch graph is the complement of the folded 5-cube. The Clebsch graph has geometric parameters $(5, 1, 3)$, and so by Proposition 5.7 the folded 5-cube has geometric parameters $(5/3, 2, 2/3)$ and is therefore of type XQ. The Clebsch graph is hence of type CN or XZ by Proposition 5.10, depending on if it has a clique of size 6 or not. It turns out to have a clique of size 5, for example the set

$$\{(0,0,0,0,0), (1,1,0,0,0), (1,0,1,0,0), (1,0,0,1,0), (1,0,0,0,1)\},$$

of vertices of the folded 5-cube, but a 6-clique is impossible. Therefore the Clebsch graph is of type XZ.

The *Shrikhande graph* and the three *Chang graphs* are the cospectral mates to $L(K_{4,4})$ and $L(K_8)$, respectively. These four graphs are the only cospectral mates to any of the triangular or square lattice graphs (see [3, Corollary 4.3 and Corollary 4.4]). They are Hoffman colorable, but their clique numbers are not equal to their Hoffman bounds. Therefore they are of type A. Their complements are therefore of type CS.

By computer search, we find the following data for the five remaining parameter sets set out at the beginning of Section 5.3 and their complements:

(n, k, a, c)	number of graphs in the database	type distribution
$(26, 10, 3, 4)$ $(26, 15, 8, 9)$	10	all XQ all CN
$(35, 16, 6, 8)$ $(35, 18, 9, 9)$	3854	180 XZ, 3652 CN, 22 CS 293 XZ, 3539 CN, 22 BN
$(36, 14, 4, 6)$ $(36, 21, 12, 12)$	180	all XQ 178 XZ, 2 CN
$(36, 15, 6, 6)$ $(36, 20, 10, 12)$	32548	32517 XZ, 19 CN, 12 CS 26678 XZ, 5858 CN, 6 A, 6 BN
$(40, 12, 2, 4)$ $(40, 27, 18, 18)$	28	1 XZ, 25 CN, 2 CS 15 XZ, 11 CN, 2 A

This gives the following list of graphs of Hoffman colorable types.

- Type A: Shrikhande graph, Schläfli graph, Chang graphs, 6 graphs with parameters $(36, 20, 10, 12)$, 2 graphs with parameters $(40, 27, 18, 18)$;
- Type BN: $L(K_6)$, $L(K_8)$, 1 conference graph on 25 vertices, 22 graphs with parameters $(35, 18, 9, 9)$, 6 graphs with parameters $(36, 20, 10, 12)$;
- Type BS: Paley(9), Paley(25), $L(K_{4,4})$, $L(K_{5,5})$, $L(K_{6,6})$.

Thus far, we have seen every type pair from Proposition 5.10 appear, except a XQ-XQ pair. There is no small example of this, but they do exist: the Higman-Sims graph is the unique strongly regular graph with parameters $(100, 22, 0, 6)$ and eigenvalues $(22, 2, -8)$, which gives geometric parameters $(11/4, 7, 3/4)$, so it is of type XQ. The complement has geometric parameters $(77/3, 2, 56/3)$, so is also of type XQ. As one can check from the table of possible parameters of strongly regular graphs in [7, Section 9.9], there is no smaller XQ-XQ pair.

Comparing our results to the content of [19, Proposition 6.1 and Table 1], we see that it matches up. With parameters $(35, 18, 9, 9)$ we indeed have no strongly regular graphs with a spread (as proved in [19, Proposition 6.1]). For parameter set $(35, 16, 6, 8)$ there is a spread originating from a linear space, for parameter set $(36, 15, 6, 6)$ there is a spread for each of the twelve Latin squares of order 6, and for parameter set $(40, 12, 2, 4)$ we have a strongly regular graph coming from a spread, as predicted in [19, Table 1]. Our table does not go as far as the table from [19] (because of the limit of the database of strongly regular graphs in magma). However, it contains data of how many of strongly regular graphs are Hoffman colorable/spreadable, and it also contains a refinement of the types.

5.4 Unique vector colorability

Recall that from Proposition 3.5 and Proposition 3.6 it follows that 2-walk regular graphs (except bipartite and regular complete multipartite graphs) have a locally injective canonical strict vector coloring with value equal to the Hoffman bound $\chi_{sv}(G) = h(G)$.

However, if a graph G is Hoffman colorable, and this coloring is not locally injective, we obtain a different strict vector Hoffman coloring, implying that G is not uniquely vector colorable. Graphs with diameter 2 do not have locally injective optimal colorings, as proven in the next lemma.

Lemma 5.14. *Let G be a graph of diameter at most 2. If $f : V(G) \to \{1, \dots, c\}$ is a locally injective coloring of G, then $|V(G)| \leq c$. Furthermore, if G is of diameter 2, then no optimal coloring is locally injective.*

Proof. Suppose that $\mathrm{diam}(G) \leq 2$ and that f is a locally injective coloring. Let u and v be distinct vertices of G. If $u \sim v$, then we must have $f(u) \neq f(v)$ because f is a coloring. If $u \not\sim v$, then u and v must be of distance 2, so they share a common neighbor. By local injectivity of f, we have $f(u) \neq f(v)$. So f assigns a unique color to every vertex, so $|V(G)| \leq c$. If $\mathrm{diam}(G) = 2$, then there must exist two vertices at distance 2. These two vertices could get the same color, which proves $\chi(G) \leq |V(G)| - 1$. This concludes the proof. □

The intersection of the class of 2-walk regular graphs (excepting bipartite and regular complete multipartite graphs) and the class of graphs with diameter 2 is precisely the primitive strongly regular graphs. We obtain the following result.

Theorem 5.15. *A Hoffman colorable primitive strongly regular graph is not uniquely vector colorable.*

Combined with [29, Corollary 4.2] (which states in particular that if a strongly regular is not a core, then it is Hoffman colorable), Theorem 5.15 is an improvement of [14, Theorem 3.12] (which states that every uniquely vector colorable imprimitive strongly regular graph is a core).

From Theorem 5.15 we see that all uniquely vector colorable primitive (UVC) strongly regular graphs are of type C or X. Comparing our table to [14, Table 1], we see that the twelve non-UVC ("loose") graphs with parameters $(36, 20, 10, 12)$ are exactly the six graphs of type A and the six graphs of type BN. These twelve graphs arise from the twelve Latin squares of order 6.

Furthermore, of the fifteen graphs with parameters $(25, 12, 5, 6)$, we found exactly two with a Hoffman coloring, and these happen to be the only graphs with these parameters that are not UVC. Also for $(16, 9, 4, 6)$ we see that exactly one graph is UVC, and this has to be the complement of the Shrikhande graph, as the Shrikhande graph is of type A and $L(K_{4,4})$ is of type BS. For many parameters sets there are no graphs that are not UVC and not Hoffman colorable. However, examples of non-UVC non-Hoffman colorable graphs do exist: $L(K_7)$ and all of the 8526 non-UVC graphs with parameters $(36, 15, 6, 6)$.

Chapter 6

The Decomposition Theorem and its consequences

In this chapter we will state and prove one of the main new result of this thesis, namely the Decomposition Theorem. We will first state and digest the theorem, after which we will provide the proof. After that, we will study various corollaries that the Decomposition Theorem has. Among other things, we will be able to characterize Hoffman colorability of cone graphs completely, and furthermore finish the characterization of Hoffman colorability of line graphs which we started in Section 4.1.4.

Theorem 6.1 (Decomposition of Hoffman colorings)**.** *Let G be a non-empty graph with positive eigenvector x and Hoffman coloring $V(G) = \bigsqcup_{i=1}^{\chi} V_i$. Let J be a subset of the colors $\{1, \dots, \chi\}$ with $|J| \geq 2$. Let H be the induced subgraph of G on the vertices $\bigcup_{j \in J} V_j$. Then the following hold.*

(i) *H is Hoffman colorable, with coloring $V(H) = \bigsqcup_{j \in J} V_j$;*

(ii) $\lambda_{\max}(H) = \dfrac{|J| - 1}{\chi - 1} \lambda_{\max}(G)$;

(iii) *The restriction $x|_{V(H)}$ is a positive eigenvector of H, and consequently its eigenvalue is $\lambda_{\max}(H)$;*

(iv) $\lambda_{\min}(H) = \lambda_{\min}(G)$.

Note that this means that the class of Hoffman colorable graphs with positive eigenvector is closed under the operation of removing color classes. This does not hold for the subclass of connected Hoffman colorable graphs; we will an example of this in Section 8.4.2.

As a sanity check, let's investigate what the Decomposition Theorem says about tensor graphs. Recall that, by Proposition 4.6, the tensor product $G \times H$ is Hoffman colorable if G is Hoffman colorable, H possibly contains self loops and furthermore $h(G) \leq h(H)$. The optimal coloring of $G \times H$ comes from G in this case. Choosing color classes for $G \times H$ is therefore equivalent to choosing color classes J for G. So if K is the induced subgraph of G on color classes picked by J, the corresponding induced subgraph of $G \times H$ will be $K \times H$. Note that $h(K) = |J|$, so $h(K) \leq$

$h(H)$ is satisfied, and $K \times H$ is indeed Hoffman colorable by Proposition 4.6. The Decomposition Theorem therefore agrees with the tensor product.

If we choose J to be of size 2 in every possible way, then we see that if we decompose a Hoffman colorable graph into its $\binom{\chi}{2}$ bipartite parts (one bipartite part H_{ij} for every pair of colors $\{i, j\} \subseteq \{1, \ldots, \chi\}$), these parts are compatible in their eigenvectors and eigenvalues.

Definition 6.2. A *collection of compatible bipartite parts* (H_{ij}) is a collection of graphs for every pair $\{i, j\} \subseteq \{1, \ldots, c\}$ (with $i \neq j$) such that a sequence of positive integers $(a_i)_{i=1}^{c}$ exists and the following hold.

- H_{ij} is bipartite with two bipartite classes of size a_i and a_j for each i, j;
- There exists a positive real number ν such that each H_{ij} has largest eigenvalue ν;
- Each H_{ij} has a positive eigenvector x_{ij} for ν;
- If i is fixed, then the projections of x_{ij} onto the bipartite class of size a_i agree for every j (that is, they are equal up to reordering and rescaling).

Moreover, we say that a graph G is *composed* of the collection of compatible bipartite parts if it has a c-coloring $V(G) = \bigsqcup_{i=1}^{c} V_i$ such that H_{ij} is given by the induced subgraph of G on $V_i \cup V_j$, in such a way that if $v \in V_i$, then $x_{ij}(v)$ is independent of j. In this case we write $x_i(v)$ instead of $x_{ij}(v)$.

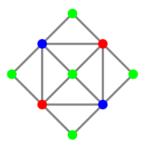
We see that Hoffman colorable graphs with a positive eigenvector decompose into collections of compatible bipartite parts, by the Decomposition Theorem.

The numbers $(a_i)_{i=1}^{c}$ can be easily found if $c \geq 3$:

$$2a_i = |V(H_{ij})| + |V(H_{ik})| - |V(H_{jk})|.$$

Further note that the two projections of an eigenvector of a bipartite graph onto the two bipartite classes have equal norm. So, given a collection of compatible bipartite parts, we could normalize every x_{ij} to be of norm $\sqrt{2}$ (so that their projections onto the classes are unit vectors). Then the "agreeing" in the fourth item of the definition now only means being equal up to reordering (as the rescaling is done already).

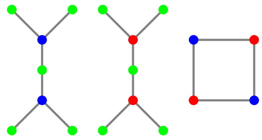
Example 6.3. To illustrate the Decomposition Theorem and collections of compatible bipartite parts, we can take a look at the following graph G:



This graph has a chromatic number of 3, and its spectrum is given by

$$\{(-2)^2, (-\sqrt{2})^2, 0^2, (\sqrt{2})^2, 4\},$$

so it is Hoffman colorable. It decomposes into the following three graphs:



A positive eigenvector of the left two graphs is given by a function $x : V \to \mathbb{R}$ giving 1 on the four leaves, and 2 on the middle three vertices (see Example 2.1). This is the Perron eigenvector, and the eigenvalue is 2. The 4-cycle is 2-regular, so a positive eigenvector can be given by a constant vector (for example constant 1). Also note that the given eigenvectors agree: for the green vertices we get the exact same vector, and for the red and blue vertices, we can see that the vectors $(2, 2)$ and $(1, 1)$ are identical up to rescaling. So these three bipartite parts are compatible. Furthermore, G is composed of this collection of three bipartite parts, as the two green vertices with eigenvector entry 2 of the two bipartite parts come from the same vertex in G (namely the middle vertex). Further note that in this case (since G is Hoffman colorable), the Perron eigenvectors of the three bipartite parts can be obtained using projections of the Perron eigenvector of G. This is because the Perron eigenvector of G can be given by a function $V(G) \to \mathbb{R}$ assigning 1 to the four green vertices of degree 2, and 2 everywhere else.

In the remainder of this chapter, we will see the proof of the Decomposition Theorem, and after that we will study various corollaries of the Decomposition Theorem. For the first set of corollaries, we will look at the algebraic properties of the eigenvalue ν from the collection of bipartite parts. More specifically, the case where ν has a minimal polynomial composed of only even powers of the variable (equivalently, if ν and $-\nu$ are Galois conjugates), turns out to be very restrictive. Secondly, we look at what we can say if the Hoffman colorable graph contains a small color class. Using this theory we classify all Hoffman colorable graphs with a positive eigenvector such that the number of vertices is less than thrice the number of colors. Thirdly, we finish the classification of Hoffman colorability of line graphs.

Before we do any of this, we state three very elementary corollaries of the Decomposition Theorem.

Corollary 6.4. *If G is a nonempty Hoffman colorable graph with positive eigenvector and H is a bipartite part, then H does not have isolated vertices.*

Proof. Since G is non-empty, $\lambda_{\max}(G) > 0$. By the Decomposition Theorem, H now has a positive eigenvector x for a positive eigenvalue ν. If $v \in V(H)$ is isolated, then $(A(H) \cdot x)(v) = 0$, while it should be $\nu x(v) > 0$. □

We will need the above corollary at various places in the coming sections. Besides that, it also has the following corollaries, which immediately follow.

Corollary 6.5. *Let G be a non-empty graph with a positive eigenvector. Then every vertex must have at least one neighbor of every other color for every Hoffman coloring.*

Corollary 6.6. *Let G be a non-empty Hoffman colorable graph with a positive eigenvector. Then every vertex has degree at least $\chi(G) - 1$.*

This means that graphs with a small minimal degree (compared to the chromatic number) cannot have a Hoffman coloring. In particular, non-bipartite Hoffman colorable graphs cannot have leaves.

6.1 Proof of the Decomposition Theorem

This section is devoted to the proof of the Decomposition Theorem. We will need two lemmas. The following lemma classifies the matrices with zeroes on the diagonal that have two eigenvalues of which one is simple and positive. It can also be used to classify the connected graphs with just two eigenvalues, namely only the complete graphs.

Lemma 6.7. *Let M be a diagonalizable matrix that has zeroes on the diagonal and has two eigenvalues of which one is simple and positive. If M only has non-negative entries, then M is a scalar multiple of $J - I$, where J is the all-ones matrix.*

Proof. Let n be the size of M. Since the trace of M is 0, we can find a positive ν such that

$$\mathrm{Spec}(M) = \{(n-1)\nu, (-\nu)^{n-1}\}.$$

Let x be an eigenvector for $(n-1)\nu$ of norm 1 such that $x_1 \geq 0$. Then x generates the eigenspace for $(n-1)\nu$, and the space of vectors orthogonal to x forms the eigenspace for $-\nu$. The projection onto the eigenspace for $(n-1)\nu$ is now given by xx^T, and hence the projection onto the eigenspace for $-\nu$ is $I - xx^T$. By spectral decomposition, we obtain

$$M = (n-1)\nu \cdot xx^T - \nu(I - xx^T) = -\nu I + n\nu \cdot xx^T.$$

At the diagonal we thus have $M_{i,i} = \nu(nx_i^2 - 1) = 0$. We conclude that $x_i^2 = 1/\sqrt{n}$, independent of i. For values off the diagonal, we get $M_{i,j} = n\nu x_i x_j$. Since n and ν are positive and M only has non-negative entries, x_i and x_j must be of the same sign. Arguing for all possible pairs of indices, and taking into consideration that x_i^2 is independent of i, we conclude that x is a constant vector. So for all i we have $x_i = x_1 > 0$. Now

$$xx^T = x_1^2 J = \frac{1}{n} J,$$

and so $M = \nu(J - I)$. □

We will use this to prove the following lemma, which is an extension of both Proposition 3.1 and Proposition 3.2. Part (i) is precisely Proposition 3.2 ([1, Proposition 5.3(i)]), and Proposition 3.1 is Parts (ii) and (iii) restricted to regular graphs. For the sake of completeness, we include the proof of [1, Proposition 5.3(i)] here as well. Moreover, along the way we will see Haemers' interlacing proof of the Hoffman bound (Theorem 1.3), see [17].

Lemma 6.8. *Let G be a non-empty graph with positive eigenvector x, and Hoffman coloring $V(G) = \bigsqcup_{i=1}^{\chi} V_i$. Then*

(i) The partition $V_1, \ldots, V_\chi$ is weight-regular;

(ii) *All irreflexive intersection numbers b^*_{ij} of this weight-regular partition equal $-\lambda_{\min}(G)$;*

(iii) *The projections of x onto the color classes have norm independent of the color class.*

Proof. Write $y(i) = \sqrt{\sum_{v\in V_i} x(v)^2}$, the norm of the projection of x onto color class V_i. Let S be the weight-characteristic matrix, which is a $|V| \times \chi$-matrix given by

$$S_{v,i} = \begin{cases} \frac{x(v)}{y(i)} & \text{if } v \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then note that $S^T S = I_\chi$, so now we can apply Theorem 2.8 to get that the eigenvalues of $B = S^T AS$ interlace those of A. The vector y with $y(i)$ defined as before is an eigenvector of B with eigenvalue $\lambda_{\max}(A)$, because $(Sy)(v) = x(v)$ for all v. Also, B has zeroes on the diagonal, since the color classes are independent sets. So the trace of B is 0, hence the sum of the eigenvalues of B is 0. By the Interlacing Theorem, all the eigenvalues of B are bounded below by $\lambda_{\min}(A)$, hence

$$0 = \operatorname{tr}(B) \geq \lambda_{\max}(A) + (\chi - 1)\lambda_{\min}(A).$$

Reordering gives the Hoffman bound, and this is exactly Haemers' proof. Since the graph G is Hoffman colorable, this inequality is an equality, so that all eigenvalues of B, except the largest one, are equal to $\lambda_{\min}(A)$. We can then conclude two things:

(1) The interlacing of B and A is tight,

(2) B satisfies the requirements of Lemma 6.7.

The first point gives weight-regularity by Corollary 2.12, as proven by Abiad in [1, Proposition 5.3(i)], and furthermore B now contains all the weight-intersection numbers. By the second point, B is a scalar multiple of $J - I$, say $B = \nu \cdot (J - I)$, where $\nu = -\lambda_{\min}(B)$, which by tight interlacing is equal to $-\lambda_{\min}(A)$, proving the second point of this lemma. The vector y is an eigenvector to the largest eigenvalue of B, which is constant, so $y(i)$ is independent of i. This is Part (iii) of this lemma. □

Note that Part (iii) is satisfied by any bipartite graph (see Section 2.1.2), but not necessarily by non-bipartite graphs. This gives some intuition behind the fact that 2-colorable graphs are necessarily Hoffman colorable, whereas graphs that need at least three colors are not necessarily Hoffman colorable.

We are now ready to prove the Decomposition Theorem.

Proof of the Decomposition Theorem. Write ν for the absolute value of the least eigenvalue of G. Note that by Lemma 6.8 for all $i, j \in \{1, \ldots, \chi\}$ with $i \neq j$ and all $u \in V_i$ we have

$$\nu x(u) = \sum_{v\in N_G(u)\cap V_j} x(v).$$

If we fix i in J and $u \in V_i$ and let j run over all other indices in J, then we get

$$\nu(|J| - 1)x(u) = \sum_{j\in J\setminus\{i\}} \sum_{v\in N_G(u)\cap V_j} x(v) = \sum_{v\in N_H(u)} x(v),$$

so $x|_{V(H)}$ is an eigenvector of H with eigenvalue $\nu(|J|-1)$. This is a positive eigenvector, so its eigenvalue is the largest of H. Since H is an induced subgraph of G, by Cauchy interlacing (see Corollary 2.9) we have

$$h(H) \geq -\nu.$$

Looking at the Hoffman bound of H, we get

$$h(H) = 1 - \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)} \geq 1 + \frac{\nu(|J|-1)}{\nu} = |J|.$$

The coloring $V(H) = \bigsqcup_{j \in J} V_j$ uses $|J|$ colors, so this must be a Hoffman coloring. Now $\lambda_{\min}(H) = -\nu = \lambda_{\min}(G)$ and by Hoffman colorability of G we get

$$\lambda_{\max}(H) = \nu(|J|-1) = \frac{|J|-1}{\chi - 1}\lambda_{\max}(G). \qquad \square$$

6.2 Algebraic properties of the Perron eigenvalue

Now that we have proven the Decomposition Theorem, we can look at some of its consequences. As we will see, the case where the least eigenvalue of a Hoffman colorable graph is a Galois conjugate to its own additive inverse ν, is very restrictive. In fact, in this case, we will prove that the graph must be bipartite, or have only disconnected bipartite parts.

The property of a real algebraic number that it is conjugate to its own additive inverse, is equivalent to its minimum polynomial being even, meaning that it contains only even powers of the variable. Eigenvectors with such an eigenvalue turn out to behave very predictably on bipartite graphs, as is the content of the following lemma.

Lemma 6.9. *Let G be a connected bipartite graph with Perron eigenvalue ν, such that the minimal polynomial of ν over $\mathbb{Q}$ is an even function. Write $V(G) = U \sqcup W$ for the bipartition of G, and x for the Perron eigenvector of G. Then there exists a real number r and polynomials with rational coefficients f_v for every $v \in V(G)$ such that the following hold.*

(i) The degree of f_v is less than the degree of ν over $\mathbb{Q}$;

(ii) If $v \in U$, then f_v is an even polynomial, and if $v \in W$ then f_v is an odd polynomial;

(iii) The entry of x corresponding to v is given by $rf_v(\nu)$.

Proof. Let $K = \mathbb{Q}[y]/(g)$, where g is the minimal polynomial of ν over $\mathbb{Q}$. Write Y for the class $y + (g)$. Since g is a factor of the characteristic polynomial of G and Y is a root of g, we have that Y is a root of the characteristic polynomial of G. So, over K, we can find an eigenvector z of G with eigenvalue Y, where we assume without loss of generality that it is 1 on some vertex $u \in U$. Write f_v for the unique polynomial of degree less than the degree of ν over $\mathbb{Q}$ such that the entry of z corresponding to vertex v is given by $f_v + (g)$.

Write Φ_ν for the field homomorphism $K \to \mathbb{Q}(\nu)$ sending Y to ν, and $\Phi_{-\nu}$ for the field homomorphism sending to Y to $-\nu$. This works because $-\nu$ is also a root of g.

The vector $\Phi_\nu(z)$ over $\mathbb{Q}(\nu)$ given by applying Φ_ν to every entry of z is now an eigenvector of G over $\mathbb{Q}(\nu)$ with eigenvalue ν. Similarly, $\Phi_{-\nu}(z)$ is an eigenvector of G over $\mathbb{Q}(\nu)$ with eigenvalue $-\nu$.

We know that $\Phi_\nu(z)$ is a Perron eigenvector of G. Now consider $\Phi_{-\nu}(z)$, this is an eigenvector with eigenvalue $-\nu$. However, if we negate the entries of $\Phi_\nu(z)$ on the bipartite class W, we get another eigenvector with eigenvalue $-\nu$. By the reflectional symmetry of the spectrum of bipartite graphs, the eigenvalue $-\nu$ is simple, and so these two vectors are actually identical (no rescaling because both vectors give 1 on u). Let $v \in V$. If $v \in U$, then this implies that

$$\Phi_\nu(f_v + (g)) = \Phi_{-\nu}(f_v + (g)),$$

so that

$$f_v(\nu) = f_v(-\nu)$$

and

$$f_v(y) - f_v(-y) \in (g(y)),$$

and because f_v has degree less than the degree of g we see that

$$f_v(y) = f_v(-y),$$

so that f_v is an even function. Similarly, if $v \in W$, then f_v will be an odd function.

Defining $r = x(u)$, it is now clear by uniqueness of the Perron eigenvector that $x = r \cdot z$ and

$$x(v) = r f_v(\nu)$$

for every v. □

The shape of the Perron eigenvector of bipartite graphs that arises from this lays a strong restriction on Hoffman colorable graphs having these as bipartite parts, as we will see in the following corollary.

Corollary 6.10. *Let G be a non-empty Hoffman colorable graph with positive eigenvector x such that the minimal polynomial of the Perron eigenvalue of G over $\mathbb{Q}$ is an even function. Then G is bipartite or every bipartite part of G is disconnected.*

Proof. Suppose the chromatic number of G is at least 3 and there exists a bipartite part of G that is connected. Without loss of generality, assume that $H_{1,2}$ is connected. Write ν for the Perron eigenvalue of $H_{1,2}$, then by the Decomposition Theorem we know that the largest eigenvalue of G is $(\chi-1)\nu$. Since $\chi-1$ is rational, the minimum polynomial of $(\chi-1)\nu$ is even if and only the minimum polynomial of ν is even. Now the previous lemma applies, so that the Perron eigenvector of $H_{1,2}$ has only even polynomials in ν on V_1 and only odd polynomials in ν on V_2.

By the Decomposition Theorem, the Perron eigenvector of $H_{1,2}$ is the restriction of the positive eigenvector x of G. Rescale x accordingly and consider color class V_3. Let a be the entry of x on some vertex v in V_3. By weight regularity of the partition (Lemma 6.8), we must have that $a\nu$ is a sum of even polynomials in ν (by the neighbors of v in V_1). Similarly, $a\nu$ must be a sum of odd polynomials in ν (by V_2). The only possibility is that $a\nu = 0$, but a and ν are both positive, concluding the proof. □

This is a very general result, but we can already see a concrete application of this, excluding the existence of a color class of size 2 and a color class of size 3 in a Hoffman coloring.

Corollary 6.11. *Let G be a Hoffman colorable graph with a positive eigenvector. If there exists a color class of size 2 and a color class of size 3, then G is bipartite.*

Proof. Let H be the induced subgraph of G onto the union of these two color classes. Then H is a bipartite graph on five vertices. By Corollary 6.4, H cannot have isolated vertices. Up to symmetry, there are just five options for H, in order of the number of vertices in the 3-class adjacent to both vertices of the 2-class:

(1) The disjoint union of a path on two vertices and a path on three vertices;

(2) The path on five vertices;

(3) The graph obtained by adding a leaf to a center vertex of a path on four vertices;

(4) The graph obtained by adding a leaf to the cycle on four vertices;

(5) The complete bipartite graph with class sizes two and three.

The first option does not have a positive eigenvector, so it cannot be H. The other four options have the following minimum polynomials of their Perron eigenvalues:

$$x^2 - 3, \quad x^4 - 4x^2 + 2, \quad x^4 - 5x^2 + 2, \quad x^2 - 6.$$

Note that these are all even polynomials, so the previous corollary applies. Since these four graphs are all connected, we must conclude that G is bipartite. □

6.3 Small color classes and small Hoffman colorable graphs

With the knowledge of the exclusion of colorings with color class sizes 2 and 3 (see Corollary 6.11), we investigate which other small bipartite parts or small color classes are possible. We will first investigate what happens when a color class is of size 1, and after that we will address color classes of sizes 2.

Definition 6.12. A *universal vertex* in a graph is a vertex connected to every other vertex. The *cone graph* over a graph G is the graph obtained by adjoining a vertex to G and connecting it to every vertex in G.

A cone graph always has a universal vertex this way. Complete graphs are an example of cone graphs, as K_{n+1} is the cone graph over K_n. Other examples are wheel graphs, which are the cone graphs over cycle graphs, and star graphs $K_{1,m}$, as $K_{1,m}$ is the cone graph over the empty graph on m.

Using the Decomposition Theorem, we can prove that if a vertex constitutes a color class on its own in a Hoffman coloring, this must be a universal vertex.

Corollary 6.13. *Suppose that $\{u\}$ is a whole color class of a Hoffman coloring of a non-empty graph G with a positive eigenvector. Then u is universal.*

Proof. Let v be distinct from u; we claim that u and v are connected. Let H be the induced subgraph of G on $X \cup \{u\}$ where X is the color class of v. Then by Corollary 6.4 we know that v cannot be an isolated vertex in H, so v must be connected to u. □

Now we have seen that graphs with a Hoffman coloring containing a class of size 1 are cone graphs, we investigate those further. Using the Decomposition Theorem, we are able to completely classify Hoffman colorability of cone graphs.

Corollary 6.14 (Hoffman colorability of cone graphs)**.** *Let G be a non-empty graph. Then the cone graph over G is Hoffman colorable if and only if there exists a positive integer ν such that the following hold simultaneously.*

(i) G is Hoffman colorable;

(ii) G has $\chi(G)\nu^2$ vertices;

(iii) G is regular with valency $(\chi(G)-1)\nu$.

Proof. Write CG for the cone graph over G. Note that $\chi(CG) = 1 + \chi(G) \geq 3$, and that CG is connected, so it has a positive eigenvector x.

Suppose that CG is Hoffman colorable, then the universal vertex u added by the cone graph operation must form a color class on its own, so by the Decomposition Theorem G is Hoffman colorable as well. Write $V(CG) = \bigsqcup_{i=0}^{\chi(G)} V_i$, where $V_0 = \{u\}$. Now consider $1 \leq i \leq \chi(G)$. Since u is universal, we must have that $H_{0,i} \cong K_{1,|V_i|}$. So by biregularity

$$\lambda_{\max}(H_i) = \sqrt{|V_i|}.$$

By the Decomposition Theorem the restriction of x onto $H_{0,i}$ must be a positive eigenvector of $H_{0,i}$, and the largest eigenvalue of $H_{0,i}$ must be independent of i. Hence all color classes of G have the same size. Write $\nu = \sqrt{|V_i|}$, then we know that G has $\chi(G)\nu^2$ vertices. Furthermore, the Perron eigenvector of $H_{0,i} \cong K_{1,\nu^2}$ is given by ν on the universal vertex, and 1 everywhere else. So we can conclude that x gives ν on u, and 1 everywhere else. By the Decomposition Theorem, G has a constant positive eigenvector, so G must be regular. The valency must be $(\chi(G)-1)\nu$ since every bipartite part of G must now be ν-regular. This implies as well that ν must be a rational number, and hence an integer as it is an algebraic integer.

Conversely, suppose that G satisfies the three given conditions. If y is an eigenvector for G that is orthogonal to the constant vector, then setting $y(u) = 0$ on the universal vertex added by the cone graph operation will result in an eigenvector for CG with the same eigenvalue. Alternatively, setting y equal to ν on u and to 1 everywhere else, we get an eigenvector with eigenvalue $\chi(G)\nu$. Lastly, setting y equal to $-\chi(G)\nu$ on u and to 1 everywhere else will result in an eigenvector for the eigenvalue $-\nu$. We obtain:

$$\mathrm{Spec}(CG) = \{\chi(G)\nu, -\nu\} \cup \mathrm{Spec}(G) \setminus \{(\chi(G)-1)\nu\}.$$

Since G is Hoffman colorable, we know that

$$\chi(G) = 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)},$$

which, after substituting $\lambda_{\max}(G) = (\chi(G)-1)\nu$ gives

$$\lambda_{\min}(G) = -\nu.$$

It is now clear that $\lambda_{\min}(CG) = -\nu$ as well and that

$$h(CG) = \chi(G) + 1.$$

Now CG is Hoffman colorable as well. □

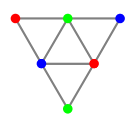
This completely classifies Hoffman colorability of cone graphs, reducing it to Hoffman colorability of the graph over which the cone graph is taken. If we choose $\nu = 1$, then this construction leads to all the complete graphs. Other examples include the cone graph over the 8-cycle, and the cone graph of the disjoint union of two 4-cycles, with $\nu = 2$.

Now we have covered color classes of size 1, we move to color classes of size 2.

Corollary 6.15. *Let G be a non-empty Hoffman colorable graph with a positive eigenvector such that at least two color classes have size 2. Then one of the following holds.*

(i) G is the path graph on four vertices;

(ii) G is isomorphic to:

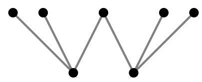


(iii) G is a disjoint union of two complete graphs of the same size;

(iv) The least eigenvalue of G is -2 and each color class of the Hoffman coloring has size 2, 5 or 8. Furthermore, each color class of size 5 has a marked vertex that is connected to every vertex in every color class of size 2, and to exactly 4 vertices among every color class of size 8.

Proof. Write $V(G) = \bigsqcup_{i=1}^{\chi} V_i$ for the Hoffman coloring and assume that $|V_1| = |V_2| = 2$. Let $H_{1,2}$ be the induced subgraph on $V_1 \cup V_2$, and write ν for its largest eigenvalue. We know that $H_{1,2}$ is bipartite with both color classes of size two, and that $H_{1,2}$ does not have isolated vertices (Corollary 6.4). There are three possibilities: a cycle, a path or two copies of K_2. We can solve the last case right away, as this would imply that $\lambda_{\min}(G) = -1$ and therefore G is a disjoint union of complete graphs by Corollary 2.10. Since G must have a positive eigenvector, the complete graphs in G must be of the same size. Since two color classes have size 2, G must now be a disjoint union of two complete graphs. For the remaining, suppose that $H_{1,2}$ is either a cycle or a path.

Let $3 \leq i \leq \chi$. Let $H_{1,i}$ be the induced subgraph of G on the set $V_1 \cup V_i$. Then ν is equal to the greatest eigenvalue of $H_{1,i}$. Write ℓ for the number of vertices of V_i that are connected to the one vertex of V_1, m the number of vertices connected to both and r the number of vertices connected to the other vertex of V_1 (ℓ for "left", m for "middle" and r for "right", since if you drew V_1 at the bottom and V_i at the top, this is how you would order the vertices of V_i). For example, $\ell = 2$, $m = 1$, $r = 2$ corresponds to:



The ℓ left vertices are symmetric, and so are the m middle vertices and r right vertices. Therefore the positive eigenvector x of G (which restricts to a positive eigenvector of $H_{1,i}$ by the Decomposition Theorem) takes at most five values on $H_{1,i}$: one for each of the vertices of V_1 (which we will call the left and right vertex of V_1 accordingly), one for the left vertices, one for the middle vertices and one for the right vertices of V_i. If x_ℓ is the value on the left vertices and x_r on the right vertices of V_i, then the left vertex of V_1 must have value νx_ℓ. Similarly, the right vertex of V_1 has value νx_r. It now follows that the middle vertices have value $x_\ell + x_r$ (since $\nu \neq 0$).

Write a for x_r/x_ℓ, the quotient of the values of the Perron eigenvector on the two vertices of V_1. Then, up to switching left and right, there are two possible values for a, one corresponding to each possibility of $H_{1,2}$. If $H_{1,2}$ is a path, then ν and a are equal to the golden ratio $\frac{1+\sqrt{5}}{2}$. If $H_{1,2}$ is a cycle, then it is 2-regular, and so $a = 1$ and $\nu = 2$.

Considering $H_{1,i}$ again, this implies without loss of generality that $x_r = a x_\ell$ with a being 1 or the golden ratio. Consider the neighbors of the left vertex of V_1: those are the ℓ left vertices of V_i and the m middle vertices of V_i. Therefore

$$\nu^2 x_\ell = \ell x_\ell + m(x_\ell + x_r) = \ell x_\ell + m(1+a)x_\ell = (\ell + m(1+a))x_\ell,$$

so that

$$\nu^2 = \ell + m + am.$$

Similarly, considering the right vertex of V_1 we obtain

$$\nu^2 = r + m + m/a.$$

It now makes sense to cover the two possibilities for a separately.

If $H_{1,2}$ is a path, the equations on ν^2, using the properties of the golden ratio, imply that

$$\phi + 1 = \ell + m + m\phi = r + m + m(\phi - 1).$$

Since the golden ratio is irrational, we read off that

$$(\ell, m, r) = (0, 1, 1),$$

so that $H_{1,i}$ is a path on 4 vertices as well. This holds for all $3 \leq i \leq \chi$, so we now know that every color class of the Hoffman coloring has size 2, and every bipartite part of the coloring is a path on 4 vertices. This implies that the Perron eigenvector of G has on each color class a 1 and a golden ratio. Write $(i, 1)$ for the vertex in V_i with Perron eigenvector entry 1, and $(i, 2)$ for the one with the golden ratio as its entry. Since the golden ratio vertices form the middle vertices of the path on 4 vertices, we can now conclude that G is the tensor product of K_χ and the lollipop graph L (K_2 with one self loop added). From Example 4.8 we see that χ is either 2 or 3, leading to first two cases of the statement.

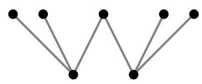
In the other case, $H_{1,2}$ is a cycle and so we have $\nu = 2$ and $a = 1$. From the equations on ν^2 we therefore get

$$4 = \ell + 2m = r + 2m.$$

We obtain the three possibilities

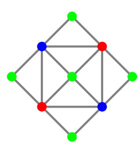
$$(\ell, r, m) \in \{(0, 2, 0), (2, 1, 2), (4, 0, 4)\},$$

so $H_{1,i}$ is the 4-cycle, two copies of $K_{1,4}$, or the graph:



This implies that $|V_i|$ is either 2, 5 or 8, and since this holds for every i, every color class must be of size 2, 5 or 8. If $|V_i| = 5$, then $(\ell, r, m) = (2, 1, 2)$ demonstrates that the positive eigenvector has entry 1 on the left and right vertices, and entry 2 on the sole middle vertex. We call this vertex "marked". By the $(2, 1, 2)$-structure, only this marked vertex is connected to every member of every color class of size 2. If $|V_j| = 8$, then by the Decomposition Theorem the induced subgraph H_{ij} of G on $V_i \cup V_j$ has to consist of $K_{1,4}$ and then either an 8-cycle or two 4-cycles, where the marked vertex of the class of size 5 is the universal vertex of $K_{1,4}$. This shows that it is connected to exactly half of the vertices of V_j, which concludes the proof. □

The last class of this corollary might sound a bit mysterious. However, we have seen two examples already. One is the regular complete multipartite graphs $K_{2,2,\ldots,2}$, and the second is the graph from Example 6.3:



At this point we know what Hoffman colorable graphs with a color class of size 1 look like, and we know what happens when two color classes are of size 2 or one color class is of size 2 and one of size 3. Combining all of these facts, we are able to classify all Hoffman colorable graphs on at most $3\chi(G) - 1$ vertices. If we furthermore assume that $|V(G)| \leq 2\chi(G) + 2$, then the classification is complete as the last case collapses to regular complete multipartite graphs with classes of size 2.

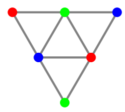
Corollary 6.16. *Let G be a non-empty Hoffman colorable graph with a positive eigenvector. Suppose that $|V(G)| < 3\chi(G)$. Then one of the following holds.*

(i) G is bipartite;

(ii) G is complete;

(iii) G is a disjoint union of two equal sized complete graphs;

(iv) G is isomorphic to:



(v) There exists an optimal coloring of G such that every color class is of size 2 or 5, and $\lambda_{\min}(G) = -2$.

Proof. Let G be a non-empty Hoffman colorable graph with a positive eigenvector. If G is bipartite, then we are in the first case. So assume that G is not bipartite (so that $\chi(G) \geq 3$). Suppose that $V(G) = \bigsqcup_{i=1}^{\chi} V_i$ is a Hoffman coloring of G, such that $|V_1| \leq |V_2| \leq \cdots \leq |V_\chi|$. Since $|V(G)| < 3\chi(G)$, we must have $|V_1| \leq 2$. We distinguish two cases.

First, suppose that V_1 contains just one element, say u. By Corollary 6.13 and Corollary 6.14 we know that $G - u$ is Hoffman colorable, $(\chi - 2)\nu$-regular and that it has $(\chi - 1)\nu^2$ vertices, while we know that $|V(G)| < 3\chi$. So

$$(\chi - 1)\nu^2 \leq 3\chi - 2$$

so that

$$(\nu^2 - 3)(\chi - 1) \leq 1.$$

Since $\chi \geq 3$ and ν is a positive integer, this implies that $\nu = 1$. Now G is complete.

Next suppose that $|V_1| = 2$. By Corollary 6.11 we cannot have that $|V_i| = 3$ anymore. If $|V_2| \geq 4$, then

$$3\chi > |V(G)| \geq 2 + 4(\chi - 1),$$

so that $2 \geq \chi$, in which case G is bipartite and is already covered by the statement. Otherwise $|V_2| = 2$, and then Corollary 6.15 applies. The first three cases from the corollary are already covered. In the last case, all color classes have size 2, 5 or 8 and $\lambda_{\min}(G) = -2$. Let U be a color class of size 2 and W a color class of size 8. Since $\lambda_{\min}(G) = -2$, by the Decomposition Theorem, the induced subgraph H of G

on $U \cup W$ has greatest eigenvalue 2 with a positive eigenvector that is constant on U. The only possibility for H is two disjoint copies of $K_{1,4}$. By interchanging the colors on one copy of $K_{1,4}$, we construct another optimal coloring of G, with one fewer class of size 8. Note that in order for $|V(G)| < 3\chi(G)$ to hold, we must have more classes of size 2 than classes of size 8. Therefore, we can repeat this process until there are no more color classes of size 8, so that G is Hoffman colorable with only color classes of size 2 and 5. □

In conclusion, using the Decomposition Theorem we are able to classify Hoffman colorability of cone graphs and of graphs with a number of vertices less than thrice the number of colors. However, there is still more to explore, namely the line graphs.

6.4 Hoffman colorability of line graphs

In Section 4.1.4 we got a long way into characterizing when exactly a connected graph G has a Hoffman colorable line graph. We found that G has to be 1-factorable if G had more edges than vertices or had an equal number of edges and vertices and an even cycle. The only two other cases left for a complete classification are the case where G is a tree, and the case where G has an equal number of vertices and edges where the unique cycle is of odd length. The Decomposition Theorem provides us with the tools to finish these two cases.

The reasoning for these remaining cases is very similar; the crucial point is that these graphs do not have many cycles, namely at most one. Before we can solve these last cases, we will state a corollary from the Decomposition Theorem that will lead to two lemmas concerning line graphs. The following corollary applies to line graphs, but is stated in the most general way possible.

Corollary 6.17. *Let G be a non-empty Hoffman colorable graph with a positive eigenvector. Suppose that G has an optimal coloring $c : V(G) \to \{1, \dots, \chi\}$ with the property that*

$$|\{u \in N(v) : c(u) = i\}| \leq 2,$$

for every vertex $v \in V(G)$ an every color i. Let $J = \{j, j'\}$ be a pair of colors: $J \subseteq \{1, \dots, \chi\}$ with $|J| = 2$. Let H be the induced subgraph of G on the set of vertices of color j or j'. Let C be a connected component of H. Then exactly one of the following holds.

(i) $\lambda_{\min}(G) = -2$ and C is an even cycle;

(ii) There exists a positive integer m such that $\lambda_{\min}(G) = -2\cos\left(\frac{\pi}{m+1}\right)$ and C is a path on m vertices.

Proof. By the assumption on the coloring, C has maximum degree 2. Then C has to be a cycle or a path. Since C is bipartite, if C is a cycle it has to be of even length.

From the Decomposition Theorem we know $\lambda_{\min}(G) = \lambda_{\min}(H)$, and by Corollary 2.3 and the symmetry of the spectrum of bipartite graphs, we have $\lambda_{\min}(H) = \lambda_{\min}(C)$. Now the only options for $\lambda_{\min}(G)$ are -2 and $-2\cos\left(\frac{\pi}{m+1}\right)$ for some m.

The value of this determines whether C is a cycle or a path, and if it is a path, the length of the path. This concludes the proof. □

The power of this corollary is that *all* of the connected components for *all* possible choices of pairs of colors for *all* optimal colorings satisfying the requirement now are cycles, or paths on a certain fixed number of vertices, as the least eigenvalue of G does not depend on the coloring or the component.

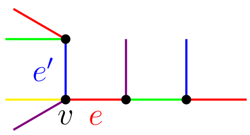
Furthermore, since edges are just incident to two vertices, and all of the edges incident to a fixed vertex must have a different color, it is clear that every coloring of a line graph satisfies the requirement in the corollary. In particular, in case G is a tree or has an equal number of vertices and edges and the unique cycle is of odd length, then even the case $\lambda_{\min}(G) = -2$ is impossible because a cycle of alternating colors in $L(G)$ gives an even cycle in G, which does not exist. We now give two lemmas, considering edges that do not lie on a cycle in G.

In Lemma 6.18 we restrict the possibilities of the degrees of vertices in a graph with a Hoffman colorable line graph. In Lemma 6.19 we restrict the sum of the degrees of two adjacent vertices. Together, they will give enough information to characterize Hoffman colorability of line graphs.

Lemma 6.18. *Let G be a connected graph with at least two edges. Let e be an edge of G that is not part of any cycle. Let v be one of the endpoints of e. If $L(G)$ is Hoffman colorable, then*

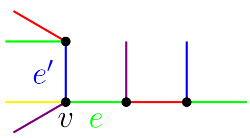
$$\deg(v) \in \{1, \chi(L(G))\}.$$

Proof. By contradiction, suppose that $2 \le \deg(v) < \chi(L(G))$. Since $2 \le \deg(v)$, there is another edge, say e' incident to v. Let c be an optimal coloring of $L(G)$, where we may assume that $c(e) = 1$, $c(e') = 2$ and there are no edges incident to v of color 3 (since $d < \chi(L(G))$. Say that color 1 is red, color 2 is blue, and color 3 is green, then:

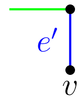


Note that $G \setminus e$ is disconnected; write F for the set of edges that are in the connected component of $G \setminus e$ not containing v. Then we can construct another valid edge coloring of G as follows:

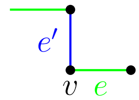
$$c'(f) = \begin{cases} 3 & \text{if } f = e, \\ 3 & \text{if } f \in F \text{ and } c(f) = 1, \\ 1 & \text{if } f \in F \text{ and } c(f) = 3, \\ c(f) & \text{otherwise.} \end{cases}$$



So what we do is we exchange the colors of 1 and 3 on $F \cup \{e\}$ (red and green). This gives a valid coloring, since there was no edge incident to v of color 3. We now apply Corollary 6.17 on colors 2 and 3 (blue and green). Let C be the component containing e' in the induced subgraph of $L(G)$ on the set of edges of G with color 2 or 3 under c, and let C' be the component containing e' in the induced subgraph of c'-colors 2 and 3. So C looks like:



And C' looks like:

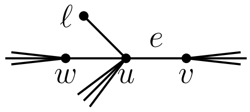


Note that in coloring c, no edge incident to v has color 3. So e' is an endpoint of C. Note furthermore that all edges that are not in F and are not e have the same color under c and c'. We conclude that $C \subseteq C'$. However, C' also contains e, since $c'(e) = 3$ and not 1. So C' actually contains more edges than C. We now get two distinct values of $\lambda_{\min}(L(G))$ from Corollary 6.17, which gives a contradiction. $\square$

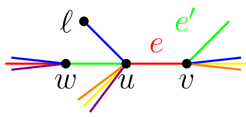
Lemma 6.19. *Let G be a connected graph with at least two edges. Let e be an edge that is not part of any cycle. Write u and v for the endpoints of e. Suppose that there is a leaf ℓ connected to u. If $L(G)$ is Hoffman colorable, then*

$$\deg(u) + \deg(v) \leq \Delta(G) + 2.$$

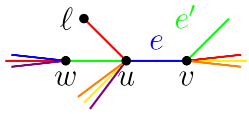
Proof. Again, we argue by contradiction. Suppose that $\deg(u)+\deg(v) \geq \Delta(G)+3$. Then u must be adjacent to at least 3 vertices, so write w for some neighbor of u other than v or ℓ. The situation is as follows.



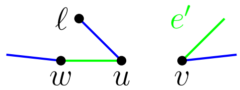
There are at least $\Delta(G) + 1$ edges incident to either u or v. By the pigeonhole principle, for every optimal edge coloring of G there must be an edge incident to u and not to v and an edge incident to v and not to u of the same color (by Vizing's Theorem, there are only at most $\Delta(G)$ colors to assign to the edges out of u and v). Without loss of generality, we consider an optimal edge coloring c with $c(u, v) = 1$, $c(u, \ell) = 2$, $c(u, w) = 3$, and some edge e' to the right of v also has color 3. If we again say that color 1 is red, color 2 is blue and color 3 is green, then we get the following optimal edge coloring:



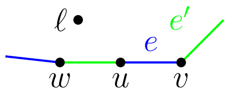
Write F for the set of edges in the connected component of $G \setminus e$ containing v. Then we construct a different edge coloring c' by switching colors 1 and 2 (red and blue) for the edges $F \cup \{e, \{u, \ell\}\}$, and it is clear that this still is a valid optimal edge coloring.



We now choose $J = \{2, 3\}$ (blue and green) in Corollary 6.17. Let C and C' be the connected components containing $\{u, w\}$ for coloring c and c' respectively. We have that C looks like:



And C' looks like:



Note that c and c' agree on all of the edges not in F, except for e and $\{u, \ell\}$. We can see that C contains e, and also e', whereas C' does not contain e. However, C' contains $\{u, \ell\}$, which C does not have. Now again C and C' are paths of different lengths, so by Corollary 6.17 we get two distinct values for $\lambda_{\min}(L(G))$, which gives a contradiction. □

We are now ready to finish the characterization of Hoffman colorability of line graphs, which we started in Section 4.1.4. We will handle the two remaining cases (the case where G is a tree and the case where G has a unique odd cycle) separately. We start with the trees.

Theorem 6.20. *Let G be a tree with at least two edges. Then $L(G)$ is Hoffman colorable if and only if G is a path graph or if $G \cong K_{1,\Delta}$.*

Proof. Since G has no cycles, by Lemma 6.18 every vertex is either of degree 1 or of degree $\chi(L(G))$. Since G has at least two edges, there exists a vertex with degree greater than 1, so that $\chi(L(G)) = \Delta(G) \geq 2$. Consider the set of maximum degree vertices $D = \{v \in V(G) : \deg(v) = \Delta(G)\}$. We make a case distinction on the cardinality of D. Note that we know that D has at least one element.

If $|D| = 1$, then there is one element of maximal degree, and every other vertex is a leaf. Immediately it follows that $G \cong K_{1,\Delta}$.

If $|D| \geq 2$, then take two vertices in D and consider the shortest path from u to v. Every vertex on this path must have degree at least 2, and so must be in D. So every vertex in D also has at least one neighbor in D. Note that since G is a tree, there must be a vertex u in D adjacent to only one other vertex v in D (otherwise the induced subgraph of G on D has minimum degree 2 hence contains a cycle). Now u must be adjacent to a leaf, and by Lemma 6.19 we get

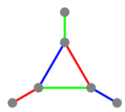
$$\Delta + \Delta \leq \Delta + 2,$$

and so $\Delta \leq 2$. Now $\Delta = 2$, and so G is a path graph.

Conversely, the line graph of a path graph is bipartite, hence Hoffman colorable and the line graph of $K_{1,m}$ is the complete graph on m vertices, hence Hoffman colorable. □

Now the trees are done, we turn to the graphs with a unique cycle which is of odd length. These graphs have an equal number of vertices and edges.

Theorem 6.21. *Let G be a connected graph with an equal number of edges and vertices. Let C be the unique cycle of G, and suppose that C is of odd length. If $L(G)$ is Hoffman colorable, then G is either isomorphic to K_3 or to K_3 with one leaf added to every vertex:*



Proof. In the case that $\Delta = 2$, we can conclude that G has to be an odd cycle, and $L(G) \cong G$. In this case, only the cycle of length 3 will work by Proposition 2.4. In the following we assume that $\Delta \geq 3$. Note that any 3-coloring of the cycle can be extended to an optimal coloring of the edges using only Δ colors, for example by a greedy algorithm running through the edges in increasing order of distance to the cycle. So now we know that $\chi(L(G)) = \Delta$, or in other words: G is of class 1.

First we use Lemma 6.18. If v is on the cycle and is only incident to the two edges from the cycle, then $\deg(v) = 2$. Otherwise, the lemma applies and $\deg(v) = \Delta \geq 3$. If v is not on the cycle, then it is incident to an edge not on the cycle and so by the lemma $\deg(v) \in \{1, \Delta\}$.

Assume that there exists a vertex u outside of the cycle that is not a leaf, then it must be of maximal degree. Without loss of generality, assume that u is as far away from the cycle as possible, so that u is adjacent to a leaf. The neighbor v of u on the shortest path to the cycle must also be of maximal degree. Now Lemma 6.19 applies and we get $\Delta + \Delta \leq \Delta + 2$, so that $\Delta = 2$, which is impossible since we assumed $\Delta \geq 3$. We conclude that outside of the cycle, there are only leaves.

Suppose the cycle has at least five vertices (and five edges). Then we can color one of the edges in C color 1, and the other edges alternatingly 2 or 3 (which both occur at least two times), and extend this to an optimal Δ-coloring of $L(G)$. Applying Corollary 6.17 to colors 2 and 3, on the component containing the edges on the cycle, we see that we get a path of length at least $|C| - 1$, which is at least 4. So every path coming from Corollary 6.17 has length at least 4. Since $|C|$ is assumed to be at least 5, we can find an edge e on the cycle of color 2 such that both neighboring edges are of color 3. Note that every vertex outside of the cycle must be a leaf. Now if we apply Corollary 6.17 on colors 1 and 2 on the component containing e, then we get a path of length at most 3. This is a contradiction, and we conclude that C must be a 3-cycle.

In that case every optimal coloring must assign three distinct colors to the three edges of the cycle. Since $\Delta \geq 3$, there must be some leaf outside the cycle. Write u, v, w for the three vertices of the 3-cycle and ℓ for the leaf, connected to, say, u.

There exists an optimal edge coloring assigning to $\{u, \ell\}$ the same color as to $\{v, w\}$, and this way we get a path of length at least 3.

However, if $\Delta \geq 4$, then there must be a vertex on the cycle incident to at least two leaves outside the cycle. Applying Corollary 6.17 on the colors of the edges to these leaves, we get a path of length just 2, which is a contradiction. We conclude that $\Delta = 3$.

There are now three possible cases: a triangle with one leaf added to one vertex, a triangle with one leaf added to two vertices, and a triangle with one leaf added to all three vertices. By Corollary 6.16, the first two cases are excluded, so we are left with the last case only. The line graph of K_3 with leaves is isomorphic to $K_3 \times L$, where L is the lollipop graph: K_2 with one self loop. This is also Hoffman colorable by Example 4.8. □

We can now gather everything we have proved both in this section and in Section 4.1.4 in one statement, completely covering Hoffman colorability of line graphs.

Theorem 6.22. *Let G be a connected graph with at least two edges. Then $L(G)$ is Hoffman colorable if and only if one of the following statements holds.*

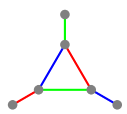
(i) G *is 1-factorable;*

(ii) $G \cong K_{1,m}$ *for some* m*;*

(iii) G *is a path graph;*

(iv) $G \cong K_3$*;*

(v) G *is isomorphic to* K_3 *with one leaf added to each vertex:*



Chapter 7

The Composition Theorem and constructions

In this chapter, we will investigate the Composition Theorem, which is in a way a converse to the Decomposition Theorem. Recall that the Decomposition Theorem allows us to decompose a Hoffman colorable graph into a collection of compatible bipartite parts. In the Composition Theorem, we start with a graph composed of a collection of compatible bipartite parts, and we want to conclude Hoffman colorability. There is one requirement needed for this: that the least eigenvalue is what it needs to be.

Recall from Definition 6.2 that a graph G is composed of a collection of compatible bipartite parts if G has a c-coloring $V(G) = \bigsqcup_{i=1}^{c} V_i$ such that the bipartite parts H_{ij}, which are the induced subgraphs of G on $V_i \sqcup V_j$, all have the same largest eigenvalue ν, all with positive eigenvectors x_{ij}, such that the vectors x_{ij} for all $j \neq i$ agree on color class V_i. In other words, for every i, we can take a vector $x_i : V_i \to \mathbb{R}$, such that the restriction of x_{ij} to V_i is x_i for all $j \neq i$. We will show that G is Hoffman colorable in this case if and only if the least eigenvalue of G is $-\nu$.

After we have stated and proven the Composition Theorem, we will apply it in various situations to find various new infinite families of Hoffman colorable graphs, many of which are irregular. We will also look at the Composition Theorem in the context of tensor graphs, which we have seen in Proposition 4.6.

Theorem 7.1 (Composition of Hoffman colorings)**.** *Let G be a graph composed of a collection of compatible bipartite parts (H_{ij}) (where c, V_i, ν, and x_i are as in Definition 6.2). Then the following hold.*

(i) Let x be the vector formed by setting

$$x(v) = x_i(v),$$

where $v \in V_i$. Then x is a positive eigenvector of G with eigenvalue $(c-1)\nu$.

(ii) Let $(r_i)_{i=1}^{c}$ be a sequence of real numbers adding to zero. Then the vector y formed by setting

$$y(v) = r_i \cdot x_i(v),$$

where $v \in V_i$, is an eigenvector for G with eigenvalue $-\nu$.

(iii) *The largest eigenvalue of G is $(c-1)\nu$, and G has eigenvalue $-\nu$ with multiplicity at least $c-1$.*

(iv) *Let z be an eigenvector of G with eigenvalue other than $(c-1)\nu$ or $-\nu$, then for all color classes i we have*

$$\sum_{v \in V_i} z(v)x_i(v) = 0.$$

(v) *If no eigenvector z of G exists with an eigenvalue less than $-\nu$, then G is Hoffman colorable and $\chi(G) = c$.*

Proof. We prove every part on its own.

(i) If $v \in V_j$ is fixed, then note that for every $i \neq j$ we know

$$\nu x(v) = \sum_{u \in N(v) \cap V_i} x(u),$$

because the projection of x onto H_{ij} is an eigenvector with eigenvalue ν. Summing over all $i \neq j$, we obtain

$$(c-1)\nu x(v) = \sum_{u \in N(v)} x(u),$$

so that x is indeed an eigenvector with eigenvalue $(c-1)\nu$.

(ii) Again, fix $v \in V_j$. Then

$$-\nu y(v) = -r_j \nu x_j(v) = \Big(\sum_{i \neq j} r_i\Big)\nu x_j(v) = \sum_{i \neq j} r_i \nu x_j(v).$$

Now again we consider bipartite part H_{ij} to get

$$= \sum_{i \neq j} r_i \sum_{u \in N(v) \cap V_i} x_i(u) = \sum_{u \in N(v)} y(v),$$

so we are done with this part as well.

(iii) Since x is a positive eigenvector of G, its eigenvalue $(c-1)\nu$ is the greatest eigenvalue of G by Corollary 2.3. The choice of the finite sequence $(r_i)_{i=1}^c$ for the vector y is a $c-1$-dimensional choice, since the only requirement is that they add up to 0. Therefore the space of these vectors y has dimension $c-1$ and G has eigenvalue ν with multiplicity at least $c-1$.

(iv) Let z be an eigenvector of G with an eigenvalue other than $-\nu$ or $(c-1)\nu$. Then z is orthogonal to x and to all choices of y. Consider y constructed from the finite sequence $(r_i)_{i=1}^c$ putting $r_1 = c-1$ and $r_i = -1$ for all other classes. Then

$$0 = \langle z, x\rangle + \langle z, y\rangle = c \sum_{v \in V_1} z(v)x_1(v),$$

and similarly for the other classes.

(v) If no such z exists with an eigenvalue less than $-\nu$, then clearly $\lambda_{\min}(G) = -\nu$, and so $h(G) = c$. Since the partition $V(G) = \bigsqcup_{i=1}^{c} V_i$ is a c-coloring, the result follows. □

In the following, we will see how the Composition Theorem applies to various new constructions of Hoffman colorable graphs. Before applying the Composition Theorem in new situations, we first investigate it in a setting that we already know, the tensor products. After that we will go over regular Hoffman colorable graphs, irregular Hoffman colorable graphs with a regular template and a construction involving block designs.

7.1 Tensor products revisited

Recall from Proposition 4.6 that if G is Hoffman colorable and K has a Hoffman bound at least that of G, then $G \times K$ is Hoffman colorable (even if K contains self loops). The Hoffman coloring of $G \times K$ will be induced by the Hoffman coloring of G. Therefore, the bipartite parts of $G \times K$ will look like $H_{ij} \times K$, where H_{ij} is a bipartite part of G. If we say that H_{ij} has largest eigenvalue ν for all i, j, and K has largest eigenvalue λ, then $H_{ij} \times K$ will have largest eigenvalue $\nu\lambda$, regardless of the choice of i, j. Furthermore, an eigenvector for $\nu\lambda$ of $H \times K$ can be obtained by taking the tensor product $x_{ij} \otimes y$ of eigenvectors x_{ij} and y for H_{ij} and K respectively. Since the restrictions of x_{ij} to V_i are all equal, also the restrictions of $x_{ij} \otimes y$ to $V_i \times K$ will be equal.

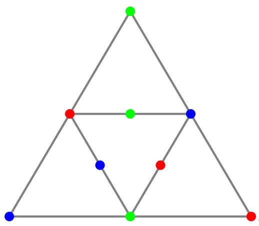
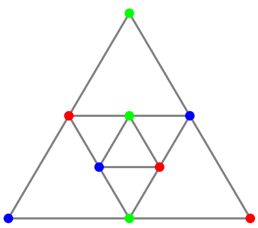
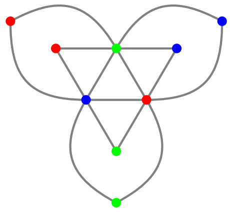
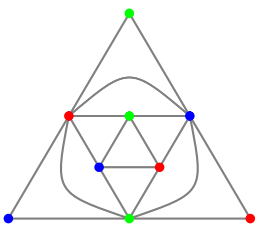
In this way, any tensor product $G \times K$ with a coloring coming from a graph G that is composed of a collection of compatible bipartite parts is itself composed of a collection of compatible bipartite parts. Note that we do not need Hoffman colorability of G, the requirement $h(G) \leq h(K)$, or even $\chi(G) = \chi(G \times K)$ in order to conclude this. However, for Hoffman colorability, we need to know the least eigenvalue of $G \times K$. If $h(G) \leq h(H)$, then we know that the least eigenvalue of $G \times K$ is equal to the elast eigenvalue of G. If G is Hoffman colorable, then in this case we know by the Decomposition Theorem that $G \times K$ is Hoffman colorable, and furthermore that $\chi(G \times K) = \chi(G)$, so that (G, K) satisfies Hedetniemi's conjecture.

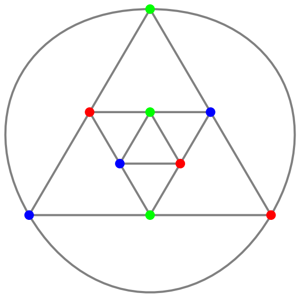
Turning to the practical perspective on tensor graphs, the subfamily most suitable for algorithmic construction is the case where G is a complete graph. Note that the case where G is a tensor of a complete graph and another graph also reduces to this case, as $(K_\chi \times H') \times H$ is isomorphic to $K_\chi \times (H' \times H)$. Note that tensor products show the symmetries of both factors, so the graphs $K_\chi \times H$ in this subfamily will contain the symmetric group of order χ as a subgroup of their automorphism group. If H is disconnected, then $G \times H$ will also be disconnected so we are only interested in connected graphs H.

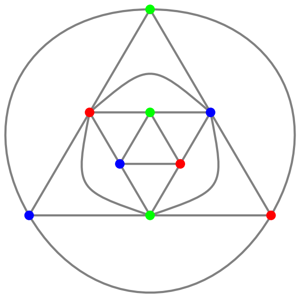
To construct connected Hoffman colorable K_χ-tensor graphs of size, say, $m\chi$ (such that every color class has size m), one can generate all graphs H with possible self loops on m vertices, calculate the Hoffman bound $h(H)$ and discard if $h(H) < \chi$. All graphs with $\chi \leq h(H)$ will give a Hoffman colorable tensor graph $K_\chi \times H$.

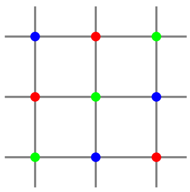
Applied to graphs on nine vertices

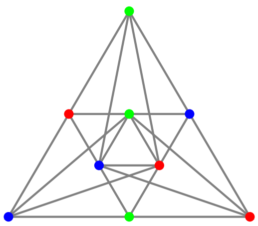
To illustrate this, we construct all 3-Hoffman colorable tensor graphs on 9 vertices. These graphs are of the form $K_3 \times H$, where H has a Hoffman bound of at least 3. Since we want H to be connected, forgetting the self loops there are just two options: the complete graph on three vertices and the path graph on three vertices. There are four possible ways to add self loops to the complete graph on three vertices (adding 0, 1, 2 or 3 self loops) and six ways of adding self loops to the path graph on three vertices (adding 0 or 3 self loops, adding 1 self loop to the middle vertex, adding 1 self loop to a leaf, adding 2 self loops, one to the middle vertex and one to a leaf, and adding 2 self loops to the two leaves). All but two of these options give a Hoffman bound of at least 3. The two exceptions are the path graph on three vertices with no self loops added (is bipartite, so has Hoffman bound 2), and with one self loop added to one of the leaves (this has Hoffman bound around 2.45). The other eight graphs H give rise to a Hoffman colorable tensor product. Three of these tensor factors are regular, which result in three regular Hoffman colorable tensors.

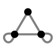
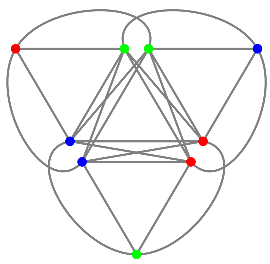
- Non-Hoffman colorable tensors:
  - H: , $K_3 \times H$: , with coloring taken from K_3 although this is not optimal,
  - H: , $K_3 \times H$: 
- Hoffman colorable tensors:
  - H: , $K_3 \times H$: 
  - H: , $K_3 \times H$: 

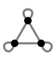
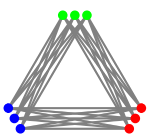
– H: , $K_3 \times H$: 

– H: , $K_3 \times H$: 

– H: , $K_3 \times H$: 

– H: , $K_3 \times H$: 

– H: , $K_3 \times H$: 

– H: , $K_3 \times H$: 

One can see very clearly from these examples the S_3-symmetry from K_3 present in all these tensor graphs. The procedure set out above can be done for other values of χ and m (where χ will be the chromatic number and m the class size). In the following table, for various values of χ and m we provide the number of (regular) connected Hoffman colorable tensor graphs $K_\chi \times H$, where H is an m-vertex graph. The cell "A/B, C%" means that there are A regular tensors, B total, and C is the percentage of the tensor graphs that are regular.

$m \setminus \chi$	3	4	5	6
3	3/8, 38%	1/3, 33%	1/2, 50%	1/2, 50%
4	4/30, 13%	4/15, 27%	1/5, 20%	1/3, 33%
5	5/204, 2%	5/78, 6%	5/28, 18%	1/8, 13%

connected Hoffman colorable tensor graphs

As χ increases, one can see that the number of suitable tensor factors H for $K_\chi \times H$ thins out. This is because for Hoffman colorability we need $\chi \leq h(H)$, so if $h(H)$ is finite, then it will not be feasible for big enough χ. The only case where $h(H) = \infty$, is when H is the fully complete graph. For big enough χ, this will be the only suitable tensor factor and therefore we will only have regular complete multipartite graphs.

7.2 Regular Hoffman colorable graphs

The next interesting class of Hoffman colorable graphs to look at through the lens of the Composition Theorem is the class of regular graphs. Indeed, if we apply the Decomposition Theorem to a Hoffman colorable regular graph with a constant eigenvector, then we get bipartite parts H_{ij} with a constant eigenvector and eigenvalue ν. This means precisely that all bipartite parts are ν-regular. Therefore we will consider graphs that are composed of a collection of compatible *regular* bipartite parts. We can pose the following definition, which is precisely the conclusion of Proposition 3.1.

Definition 7.2. A graph G is *composed of a collection of compatible regular bipartite parts* if G has a c-coloring $V(G) = \bigsqcup_{i=1}^{c} V_i$, such that a positive integer ν exists such that every vertex u of G is adjacent to precisely ν vertices of every color different from the color of u.

Note that in this case the color classes must all be of the same size, say m. We would like to know when a graph composed of a collection of compatible regular bipartite parts is Hoffman colorable. This is not an easy question to answer. However, using color complements, we are able to prove Hoffman colorability of a specific subclass of graphs composed of compatible regular bipartite parts using the Composition Theorem, namely those that have valency above the bound set out in Proposition 7.7.

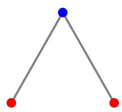
In principle, given ν and the color class size m, it is possible to generate all possible bipartite graphs on $2m$ vertices that are ν-regular, and to consider all of the different possible ways to compose them together. In these cases, one needs to check the least eigenvalue in order to have Hoffman colorability by the Composition Theorem. As said before, Proposition 7.7 gives a requirement with which the least eigenvalue is indeed what it needs to be.

7.2.1 Color complement

Definition 7.3. Let G be a graph with a coloring $V(G) = \bigsqcup_{i=1}^{\chi} V_i$. Then the *color complement* $\overline{G}_{\text{color}}$ is the graph on the same vertex set such that two vertices are adjacent in $\overline{G}_{\text{color}}$ if and only if they were not adjacent in G and belong to different color classes.

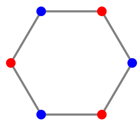
Equivalently, the color complement can be obtained by first taking the standard complement, and then removing the edges of the cliques that come from the color classes. The color complement of a graph composed of a collection of compatible regular bipartite parts is again composed of a collection of compatible regular bipartite parts, just by taking color complements of every bipartite part. The coloring of G used for the color complement is also a valid coloring for the color complement, by construction.

Example 7.4. If we take $G \cong P_3$ to be path graph on three vertices colored optimally,

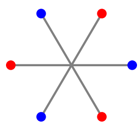


then its color complement $\overline{(P_3)}_{\text{color}}$ is the empty graph. So even though the coloring for P_3 was optimal, for the color complement this was not the case.

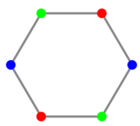
Example 7.5. As an example for a graph composed of a collection of compatible regular bipartite parts, we can use a bipartite part itself, like the 6-cycle:



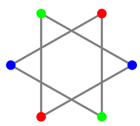
The color complement is:



However, we can also color the 6-cycle with three colors:



In this case, the color complement is:



This shows that the color complement depends on the coloring chosen.

Like the spectrum of a regular graph determines the spectrum of its complement (see [7, Section 1.3.2]), the spectrum of a graph composed of a collection of compatible regular bipartite parts determines the spectrum of its color complement.

Lemma 7.6. *Let G be a graph composed of a collection of compatible regular bipartite parts. Let $\overline{G}_{color}$ be the color complement of G given this coloring (which has classes of size m). Then G and $\overline{G}_{color}$ have a common eigenbasis. Furthermore, there exists a multiset of real numbers E such that*

$$\begin{aligned}\operatorname{Spec}(G) &= E \cup \{(c-1)\nu, (-\nu)^{c-1}\}\\ \operatorname{Spec}(\overline{G}_{color}) &= -E \cup \{(c-1)(m-\nu), (\nu-m)^{c-1}\},\end{aligned}$$

where $-E = \{-e : e \in E\}$.

Proof. From the Composition Theorem, we get that the constant eigenvectors give eigenvalue $(c-1)\nu$ for G and $(c-1)(m-\nu)$ for $\overline{G}_{\text{color}}$. Furthermore, there is a space of dimension $c-1$ with eigenvectors giving eigenvalue $-\nu$ for G and $\nu - m$ for $\overline{G}_{\text{color}}$.

Let x be an eigenvector for G that is orthogonal to the χ eigenvectors constructed above. Then, as in the proof of the Composition Theorem, we know that for every color class i we have

$$\sum_{v \in V_i} x(v) = 0.$$

Let λ be the eigenvalue corresponding to this eigenvector. So for all $v \in V$ we have

$$\lambda x(v) = \sum_{i=1}^{\chi} \sum_{u \in N_i(v)} x(u),$$

where $N_i(v)$ is the set of neighbors of v of color i. We assume without loss of generality that $v \in V_1$ so that $N_1(v) = \emptyset$. We now use that the sum over all the vertices of a fixed color class of the eigenvector is 0, to get

$$\lambda x(v) = \sum_{i=2}^{\chi} - \sum_{u \in V_i \setminus N_i(v)} x(v).$$

This precisely says that x is also an eigenvector of $\overline{G}_{\text{color}}$, with eigenvalue $-\lambda$. This concludes the proof. □

As mentioned earlier, if the valency ν of the bipartite parts is big enough, then we can guarantee Hoffman colorability. This is the content of the following proposition, which is based on color complements in its proof.

Proposition 7.7. *Let G be a graph composed of a collection of compatible regular bipartite parts. Let m be the size of the classes in the coloring. If ν is at least*

$$m \cdot \frac{c-1}{c},$$

then G is Hoffman colorable and $\chi(G) = c$.

Proof. Note that since G has a c-coloring, we have $\chi(G) \leq c$. Write E as in the above lemma. The absolute value of every $e \in E$ is bounded by $(c-1)(m-\nu)$, since this eigenvalue of $\overline{G}_{\text{color}}$ was obtained by a positive eigenvector. Since

$$m \cdot \frac{c-1}{c} \leq \nu,$$

we have

$$m(c-1) \leq \nu c$$

and so

$$(m-\nu)(c-1) \leq \nu,$$

so that every $e \in E$ has absolute value bounded by ν. Then $-\nu$ must be the least eigenvalue of G. The Composition Theorem now gives that G is Hoffman colorabile and that $\chi(G) = c$. □

This result can be used to prove Hoffman colorability of regular graphs that satisfy the conclusion of Proposition 3.1 (a coloring giving an equitable partition), given that the valency is high enough.

For example, for 3-colorable graphs, if the valency of the three bipartite parts is at least two thirds of the color class size, then the composition of the three bipartite parts will be Hoffman colorable.

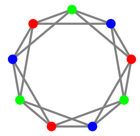
Applied to graphs on nine vertices

To demonstrate how to construct Hoffman colorable graphs this way, we construct all regular 3-Hoffman colorable graphs on nine vertices. Note that we now have $m = 3$, so $0 \leq \nu \leq 3$. By the above proposition all of the graphs with $2 \leq \nu$ will be Hoffman colorable. These are all the color complements of the graphs with $\nu \leq 1$, which will be easier to enumerate.

If $\nu = 0$, then we get an empty graph. The color complement is $K_{3,3,3}$, which is the first regular Hoffman colorable graph.

If $\nu = 1$, then we are looking for a nine-vertex graph G with three color classes, each containing three vertices, such that every vertex v is connected to a unique vertex in the two color classes not containing v. So G is 2-regular, so G consists of cycles. The distance between two vertices of the same color cannot be 2 as the vertex in between will have two neighbors of the same color. This leaves only three options for G: the 9-cycle, the disjoint union of a 6-cycle and a 3-cycle, and the disjoint union of three 3-cycles. The 9-cycle is not Hoffman colorable, as we have seen before by Proposition 2.4. The disjoint union of a 6-cycle and a 3-cycle is not Hoffman colorable by Proposition 4.2. The only Hoffman colorable graph with $\nu = 1$ is therefore the three disjoint 3-cycles. However, the color complements of these three graphs will all be Hoffman colorable, since they will have $\nu = 2$.

So we have found that there are five regular Hoffman 3-colorable graphs on nine vertices. Four of these turn out to be tensor graphs. Three of these are connected and we have already seen them in the previous section. The one disconnected tensor is the disjoint union of the 3-cycles. The one non-tensor graph is the following:



It can be concisely described as the Cayley graph for $\mathbb{Z}/9\mathbb{Z}$ with generating set $\{1, 2, 7, 8\}$.

7.3 Regular templates

In this section we will investigate graphs with a very special property, namely that they are almost regular, with just one color class falling short. This class will encompass many of the small examples of Hoffman colorable graphs that do not have all color classes of one size.

Definition 7.8. Let G be a graph with a coloring $V(G) = \bigsqcup_{i=1}^{c} V_i$ such that $T := G \setminus V_c$ is regular. Then we call T the *regular template* of G, and we will call the vertices of V_c the *decorations*.

The reason why this class of graphs is interesting with respect to Hoffman colorability is because we can tell a lot about the structure of those graphs, with the one requirement that G has a positive eigenvector that restricts to a constant vector on T. If T is connected, this is automatic as all positive eigenvectors are then constant. In Section 8.4.2 we will meet a graph with a regular template that does not have this nice property. Still, many graphs with a regular template do have this property.

Proposition 7.9. *Let G be a graph with a regular template with corresponding coloring $V(G) = \bigsqcup_{i=1}^{c} V_i$. Suppose that with respect to this coloring G decomposes into a collection of compatible bipartite parts (H_{ij}), where the compatibility vectors x_i are constant if $i \neq c$. Then there exist integers m, ν, $(d_v)_{v \in V_c}$ with the following properties of the bipartite parts.*

(i) *For every $1 \leq i < c$, the size of class i is m;*

(ii) *For every pair $i, j < c$, H_{ij} is ν-regular;*

(iii) *For every $1 \leq i < c$, the graph H_{ic} satisfies these two requirements:*

- *The degree of every decoration vertex $v \in V_c$ is d_v;*
- *For every vertex $u \in V_i$, the sum of the degrees of the decoration neighbors of u equals ν^2;*

(iv) *It holds that*

$$\sum_{v \in V_c} d_v^2 = m\nu^2.$$

Proof. Suppose G satisfies the requirements, then for $i, j < c$, the bipartite part H_{ij} will be regular of valency ν because x_{ij} is constant. As a consquence, the sizes of the V_i with $i < c$ must all be equal, say to m. We can rescale every x_{ij} to be of norm $\sqrt{m\nu}$ such that $x_i(u) = \nu$ for every template vertex u.

Now let $1 \leq i < c$ and consider H_{ic}. This graph must have a positive eigenvector x_{ic}, and we already set $x_i(u) = \nu$ for $u \in V_i$. Let v be a decoration vertex. If we write $N_i(v)$ for the set of neighbors of v in class i, then we know

$$\nu x_c(v) = \sum_{u \in N_i(v)} x_i(u) = \nu |N_i(v)|,$$

so that $x_c(v)$ is equal to the number of neighbors of v in V_i. This argument works for all i, so we can define d_v to be $x_c(v)$. Now, since the norm of x_c must be equal to the norms of the x_i for $1 \leq i < \chi$, we get that

$$\sum_{v \in V_c} d_v^2 = m\nu^2.$$

Lastly, if we take $u \in V_i$, then we must have

$$\nu x_i(u) = \sum_{v \in N_c(u)} x_c(v) = \sum_{v \in N_c(u)} d_v,$$

and since $x_i(u) = \nu$, the sum of the degrees of the decoration neighbors of u equals ν^2, with which we are done. □

If a Hoffman colorable graph has a regular template and a positive eigenvector restricting to a constant eigenvector on the template, we know two things from the Decomposition Theorem. Firstly, the graph decomposes into a collection of compatible bipartite parts, so Proposition 7.9 applies, and secondly, the template must be Hoffman colorable as well. Therefore, we take this as the starting point for our construction.

Construction 7.10. Let T be a regular Hoffman colorable graph with every color class of size m. Write ν for the valency of T divided by $\chi(T) - 1$. Let (d_v) be a finite sequence of integers such that the sum of the squares of d_v is equal to $m\nu^2$. Then G is constructed from regular template T by adjoining decoration vertices to the template, connecting decoration v to exactly d_v vertices in every color class of the template, such that for every template vertex, the sum of the degrees of its decoration vertices is equal to ν^2.

By Proposition 7.9 and the Composition Theorem, this construction will result in a Hoffman colorable graph G if and only if the least eigenvalue of G is equal to $-\nu$..

We will first investigate some general solutions to d_v of the equation

$$\sum_{v} d_v^2 = m\nu^2.$$

After that, we will see how we can generate all Hoffman colorable graphs on nine vertices with a regular template this way.

Trivial decoration degrees

For Construction 7.10, we start with a regular Hoffman colorable graph T as a template, which gives the data (ν, m), and we need a finite sequence of integers (d_v) such that the sum of squares is equal to $m\nu^2$. There are some trivial general solutions for this sequence (d_v). In this section, we will investigate three general solutions, which we decided to call the *regular solution*, the *universal solution* and the *leaf solution*. We will investigate what kinds of graphs come out of Construction 7.10 this way.

Before this, we briefly cover what happens when every color class of T is of size 2, as this connects to Corollary 6.15. In fact, Construction 7.10 sheds some new light on the last case of Corollary 6.15. In this case, we have a regular template with $m = 2$ and $\nu = 2$. So $m\nu^2 = 8$, which has the following solutions for (d_v).

- $8 = 2^2 + 2^2$;
- $8 = 2^2 + 1^2 + 1^2 + 1^2 + 1^2$;
- $8 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$.

These three cases exactly line up with the size 2, 5 or 8 for the decoration class that we get from Corollary 6.15.

We will now cover the three general solutions mentioned previously.

Regular solution

First, a sequence of length m with $d_v = \nu$ trivially solves this equation. However, now every decoration vertex has the same degree as the template vertices, so G is actually regular. In the class of Hoffman colorable graphs with a regular template, the regular graphs form exactly those graphs with this trivial solution, which we will therefore call the "regular solution".

Universal solution

A second trivial solution occurs when m is a divisor of ν^2. Then we can take $d_v = m$ for all v. We need $\frac{\nu^2}{m}$ decoration vertices in that case. Every decoration vertex is now connected to every vertex of every color class $1 \leq i < \chi$, so therefore we will call this the "universal solution". If $m = \nu$, then this results in the regular complete multipartite graphs. If $m = \nu^2$, then this results in just adding one universal vertex, which was covered in Corollary 6.14 so we have Hoffman colorability automatically. We can actually pose a generalization to cone graphs, the following way.

Definition 7.11. The *k-cone graph* over a graph G is the graph obtained by adding k vertices to G and connecting them to every vertex of G, but not to each other

For $k = 1$ we get the usual definition of cone graphs (Definition 6.12) back. The following proposition is a generalization of Corollary 6.14, which characterizes Hoffman colorability of k-cone graphs completely. This also shows that the universal solution will always lead to a Hoffman colorable graph

Proposition 7.12. *Let G be a non-empty graph. Then the k-cone graph over G is Hoffman colorable if and only if there exists a positive integer ν such that the following hold simultaneously.*

(i) k is a divisor of ν^2;

(ii) G is Hoffman colorable;

(iii) G has $\chi(G)\nu^2/k$ vertices;

(iv) G is regular with valency $(\chi(G)-1)\nu$.

Proof. Write CG for the k-cone graph over G. Note that $\chi(CG) = 1 + \chi(G) \leq 3$, and that CG is connected.

Suppose that CG is Hoffman colorable. Consider the optimal coloring $V(CG) = \bigsqcup_{i=0}^{\chi(G)} V_i$ where V_0 contains all the new vertices, and write $\nu = -\lambda_{\min}(CG)$. Let x be the Perron eigenvector of CG. Consider for $1 \leq i \leq \chi(G)$ the bipartite part $H_{0,i}$ of CG. This is a complete bipartite part, so $\nu = \sqrt{k|V_i|}$ by the Decomposition Theorem, and x must be constant on V_i. Every bipartite part H_{ij} of G now has a constant eigenvector x_{ij}, and so H_{ij} is ν-regular. Now ν must be a positive integer and G must be regular with valency $(\chi(G)-1)\nu$. By $\nu = \sqrt{k|V_i|}$ we also see that every color class is of size ν^2/k, so that k must divide ν^2 and G must have $\chi(G)\nu^2/k$ vertices.

Conversely, suppose G satisfies all the requirements. Every eigenvector of G orthogonal to the constant vector can be extended to an eigenvector of CG by setting 0 on the vertices in V_0. If $(r_i)_{i=1}^k$ is a sequence of real numbers adding to zero, then assigning r_i to the i'th vertex of V_0 and 0 everywhere else will result in an eigenvector of eigenvalue 0. Lastly, putting ν on every vertex of G and ν^2/k on V_0 gives an eigenvector of eigenvalue $\chi(G)\nu$ and the vector putting ν on every vertex of G and $-\chi(G)\nu^2/k$ on V_0 gives an eigenvector of eigenvalue $-\nu$. Therefore we have

$$\mathrm{Spec}(CG) = \{\chi(G)\nu, 0^{k-1}, -\nu\} \cup \mathrm{Spec}(G) \setminus \{(\chi(G)-1)\nu\}.$$

Since G was Hoffman colorable, its least eigenvalue is also $-\nu$. We obtain that CG is Hoffman colorable. □

This proposition proves that the universal solution will always lead to a Hoffman colorable graph, and furthermore that every Hoffman colorable k-cone graph arises this way.

For a new case to be covered by this proposition, we must have m to not be equal to ν or ν^2. The smallest new case is $m = 8$ and $\nu = 4$. So if we start with a Hoffman colorable template graph where every color class is of size 8 and every vertex is adjacent to 4 vertices in every class other than its own (for example two disjoint copies of $K_{4,4}$), and we add $4^2/8 = 2$ decoration vertices and connect them to every vertex in the template, we must end up with a Hoffman colorable graph.

Leaf solution

A third trivial solution is where we take $d_v = 1$ for all v. We need $m\nu^2$ decoration vertices in this case, which are connected to exactly one vertex of each color class. Now every decoration will be a leaf in every bipartite part, hence the name. In principle, it is possible to consider all possible ways of adding decorations to the template this way and checking the least eigenvalue. However we will consider two special cases only here.

If $(\nu, m) = (1, 1)$, then the template graph is complete, and this construction will add one vertex connecting to every other vertex, resulting in yet again a complete graph.

If $\chi = 3$, then there exist a trivial way to add these leaf decorations. Namely, there are $m\nu$ edges in T (which is now bipartite and ν-regular), so to add the $m\nu^2$ decorations, we add ν to each edge in T. Here, "adding decoration v to an edge e" means connecting v to both end vertices of e by an edge, adding a small "flag" onto e. We will see in this case that we always get Hoffman colorability. However, because the reasoning applies more generally outside of the setting of regular templates as well, we will introduce the following definition.

Definition 7.13. Let T be a ν-regular graph. The *flag decoration* of T is the graph F with vertices

$$V(F) = V(T) \cup (E(T) \times [\nu]),$$

where $[\nu]$ denotes the set $\{1, \ldots, \nu\}$, such that for $u, v \in V(T)$ we have $u \sim_F v$ if and only if $u \sim_T v$ and we connect $v \in V(T)$ to $(e, j) \in E(T) \times [\nu]$ if and only if v is on the edge e.

We will prove that if T is tripartite, then the flag decoration is Hoffman colorable. The template graph T does not have to be Hoffman colorable. The case where $\chi(T) = 3$ actually does not fit in the construction stated in this section, as the added vertices (e, j) will share colors with vertices from T. However, since the reasoning is the same, we include these graphs here as well.

In order to prove Hoffman colorability, we will need a couple of lemmas.

Lemma 7.14. *Let T be a ν-regular graph. Let F be its flag decoration. Then the 3-colorings of T correspond bijectively to the 3-colorings of F.*

Proof. The restriction of a 3-coloring of F gives a 3-coloring of T. Conversely, if we have a 3-coloring of T, we can extend it uniquely to a 3-coloring of F by giving every decoration (e, j) the color that is distinct from the colors of the end vertices of e. These two operations are inverse to each other. □

Lemma 7.15. *Let T be a ν-regular graph and F its flag decoration. Then*

$$\mathrm{Spec}(F) = \{(-\nu)^{|V(T)|}, 0^{(\nu-1)|E(T)|}\} \cup \Big(2 + \mathrm{Spec}(L(T))\Big).$$

Furthermore, the greatest eigenvalue of F is 2ν, the least is $-\nu$ and so its Hoffman bound is equal to 3.

Proof. Let $v \in V(T)$. Consider the vector x given by

$$x(u) = \begin{cases} \nu & \text{if } u = v, \\ 0 & \text{if } u \neq v, \end{cases}$$

$$x(e,j) = \begin{cases} -1 & \text{if } v \in e, \\ 0 & \text{if } v \notin e. \end{cases}$$

One can check that this is an eigenvector of eigenvalue $-\nu$. These vectors generate a space S_1 of dimension $|V(T)|$. Next, let $e \in E(T)$ and choose $(r_j)_{1\leq j\leq \nu}$ summing to 0. Consider the vector x given by $x(e,j) = r_j$ and 0 everywhere else. One can verify that this is an eigenvector of eigenvalue 0. This gives a space S_2 of dimension $(\nu-1)|E(T)|$.

Choose an orthogonal basis for the two aforementioned spaces, and extend it to an orthogonal eigenbasis for F. Consider an eigenvector x of this basis orthogonal to S_1 an S_2. By orthogonality to S_2, $x(e,j)$ is independent of j, so we can just write $x(e)$. By orthogonality to S_1, we get for $v \in V(T)$ that

$$\nu x(v) = \sum_{e\in E(T):v\in e} \sum_{j=1}^{\nu} x(e,j) = \nu \sum_{e\in E(T)} x(e),$$

so that $x(v)$ is the sum of all eigenvector entries of the edges incident to v in T. Now write λ for the eigenvalue belonging to x. Choose any $e = \{u,v\} \in E(T)$, then

$$\begin{aligned} \lambda x(e) = x(u) + x(v) &= \sum_{e'\in E(T):u\in e'} x(e') + \sum_{e''\in E(T):v\in e''} x(e'') \\ &= 2x(e) + \sum_{f\in E(T):|e\cap f|=1} x(f), \end{aligned}$$

so that

$$(\lambda - 2)x(e) = \sum_{f\in N_{L(T)}(e)} x(f),$$

so that x is an eigenvector of $L(T)$ with eigenvalue $\lambda - 2$.

Conversely, any eigenvector y of the line graph of T can be made into an eigenvector x of F orthogonal to the spaces S_1 and S_2 by setting

$$x(v) = \sum_{f\in E(T):v\in f} y(f),$$

$$x(e,j) = y(e),$$

for $v \in V(T)$, $e \in E(T)$ and $1 \leq j \leq \nu$. If y was an eigenvector of eigenvalue λ, then x is an eigenvector of eigenvalue $\lambda + 2$.

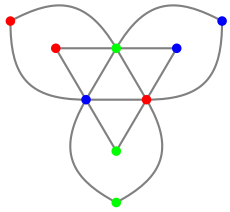
Now the eigenvectors of $L(T)$ correspond bijectively to the eigenvectors of F orthogonal to S_1 and S_2. The stated spectrum of F follows.

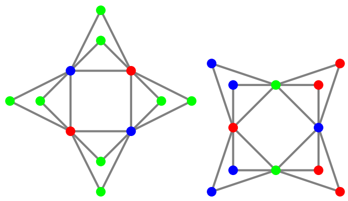
Note that all eigenvalues of a line graph are at least -2. So $2 + \lambda \geq 0$ for $\lambda \in \mathrm{Spec}(L(T))$, so that the least eigenvalue of F is $-\nu$. Furthermore, since T is ν-regular, the greatest eigenvalue of $L(T)$ is $2\nu - 2$, so the greatest eigenvalue of F is 2ν. We obtain a Hoffman bound equal to 3. □

Theorem 7.16. *Let T be a regular graph and F its flag decoration. Then F is Hoffman colorable if and only if $\chi(T) \leq 3$.*

Proof. From the second lemma we know that the Hoffman bound of F is 3. Now F is Hoffman colorable if and only if $\chi(F) = 3$. From the first lemma, we know that $\chi(F) = 3$ if and only if $\chi(T) \leq 3$, which concludes the proof. □

The smallest example of a flag decoration is K_3, being the flag decoration of K_2. Other small examples include the following graphs:





The first graph is the flag decoration of the 3-cycle, which is also a tensor product (we have seen it in Section 7.1). In fact, if you use $T \cong K_3 \times H$ for some k-regular graph with possible self loops H (self loops count one to the degree of a vertex), then F will be $K_3 \times G$ where G is formed by adding $2k$ leaves to every vertex in H. The 3-cycle is $K_3 \times FK_1$, where FK_1 is the fully complete graph on one vertex, which has valency 1. The flag decoration therefore is the tensor product of K_3 and the path graph on 3 vertices with a self loop on the middle vertex. This graph does not fit in the construction from regular templates, as the decorations have more than one color.

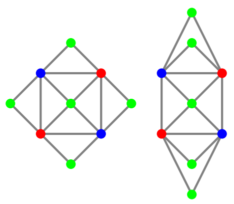
The second graph is the flag decoration of the 4-cycle. The two different optimal colorings of the 4-cycle show up in the flag decoration as well, just as the first lemma of this section predicts. We can in the same way have flag decorations of the 5-cycle, 6-cycle et cetera. these will all have multiple optimal colorings.

Applied to graphs on at most nine vertices

To find all Hoffman colorable graphs with a regular template on at most nine vertices, we know from Corollary 6.16 that we only have to look for 3-chromatic graphs, so bipartite templates (which are automatically Hoffman colorable). We will have to consider different pairs (ν, m), where ν is the valency of the template, and m half the number of vertices of the template. To end up with a graph on at most 9 vertices we need $m \leq 4$, so that we have $1 \leq \nu \leq m \leq 4$.

If $m = 1$, then we get K_3.

If $m = 2$, then there can be at most five decoration vertices. If $\nu = 1$, then $(d_v)_{v \in V_3} = (1, 1)$ and we obtain the 6-cycle, which is bipartite, or the disjoint union of two 3-cycles. If $\nu = 2$, we are in a case considered before. For at most five decorations, there are two solutions for $(d_v)_{v \in V_3}$, namely $(2, 2)$ and $(1, 1, 1, 1, 2)$. The first solution gives $K_{2,2,2}$. The second solution gives two possible compositions, which both are Hoffman colorable:

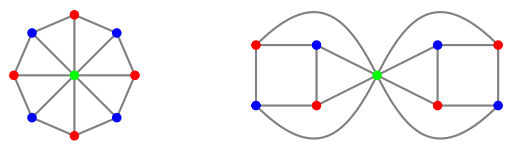


If $m = 3$, then we can have up to three decoration vertices. We aim for

$$\sum_{v \in V_3} d_v^2 = 3\nu^2,$$

which has the trivial solution $d_v = \nu$ for all $v \in V_3$. This will lead to the regular Hoffman colorable graphs on nine vertices, which we have covered before in Section 7.2. For the three options $\nu \in \{1, 2, 3\}$, there are actually no other solutions with at most three decoration vertices than the trivial regular solution.

Lastly, if $m = 4$, then we have just one decoration vertex v, with $d_v = 2\nu$. By Corollary 6.13, this decoration vertex needs to be universal. Now $\nu = 2$. Considering the template, it is a 2-regular bipartite graph on 8 vertices. There are just two options: the 8-cycle or two copies of the 4-cycle. This gives the following two Hoffman colorable graphs:



In total we have now seen 13 connected Hoffman colorable graphs on 9 vertices, and we will see in Chapter 8 that these are all.

7.4 The block design construction

Using (trivial) block designs, we can construct Hoffman colorable graphs with three colors. The significance of this construction is that it gives infintely many Hoffman colorable graphs with three distinct class sizes, whereas all the other constructions have at least two color classes of the same size. The reason we need block designs, is that they capture the requirements for the proof to work precisely. The regularity and uniformity parameters of block designs (r and k) will ensure that the bipartite parts are biregular after which we can write down the largest eigenvalue. The λ-parameter will ensure that the least eigenvalue of the graph will not be too low, so that the Composition Theorem can give Hoffman colorability. Tha a_0, a_1, and a_2 are needed in the construction in order to ensure compatibility of the bipartite parts.

Construction 7.17. For $i = 1, 2$, let (P_i, B_i, I_i) be $(v_i, b_i, r_i, k_i, \lambda_i)$-block designs. Let a_0, a_1, a_2 be positive integers such that

$$v_1 b_2 a_1 = b_1 r_2^2 a_0, \quad v_2 b_1 a_2 = b_2 r_1^2 a_0.$$

Now construct a 3-chromatic graph $G = (V, E)$ as follows.

- For the zeroeth color class, take

  $$V_0 = B_1 \times B_2 \times [a_0],$$

  where $[n]$ is the set of integers 1 up to and including n.

- For $i = 1, 2$, take

  $$V_i = P_i \times [a_i]$$

  .

- Declare two vertices from V_1 and V_2 adjacent without any conditions, so that $G[V_1 \sqcup V_2]$ is a complete bipartite graph.

- Let $i = 1, 2$. Let $(q_1, q_2, m) \in V_0$ and $(p, \ell) \in V_i$. They be adjacent if and only if $p I_i q_i$.

Note that the indices ℓ and m have no influence at all on adjacency. This means that (p, ℓ) and (p, ℓ') and (q_1, q_2, m) and (q_1, q_2, m') have the same neighbors.

Further, there are infinitely many possible choices for a_0, a_1, a_2, since if a_0, a_1, a_2 is a suitable choice, also ma_0, ma_1, ma_2 is suitable, as only the ratios matter. There is however a primitive choice, the one where $\gcd(a_0, a_1, a_2) = 1$. This is obtained by setting

$$a_0 = \operatorname{lcm}\left(\frac{v_1 b_2}{\gcd(v_1 b_2, r_2^2 b_1)}, \frac{v_2 b_1}{\gcd(v_2 b_1, r_1^2 b_2)}\right).$$

We need a_0 to be a multiple of the two integer in the lowest common multiple, to ensure that a_1 and a_2 are integers.

Theorem 7.18. *The graph G from Construction 7.17 is Hoffman colorable. If $\chi(G) = 3$, then the greatest eigenvalue is 2ν and the least is $-\nu$, where $\nu = a_0 r_1 r_2$.*

Proof. First of all, if G is actually bipartite, then it is automatically Hoffman colorable. So from now we assume that $\chi(G) = 3$. This implies that $H_{0,1}$ and $H_{0,2}$ are non-empty, and so r_1, k_1, r_2, k_2 are non-zero. This is important because we will divide by k_1 and k_2 later in the proof.

Note that $H_{1,2}$ is complete bipartite, where one color class has size $a_1 v_1$, and the other has size $a_2 v_2$. So

$$\lambda_{\max}(H_{1,2}) = \sqrt{a_1 v_1 a_2 v_2} = \sqrt{\frac{a_0 r_2^2 b_1}{v_1 b_2} \cdot v_1 \cdot \frac{a_0 r_1^2 b_2}{v_2 b_1} \cdot v_2} = a_0 r_1 r_2,$$

which is what we called ν.

The other bipartite parts $H_{0,1}$ and $H_{0,2}$ are also biregular. We do the calculation only for $H_{0,1}$, since the one for $H_{0,2}$ follows analogously. Let $(p, \ell) \in V_1$. Then it is adjacent to all (q_1, q_2, m) such that pIq_1. There are r_1 choices for q_1, b_2 choices for q_2 and a_0 choices for m. So (p, ℓ) is connected to $r_1 b_2 a_0$ vertices in V_0. Conversely, let (q_1, q_2, m) be a vertex in V_0. It is connected to all (p, ℓ) with pIq_1. There are k_1 choices for p, and a_1 choices for ℓ, so it is connected to $a_1 k_1$ vertices from V_1. By the rules for block designs, $k_1 = \frac{v_1 r_1}{b_1}$. Now since $H_{0,1}$ is biregular, we have

$$\lambda_{\max}(H_{0,1}) = \sqrt{a_0 r_1 b_2 a_1 k_1} = \sqrt{a_0 r_1 b_2 \cdot \frac{a_0 r_2^2 b_1}{v_1 b_2} \cdot \frac{v_1 r_1}{b_1}} = a_0 r_1 r_2 = \nu.$$

Since $H_{0,1}, H_{0,2}, H_{1,2}$ are all biregular, they have eigenvectors for their largest eigenvalues that are constant on both color classes (and hence positive). Since we can scale eigenvectors of bipartite graphs in such a way that they have norm 1 on both color classes, these eigenvectors are compatible, such that (H_{ij}) forms a collection of compatible bipartite parts. By the Composition Theorem we only have to prove that the least eigenvalue of G is $-\nu$.

Let x now be an eigenvector for G with eigenvalue λ. Suppose that λ is not equal to $-\nu$, 0 or 2ν. Then firstly, since $(p, \ell), (p, \ell')$ have the same neighbors, we have

$$\lambda \cdot x(p, \ell) = \sum_{v \in N(p,\ell)} x(v) = \lambda \cdot x(p, \ell'),$$

and therefore we can write $x(p)$ instead of $x(p, \ell)$ (using that $\lambda \neq 0$ so we can divide it out). Similarly we can write $x(q_1, q_2)$ instead of $x(q_1, q_2, m)$. By the Composition Theorem we also see that x must sum to 0 on each of the three classes. Combining these facts, we see

$$\begin{cases} \sum_{p \in v_1} x(p) = 0, \\ \sum_{p \in v_2} x(p) = 0, \\ \sum_{q_1 \in B_1} \sum_{q_2 \in B_2} x(q_1, q_2) = 0. \end{cases}$$

We now look at the size of λ^2, which we would like to bound above by ν^2. We would like to consider an entry of x that is non-zero, on one of the color classes V_1 or V_2. Luckily, eigenvectors are non-zero, so some vertex has a non-zero value. If this vertex u is in V_0, then consider the fact that $\lambda \neq 0$. If x is zero on V_1 and V_2, then the sum of the entries of x on the neighbors of u is 0, and so $\lambda x(u) = 0$, and so $x(u) = 0$. Therefore we may assume, without loss of generality, that we found a $p \in V_1$ such that $x(p) \neq 0$.

Note that $(A^2)_{uv}$ records the number of walks from vertex u to vertex v of length 2. Consider $(p, 1) \in V_1$. If $v \in V_0$, then the number of walks from $(p, 1)$ to v is equal to the number of neighbors of v in V_2 (as $H_{1,2}$ is complete), which is independent of the vertex v. Since x sums to zero on the color classes, this means that the contribution of these walks in $(A^2 x)(p, 1)$ is 0. Furthermore, if $v = (p', \ell) \in V_2$, then the number of walks from $(p, 1)$ to v of length 2 (so through V_0) is $a_0 r_1 r_2$, since there are a_0 choices for m, r_1 choices for q_1 and r_2 choices for q_2 for the middle vertex (q_1, q_2, m) in the walk. So also this contribution to $(A^2 x)(p, 1)$ is 0. Furthermore, if $v \in V_1$ as well, then the number of walks from $(p, 1)$ to v through V_2 is constant. So for $(A^2 x)(p, 1)$ we only have to consider walks from $(p, 1)$ to $v \in V_1$ through V_0:

$$(A^2 x)(p, 1) = \sum_{v \in V_1} x(v) |\{w \in V_0 : u \sim w \sim v\}|.$$

If $v = (p, \ell)$ for some ℓ, then the number of walks from $(p, 1)$ to ℓ through V_0 is $a_0 r_1 b_2$. If $v = (p', \ell)$ for some p' different from p, then this number of walks is $a_0 \lambda_1 b_2$. So we get

$$= a_0 \lambda_1 b_2 \Big(\sum_{v \in V_1} x(v) \Big) + a_0 (r_1 - \lambda_1) b_2 \Big(\sum_{\ell=1}^{a_1} x(p) \Big).$$

The first sum is 0, so we know

$$= a_0 a_1 (r_1 - \lambda_1) b_2 x(p).$$

Since we know that x is an eigenvector with eigenvalue λ, we must have

$$= \lambda^2 x(p),$$

and since we know $x(p) \neq 0$, we obtain

$$\lambda^2 = a_0 a_1 (r_1 - \lambda_1) b_2 \leq a_0 a_1 r_1 b_2.$$

Substituting the definition of a_1, we get

$$\lambda^2 \leq a_0 r_1 b_2 \cdot \frac{a_0 r_2^2 b_1}{v_1 b_2} = a_0^2 r_2^2 r_1 \cdot \frac{b_1}{v_1} = \frac{a_0^2 r_1^2 r_2^2}{k_1} \leq \nu^2,$$

where we used that $r_1 v_1 = k_1 b_1$. Since $\lambda^2 \leq \nu^2$, we must have $-\nu < \lambda$, and so $\lambda_{\min}(G) = -\nu$ and G is Hoffman colorable by the Composition Theorem. □

7.4.1 Relating to other constructions

Let's try to relate the block design construction to the constructions we have already seen. Along the way we will see some examples.

In order for a 3-chromatic graph to be a tensor product, or to have a regular template, we need to have color classes of the same size. By construction, the three bipartite parts are biregular, so if two of the classes are of the same size we immediately have a regular template. Consequently, if the Block Design Construction results in a tensor graph, it will automatically be a regular graph. We are therefore only interested to see when this construction leads to a regular graph, and when to a regular template.

In order to do this, it is vital to study the class sizes more closely. By construction, we have $|V_0| = a_0 b_1 b_2$, $|V_1| = a_1 v_1$, $|V_2| = a_2 v_2$, where

$$v_1 b_2 a_1 = b_1 r_2^2 a_0, \quad v_2 b_1 a_2 = b_2 r_1^2 a_0,$$

so that

$$\begin{cases} |V_0| = a_0 b_1 b_2, \\ |V_1| = a_0 b_1 r_2^2 / b_2, \\ |V_2| = a_0 b_2 r_1^2 / b_1. \end{cases}$$

Now in order for V_1 and V_2 to be of the same size, we need $r_1/b_1 = r_2/b_2$, or, equivalently, points in the two block designs need to be incident to the same proportion of all blocks. In order for V_0 and V_1 to be of the same size, we need $r_2/b_2 = 1$ (and symmetrically for V_0 and V_2 to be of the same size we need $r_1/b_1 = 1$), meaning that every point should be incident to every block, which would be a complete design. We can therefore distinguish four cases now.

(1) Both block designs are complete;

(2) Only one block design is complete;

(3) None of the two designs are complete, and the proportions r_i/b_i are equal;

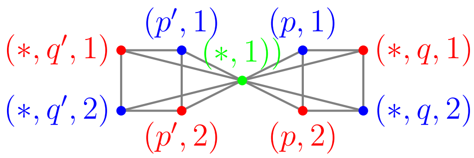
(4) None of the two designs are complete, and the proportions r_i/b_i are different.

The first case contains the regular graphs coming from the Block Design Construction. However, in this case all three bipartite parts have to be complete bipartite, and the graph is just a regular complete tripartite graph. The second and third cases contain the irregular graphs with a regular template formed from the construction. Lastly, the fourth case contains the graphs with three classes of distinct sizes. Theoretically, the fourth case is most interesting to us, because it is an infinite source of graphs of this type.

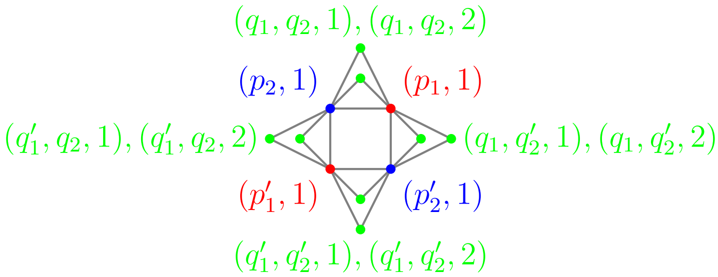
Regular templates

As we have seen before, there are two flavors of regular templates from the Block Design Construction: taking one design complete, or taking two incomplete designs with the same proportion r_i/b_i.

In the first case, we have one vertex class having complete bipartite parts with both other classes, so we are in the situation of Proposition 7.12. The simplest case is as follows: take as the first design the trivial (1,1,1,1,0)-design (just one point and one block incident to each other), and as the second design the (2,2,1,1,0)-design (two points and two singleton blocks pairwise incident). We can then choose a_0 to be 2, so that $a_1 = 1$ and $a_2 = 2$, implying $|V_0| = 4$, $|V_1| = 1$, and $|V_2| = 4$. If the unique point and block of the first design are denoted $*$ and the two points of the second block design are denoted p and p' and the two blocks q and q', then we see that we actually get the cone graph over the disjoint union of two cycles:



In the second case, we take two block designs with the same ratio r_i/b_i (but not equal to 1). The simplest case is to take this ratio to be 1/2, and to consider two (2,2,1,1,0)-block designs. If we choose $a_0 = 2$, then $a_1 = 1$ and $a_2 = 1$, and $|V_0| = 8$, while $|V_1| = |V_2| = 2$. If we denote the two points of the i'th block design p_i and p'_i (and similarly for the blocks), we see we get the flag decoration of the 4-cycle:



Three distinct-sized color classes

As nice as the examples above are, they do not add anything new. The relevance of the Block Design Construction is therefore in this last case, where we find Hoffman colorable graphs with three color classes of distinct sizes.

For this case we need two block designs that have distinct ratios r_i/b_i both less than 1. For small examples, we want to choose b_i and r_i coprime, and furthermore $v_i = b_i$. Two simple options are taking the proportion $1/m$, and considering a (m,m,1,1,0)-design made from singletons, and taking the proportion $(m-1)/m$, with a (m,m,m-1,m-1,m-2)-design (obtained by taking complements of the blocks from the singleton design). These examples were covered in Section 2.4.1.

The following table contains the number of vertices from the Block Design Construction with different proportion pairings. Each time we take the block design as described above and we take the least a_0 possible.

Proportions	$\lvert V_0\rvert$	$\lvert V_1\rvert$	$\lvert V_2\rvert$	Total number of vertices	$\nu = -\lambda_{\min}(G)$
1/2, 1/3	36	4	9	49	6
1/2, 2/3	36	16	9	61	12
1/3, 2/3	27	12	3	42	6
1/2, 1/4	32	2	8	42	4
1/2, 3/4	32	18	8	58	12
1/3, 1/4	144	9	16	169	12
1/4, 3/4	64	36	4	104	12
...					

graphs from the Block Design Costruction with three classes of distinct sizes

The two graphs on 42 vertices are most probably the smallest graphs of this subclass of the Block Design Construction. Note that they are not isomorphic, because they do not have the same least eigenvalue.

This table shows as well that the Block Design Construction is very rich, giving many different Hoffman colorable graphs, even when we only supply trivial block designs. Using more complicated block designs, such as the one related to the Fano plane, one can obtain Hoffman colorable graphs with even more intricate structure.

Chapter 8

Algorithm

This chapter covers an algorithm for computing every connected Hoffman colorable graph given a number of vertices and a number of colors. The algorithm is generative in nature and is based on the Decomposition and Composition Theorems. We first present some results. After that we will address the specifics of the algorithm and the computation time. Lastly, we will study some of the graphs.

The general idea of the algorithm is to generate bipartite parts, to gather into collections of compatible bipartite parts, and to compose these into graphs where we can check Hoffman colorability using the Composition Theorem. The Decomposition Theorem ensures that every graph arises this way.

The algorithm has been implemented in computer algebra system Magma ([6], web page), version V.28-8. We shared the code via GitHub[1].

8.1 Results

We provide four tables to present the results of the algorithm, for three, four, five and six colors. For every choice of input of a number of colors and a number of vertices, we provide the number of connected Hoffman colorable graphs (up to isomorphism) and also the number and proportion of the graphs coming from several of the constructions discussed in the previous chapter.

Some cells in the tables have a $\geq$-sign. In these cases it is possible that more connected-Hoffman colorable graphs exists. These graphs have the special property that at least one of the bipartite parts is disconnected. This disconnectedness create an issue in the algorithm. This issue is addressed in Section 8.2.2. The algorithm always outputs these disconnected cases for human intervention to take place, but we did not include these outputs in these tables.

In the tables we can very clearly see Corollary 6.16 at work: up to 3χ vertices we only find graphs on χ vertices, on 2χ vertices, and on $2\chi + 3$ vertices.

[1] https://github.com/tjvveluw/connected-Hoffman-colorable-graphs

#vertices	#graphs	#regulars	#regular templates	#tensors	#regular tensors
3	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
4	0	-	-	-	-
5	0	-	-	-	-
6	2	1 (50%)	1 (50%)	2 (100%)	1 (50%)
7	0	-	-	-	-
8	0	-	-	-	-
9	13	4 (31%)	8 (62%)	8 (62%)	3 (23%)
10	3	-	3 (100%)	-	-
11	2	-	0 (0%)	-	-
12	67	16 (24%)	23 (34%)	30 (45%)	4 (6%)
13	14	-	3 (21%)	-	-
14	46	-	8 (17%)	-	-
15	≥ 1634	≥ 900	≥ 1056	204	5

connected Hoffman colorable graphs with three colors

Above is the table for three colors. We see eight graphs with a regular template on nine vertices, and eight tensor graphs, with an overlap of three. That means that all of the Hoffman colorable graphs on nine vertices fall into one of the constructions discussed in last chapter (tensor graphs, Section 7.1, and graphs with a regular template, Section 7.3). Eleven is the smallest number of vertices for which a graph exists not belonging to one of the constructions. The multiples of three seem to give a lot of Hoffman colorable graphs with that number of vertices. This has to do with the integer partition (m, m, m) that is possible in these cases, giving many examples, including the tensor graphs and the regular graphs. Especially the number of fifteen-vertex graphs blows up greatly.

#vertices	#graphs	#regulars	#regular templates	#tensors	# regular tensors
4	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
5	0	-	-	-	-
6	0	-	-	-	-
7	0	-	-	-	-
8	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
9	0	-	-	-	-
10	0	-	-	-	-
11	2	-	2 (100%)	-	-
12	8	5 (63%)	5 (63%)	3 (38%)	1 (13%)
13	17	-	17 (100%)	-	-
14	5	-	2 (40%)	-	-
15	10	-	7 (70%)	-	-
16	≥ 167	≥ 92	≥ 98	15	4
17	8	-	7 (88%)	-	-
18	≥ 380	-	≥ 232	-	-

connected Hoffman colorable graphs with four colors

Above is the data for four colors. We see a peak in the number of Hoffman colorable graphs at thirteen vertices. This has to do with the integer partition

$$13 = 1 + 4 + 4 + 4,$$

so this case contains many (in fact, eleven) cone graphs. We also have the partition

$$13 = 3 + 3 + 3 + 4$$

providing some more Hoffman colorable graphs. In every case we see a pretty sizable number of graphs having a regular template. Only for fourteen vertices it is not a majority. This has to do with the integer partition

$$14 = 2 + 2 + 5 + 5,$$

that gives some Hoffman colorable graphs included in the last class of Corollary 6.15. Because we have two class sizes appearing twice, we do not have a regular template and this explains the dip in percentage for this number of vertices.

#vertices	#graphs	#regulars	#regular templates	#tensors	# regular tensors
5	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
6	0	-	-	-	-
7	0	-	-	-	-
8	0	-	-	-	-
9	0	-	-	-	-
10	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
11	0	-	-	-	-
12	0	-	-	-	-
13	2	-	2 (100%)	-	-
14	0	-	-	-	-
15	10	7 (70%)	7 (70%)	2 (20%)	1 (10%)
16	16	-	11 (69%)	-	-
17	34	-	34 (100%)	-	-
18	≥ 5	-	≥ 5	-	-
19	7	-	7 (100%)	-	-

connected Hoffman colorable graphs with five colors

Above is the table for five colors. The peak of thirty-four Hoffman colorable graphs on seventeen vertices is explained by the fact that we have cone graphs ($17 = 1 + 4 + 4 + 4 + 4$). On sixteen vertices, we also have relatively many Hoffman colorable graphs. We have two integer partitions at work here:

$$16 = 2 + 2 + 2 + 5 + 5$$
$$16 = 3 + 3 + 3 + 3 + 4.$$

The first partition gives Hoffman colorable graphs that do not have a regular template, which explains the decrease in the regular template percentage.

#vertices	#graphs	#regulars	#regular templates	#tensors	# regular tensors
6	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
7	0	-	-	-	-
8	0	-	-	-	-
9	0	-	-	-	-
10	0	-	-	-	-
11	0	-	-	-	-
12	1	1 (100%)	1 (100%)	1 (100%)	1 (100%)
13	0	-	-	-	-
14	0	-	-	-	-
15	1	-	1 (100%)	-	-
16	0	-	-	-	-
17	0	-	-	-	-
18	10	5 (50%)	5 (50%)	2 (20%)	1 (10%)
19	8	-	8 (100%)	-	-
20	0	-	-	-	-
21					
22	5	-	5 (100%)	-	-
23	17	-	17 (100%)	-	-

connected Hoffman colorable graphs with six colors

Above is the table for six colors. The simultaneous peak in number of Hoffman colorable graphs and the drop in regular template percentage at eighteen vertices is explained by the integer partition

$$18 = 2 + 2 + 2 + 2 + 5 + 5.$$

In general, the integer partitions using only 2 and 5 seem to be a fruitful environment for Hoffman colorability. Also regular templates with color classes of size 3, decorated with 4 vertices seem to give many Hoffman colorable graphs (in this case all of the eight Hoffman colorable graphs on nineteen vertices).

8.2 Outline of the algorithm

We first list the different steps of the process, with small explanations what the point of every step is. After that we explain the steps in great detail. Next, we will list the different steps of the algorithm with a very general description. The first three steps are for generating every set of compatible bipartite parts. Here the "b"-steps are the real meat of the algorithm, here the bipartite parts are considered. The "a"-steps concern the integer partitions, and they are intended to speed up the process by reducing the number of cases to be considered for the "b"-step.

Algorithm

Input: a number of vertices n and a number of colors χ.

Step 1: Eigenvalues:

Step 1a: Form all viable integer partitions of n into χ parts;

Step 1b: Generate all bipartite parts of relevant sizes and sort by largest eigenvalue.

Step 2: Eigenvectors:

Step 2a: Eliminate integer partitions for which there exists a pair with no possible bipartite parts;

Step 2b: Filter out the disconnected bipartite parts, and for the connected bipartite parts, compute the Perron eigenvectors and sort.

Step 3: Compatibility:

Step 3a: Check for which partitions one of the possible bipartite parts is disconnected and take those aside;

Step 3b: Form every possible collection of compatible bipartite parts.

Step 4: Composing:
For every collection of compatible bipartite parts, generate every possible way of composing the bipartite parts together and check Hoffman colorability.

Output: A sequence of the Hoffman colorable graphs, and a sequence of disconnected cases that come out of Steps 2b and 3a.

8.2.1 Steps in more detail

We would now like to explain further what goes into the different steps for making the collections of compatible bipartite parts and the composing step.

Step 1a: Viable partitions

Inputs: n and χ. Outputs: all "viable" integer partitions of n into χ parts. These will signify the sizes of the color classes of the Hoffman colorable graph.

What do we mean by "viable"? We exclude all partitions that contain some "forbidden" substructure. We also treat separately some theoretical cases to speed up the process, more on that in Section 8.2.3. We have the following theoretical restrictions derived from several results in Chapter 6.

- Any partition with a 1 is excluded (Corollary 6.13), as Hoffman colorable cone graphs will be treated separately (see Section 8.2.3);
- Any partition containing a 2 and a 3 is excluded (Corollary 6.11);

- Any partition containing two 2's is excluded (Corollary 6.15), as these graphs must then be colorable either with color classes of size 2 and 5, or with color classes of size 5 and 8. Graphs with color classes of size 2 and 5 will be treated separately (see Section 8.2.3), and the graphs with classes of size 5 and 8 will come up in a different partition.

We also have some empirical restrictions.

- Any partition containing 2-4-4 is excluded;
- Any partition containing 3-4-4 is excluded;
- Any partition with three distinct parts summing to at most 14 is excluded;
- Any partition containing 3-3-3-5 is excluded.

These exclusions are empirically justified. For each of the four exclusions we have run the algorithm without the exclusions with input of relevant size (for the first case $n = 10,\ \chi = 3$), and finding no Hoffman colorable graphs this way. However, we need to be careful about disconnected parts of connected Hoffman colorable graphs. Respectively, we got the following from the outputs.

- $n = 10,\ \chi = 3$: no connected Hoffman colorable graphs with class sizes 2-4-4, and no disconnected cases.
- $n = 11,\ \chi = 3$: no connected Hoffman colorable graphs with class sizes 3-4-4, and the one disconnected case did not have any disconnected graphs with class sizes 3-4.
- $n = 11$ up to 14, $\chi = 3$: no connected Hoffman colorable graphs with three classes of distinct size, and only one disconnected case with a partition with three distinct class sizes comes up, namely 3-5-6 with least eigenvalue -2, but no disconnected bipartite parts with class sizes 3-5 and this eigenvalue exists. It is sufficient to start at 11, since by Corollary 6.13 and Corollary 6.11 the smallest case would be 2-4-5.
- $n = 14,\ \chi = 4$: no connected Hoffman colorable graphs with class sizes 3-3-3-5, and no disconnected cases.

Step 1b: Sorting by largest eigenvalue

In this step, we generate all bipartite graphs, using the graph generation program geng due to McKay [28]. We only proceed with the bipartite graphs that satisfy the following conditions.

- The class sizes must be part of some viable partition from Step 1a;
- The graph must have a positive eigenvector to be in a collection of compatible bipartite parts, so the largest eigenvalues of the components must be equal (Corollary 2.3);

- If the bipartite graph is connected, then the largest eigenvalue cannot have an even minimum polynomial (Corollary 6.10).

In order to do this, the algorithm computes the class sizes and the Perron eigenvalues of every component of every bipartite graph, and sorts it by eigenvalue.

We compute the eigenvalues ν algebraically. That is, we represent the eigenvalues with their minimum polynomial over $\mathbb{Q}$, rather than with a real approximation. We do this to do computations over the number field $\mathbb{Q}(\nu)$ and to easily check equality.

To do this, we factor the characteristic polynomial φ of the adjacency matrix of a connected graph and we apply Newton's approximation method to decide which factor contains the largest root of φ. Doing this for all components of a disconnected graph, we can decide if the graph has a positive eigenvector.

Step 2a

Given an eigenvalue ν, this step checks which of the viable partitions are still viable for ν. It might be, and often happens that there is only a small number of bipartite graphs with ν as the largest eigenvalue, and some viable partitions might be impossible to make.

Step 2b: Sorting by eigenvector

Given a largest eigenvalue ν, in this step all the relevant bipartite parts (from Step 2a) are sorted by the eigenvector for this eigenvalue. This can only be done for the connected bipartite parts, as the disconnected bipartite parts have an eigenspace for ν of dimension at least 2, so we cannot pick one representative eigenvector for the whole space.

For connected graphs, we can compute the eigenspace of the adjacency matrix A for eigenvalue ν over the field $\mathbb{Q}(\nu)$. This eigenspace will be of dimension 1 and we can take any non-zero vector as a representative.

The disconnected bipartite parts are stored separate from the connected bipartite parts, as only the connected ones are sorted by eigenvector.

We split the Perron eigenvectors we obtained up according to the two bipartite classes, and sort the graphs accordingly. It is sometimes necessary to rescale an eigenvector for a later graph, to match it up with a graph that already came up.

Step 3a

In this step, given an eigenvalue ν, we decide for every partition that survived Step 2a the following.

- Whether there exists a substructure (a, b) in the partition, such that there exists a disconnected bipartite part with bipartite class sizes a and b;

- Whether for every substructure (a, b) of the partition there still exist connected bipartite parts.

Partitions of the first kind are stored separately for human intervention, together with all the necessary information. Partitions of the second kind are passed on to the next step.

Note that every partition that survived Step 2a is of at least one of the two kinds. There might however be an overlap. For these cases there does exist a disconnected bipartite part, but the algorithm can still proceed with just the connected bipartite parts, and it does so.

Step 3b: Collecting compatible bipartite parts

In this step, given a partition and an eigenvalue, and a survey of connected bipartite parts sorted by eigenvector, we want to form every possible collection of compatible bipartite parts possible. We do this inductively, with respect to the length of the partition.

For the inductive basis, the partition consists of two parts, so we can take a list of the bipartite parts present, categorized along compatibility of the eigenvectors (which we can read off).

For the inductive step, we consider a partition of which we already found the collections of compatible sets for the pruned partition (the partition after removing the last entry). The algorithm runs over every present eigenvector projection x for the new class, and checks which of the bipartite parts have eigenvector compatible with x and with the vectors of the bipartite parts already present in the collection. If we find bipartite parts compatible with every class from the pruned partition, we extend the collection of compatible sets to the new class.

Step 4: Composing

In this step, given a collection of compatible sets, we are tasked with finding every possible way to paste these together. This step is set up inductively as well, with respect to the length ψ of the partition.

For the inductive basis, we only have two classes in the partition, so there is nothing to paste and we just return the bipartite parts that were in the compatible collection.

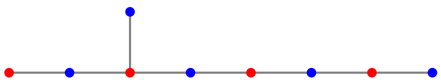
For the inductive step, we calculate every possible way of identifying vertices from the to be added bipartite parts, to the graph of classes 1 up to ψ, preserving the entries of the prescribed eigenvectors. Also we compute every way of identifying the vertices of the to be added bipartite parts for creating the new color class $V_{\psi+1}$. While we do this, we take into account the symmetries of the bipartite parts by computing transversals of automorphism groups in permutation groups. After we have pasted in every possible way, we check isomorphism and remove duplicates. After that we approximate the least eigenvalue of the graph using Newton's approximation method to check Hoffman colorability.

8.2.2 Disconnected bipartite parts

As seen in Step 2b, the step where we would like to sort the bipartite parts by eigenvector, the disconnected bipartite parts form a problem: the space of positive eigenvectors has dimension at least two. It is therefore algorithmically difficult to check compatibility of different bipartite parts, so the algorithm automatically filters out these cases. At the end, the algorithm returns these cases for human intervention to take place. It needs to be decided how to make collections of compatible bipartite parts out of both the connected and disconnected graphs that are provided (if they exist), and then these collections of compatible bipartite parts should be fed into Step 4, the composing step.

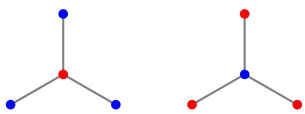
This disconnectedness issue is the reason why in the tables some entries have $\geq$-signs. For the smallest cases, we were able to solve these cases manually. To illustrate how this is done, we solve the disconnected cases for the inputs $n = 17,\ \chi = 4$.

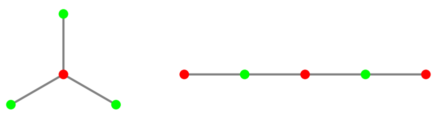
The algorithm outputs two disconnected cases. First we have a disconnected case with the data $\nu = 2$ and the class size partition $(4, 4, 4, 5)$. The algorithm gives just one possible disconnected bipartite part, namely the disjoint union of two 4-cycles. For class sizes 5-4, only one connected bipartite part is given, so it has to be part of the collection of compatible bipartite parts thrice:



On the class of size 4 (the red class), we therefore have to have a scalar multiple of the vector $(1, 1, 2, 3)$ coming from the Perron eigenvector of this bipartite part (putting 3 on the top vertex, and on the bottom row from left to right 2-4-6-5-4-3-2-1). It is impossible to have this as half of a positive eigenvector of two disjoint 4-cycles, since this space is of the form (a, a, b, b) (the entries belonging to vertices of the same component must be the same by regularity of the components). In other words, the two bipartite parts are incompatible. Therefore we get no new collections of compatible bipartite parts and no new Hoffman colorable graphs out the first disconnected case.

The second disconnected case concerns $\nu = \sqrt{3}$ with the same integer partition $(4, 4, 4, 5)$. By Corollary 6.10 all bipartite parts must be disconnected. There are two disconnected bipartite parts: the disjoint union of two $K_{1,3}$ (for two classes of size 4, so we order the components in such a way that the leaves of the one component and the leaves of the other component are in different classes), and the disjoint union of $K_{1,3}$ and the path on five vertices (again ordered so that the leaves of both components come in different classes, 4-5):





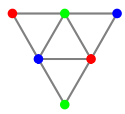
Call the left graph $G_{4,4}$ and the right graph $G_{4,5}$. Consider the class of four red vertices. From $G_{4,4}$ we read off that three of the entries of the positive eigenvector x_{red} must be equal. From $G_{4,5}$ we now see that the entry on the center vertex of the $K_{1,3}$-component must be equal to the entry on the leaves of the path, so we might choose the vector for $G_{4,5}$ composed of $(\sqrt{3}, 1, 1, 1)$ for the component isomorphic to $K_{1,3}$ and $(\sqrt{3}, 3, 2\sqrt{3}, 3, \sqrt{3})$ on the component isomorphic to the path. Now we must have (up to rescaling) $x_{\text{red}} = (1, 1, 1, 2)$, where the red vertex with entry 2 is the middle vertex of a $K_{1,3}$ in $G_{4,4}$ and the center vertex of the path in $G_{4,5}$. However, if we consider the blue-green bipartite part (which also has to be isomorphic to $G_{4,5}$) as well, we get the same entries for $x_{\text{blue}} = (1, 1, 1, 2)$. But x_{blue} and x_{red} together do not form a positive eigenvector of $G_{4,4}$. Therefore these bipartite parts are also incompatible, so also from the second disconnected case we do not get any more Hoffman colorable graphs.

8.2.3 Cone graphs and 2-5-graphs

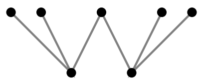
As said before, the partitions leading to cone graphs and 2-5-graphs are treated separately. This is because we already have a lot of information about the bipartite parts, and we do not have to do the time consuming operations of generating every bipartite graph of some size.

For the cone graphs, we only need to compute the graph over which we take the cone graph. We know from Corollary 6.13 what those look like: they need to be regular, Hoffman colorable, and of certain sizes. Regular graphs have a constant eigenvector, so we just generate all regular bipartite graphs of the right size and apply the “compose” step. We do not have to set aside the disconnected parts now because we already know the eigenvectors to be constant.

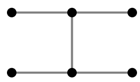
For the 2-5-graphs, we know that we have the exceptional graph



which we include in the algorithm manually, and graphs that have least eigenvalue -2. The bipartite part of two classes of size 2 will be the 4-cycle, the bipartite part of two classes, one of size 2 and the other of size 5, will be



and the bipartite part of two classes of size 5 is either the disjoint union of two copies of $K_{1,4}$, or the disjoint union of a 4-cycle and the H-graph:



We gather this information into a collection of compatible bipartite parts, and apply the “compose” step immediately.

8.3 Computation time

In this table, the computation times of our implementation of the algorithm on different inputs is presented. The computations were run on a single 64-bit 1500 MHz AMD processor. The figures are rounded up to the nearest time unit used. The cases with DNF did not finish, either because of memory constraints (for the input $n = 16,\ \chi = 3$), or because it did not finish in a week.

$n \setminus \chi$	3	4	5	6
15	18 min	2 sec	350 msec	160 msec
16	DNF	4 min	890 msec	-
17		5 sec	4 hrs	-
18		3 min	4 hrs	3 min
19		DNF	2 min	35 min
20			DNF	10 msec
21				DNF
22				18 min
23				3 hrs

computation times for various inputs to the algorithm

We can compare these computation times to those of a more "naive" algorithm. Namely, given input (n, χ), we could generate every connected graph on n vertices, compute the Hoffman bound and the chromatic number and check Hoffman colorability. We tried this approach as well, with some algebraic optimizations involving among other things Proposition 3.7. For the case $(11, 3)$, this approach took already about 16 hours. The generative algorithm set out in this chapter solved this case almost instantly.

As can be seen from the table, the computation times of the algorithm given different inputs is neither increasing in n, nor increasing in χ. The computation times given an input seems to rely greatly on the specifics on the case. A reason for a longer computation time might be a large number of viable partitions, a large number of collections of compatible bipartite parts, or a large size of the transversals used in Step 4 (see Section 8.2.1).

Although the computation time of a given input is difficult to predict, since it depends on many factors, we can say something about the complexity. More specifically, we consider the case where n is a multiple of $\chi \geq 3$, say $n = m \cdot \chi$. For this input, we will always encounter the collection of compatible bipartite parts, where every bipartite part is a cycle of length $2m$. Since $\chi \geq 3$, we have to do at least one inductive step of Step 4. In this inductive step, we need a transversal of some automorphism group in a permutation group. This permutation group is the full symmetric group S_m, since the Perron eigenvector of the cycle of length $2m$ is constant. The automorphism group (the group of automorphisms preserving the bipartition) is of size $2m$ (half of the whole automorphism group, which is dihedral). Therefore, the transversal is of size $\frac{1}{2}(m-1)!$, which means that the computation time of this input is $\Omega((m-1)!)$ as m goes to infinity.

8.4 Analyzing the graphs

In this section, we will take a closer look at the graphs that come out of the algorithm. We will look at the graphs of the smallest output classes. After that, we will mention some graphs of larger classes that have some interesting property.

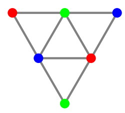
8.4.1 Some complete lists

In this subsection we will list some of the smaller cases (small in terms of number of graphs outputted and number of vertices), and along the way group some of the graphs we have constructed in the previous chapter.

Three colors

From the table, we know that there are no connected Hoffman colorable graphs with three colors and at most nine vertices that we haven't yet constructed. We list the following graphs.

- $K_{2,2,2}$;
- The graph from Example 4.8, which is an exceptional case in Corollary 6.15 and Theorem 6.22:



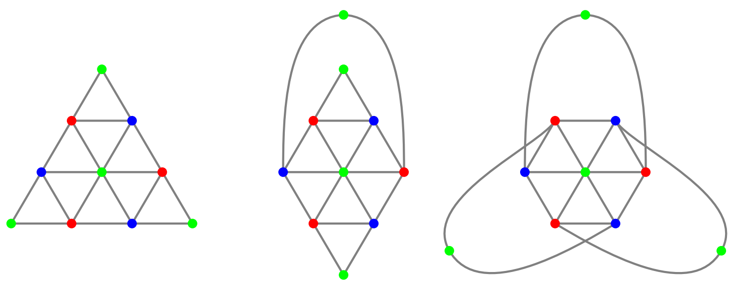
- The eight tensor graphs from Section 7.1 (including $K_{3,3,3}$);
- The one non-tensor regular graph from Section 7.2 (the Cayley graph of $\mathbb{Z}/9\mathbb{Z}$ with generating set $\{1, 2, 7, 8\}$);
- The four irregular graphs with a regular template from Section 7.3.

So we have two graphs with three colors and six vertices (both of which are tensor graphs), and thirteen on nine vertices.

On ten vertices, we have three graphs, based on

$$10 = 3 + 3 + 4.$$

The three graphs are *siblings* in the following way: they decompose into the same collection of compatible bipartite parts. Furthermore, the graphs have a regular template, namely the 6-cycle:

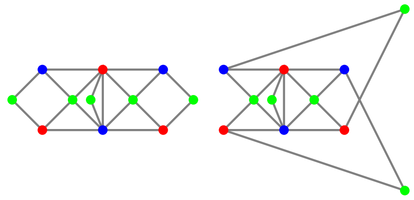


For this regular template we have $\nu = 2$ and $m = 3$, and $(d_i)_{i=1}^4 = (3, 1, 1, 1)$. Note that

$$\sum_{i=1}^{4} d_i^2 = 12 = m\nu^2.$$

We have one green vertex adjacent to every vertex of the 6-cycle template, and the remaining three vertices to one red and one blue vertex. Up to isomorphism, these three are all the ways to achieve this, and they all result in a Hoffman colorable graph.

On eleven vertices, we have just two graphs. These are interesting, because they are the smallest examples of a Hoffman colorable graph that is irregular, not a tensor, and also does not have a regular template. The two graphs are again siblings.

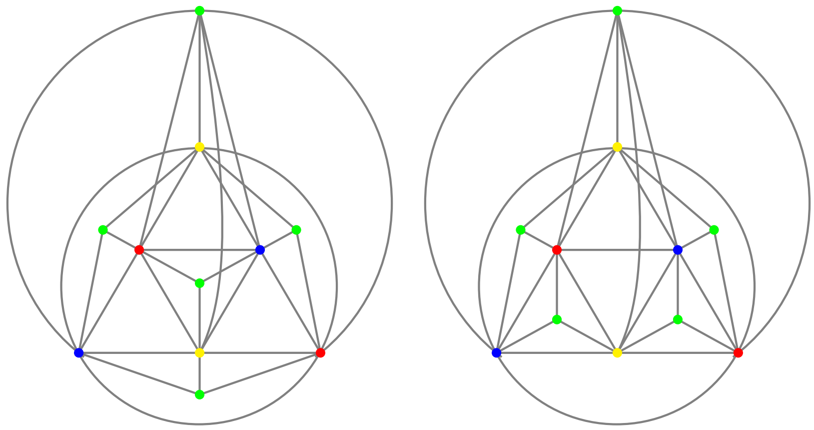


More than three colors

The two graphs above are not the only non-trivial Hoffman colorable graphs on eleven vertices. Indeed, for four colors we see

$$11 = 2 + 2 + 2 + 5,$$

so we get graphs from the last type of Corollary 6.15. There are two of them:

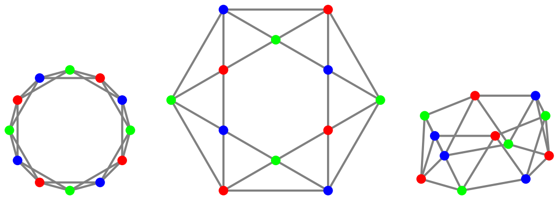


These graphs have a regular template, namely the octahedral graph ($K_{2,2,2}$) as regular template, and the green vertices as decorations. There are no other ways of decorating the octahedron this way. There are no other Hoffman colorable graphs with four colors and eleven vertices.

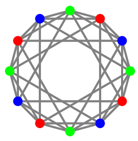
8.4.2 Other interesting graphs

Here we will mention and discuss some interesting graphs from the larger outputs.

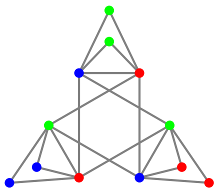
Regarding the unique non-tensor regular graph with three colors and nine vertices (the Cayley graph of $\mathbb{Z}/9\mathbb{Z}$ with generating set $\{1, 2, 7, 8\}$) from Section 7.2, we can pose two generalizations to twelve vertices. Firstly, we could look at the Cayley graph of $\mathbb{Z}/12\mathbb{Z}$ with generating set $\{1, 2, 10, 11\}$. We indeed get a Hoffman colorable graph, and this is the hexagonal antiprism (here drawn in three ways):



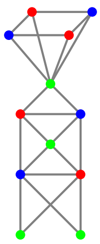
Another way to generalize, is to note that it is the color complement of the 9-cycle colored with three colors cyclically. This can be generalized to all multiples of three, and it will always result in a Hoffman colorable graph by Proposition 7.7. Here is the color complement of the 12-cycle (with the cyclic 3-coloring), which can also be described as the Cayley graph of $\mathbb{Z}/12\mathbb{Z}$ with generating set $\{1, 2, 4, 8, 10, 11\}$:



There are three other interesting Hoffman colorable graphs on twelve vertices and three colors.

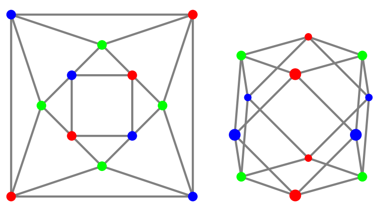


Although this graph has 3-dihedral symmetry, like all K_3-tensor graphs, this graph is not a tensor graph. We also have the following Hoffman colorable graph.



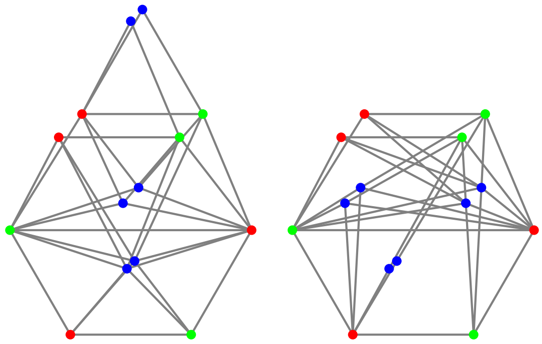
This graph has a regular template (two 4-cycles), but Proposition 7.9 does not apply. This is because the compatibility vectors are not constant on the red/blue vertices, because of the Perron eigenvector of the red-green and blue-green bipartite parts. It is the smallest Hoffman colorable graph with a regular template that does not comply with the requirements of Proposition 7.9.

On the topic of disconnected bipartite parts, we turn to the following Hoffman colorable graph, the cuboctahedron.

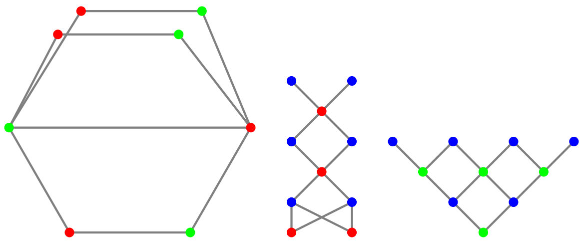


All three of the bipartite parts are two disjoint copies of the 4-cycle, even though the graph itself is connected. It is the smallest graph with this property. Note however that it is not uniquely colorable as we could rotate the colors of one of the square faces of the cuboctahedron to get a new optimal coloring that is not isomorphic.

On fourteen vertices and three colors, we find two very intriguing Hoffman colorable siblings:

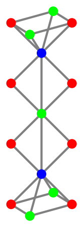


They decompose into three distinct bipartite parts, and are the smallest graphs to do so:

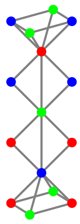


But there is more. Even though the bipartite parts are all symmetric, the left graph has no symmetries at all. It is the smallest Hoffman colorable graph with this property.

On fifteen vertices we find the smallest graph with three color classes of different sizes, namely one of size 2, one of size 5 and the last of size 8. It kind of looks like a paper lantern.



The least eigenvalue of this graph is -2 so it is very similar to the graphs of the last case in Corollary 6.15, it only has just one color class of size 2. However, the bipartite part from the classes on 2 and 8 vertices is disconnected (two copies of $K_{1,4}$), so the colors of one component could be flipped to get a different optimal coloring. In this case, this will lead to three classes of size five.



Chapter 9

Concluding remarks

Recall from the introduction that the ultimate, but infeasible goal of this thesis was to completely characterize when a graph is Hoffman colorable, and to describe the structure of Hoffman colorings. As mentioned before, completely achieving this goal seems to be far out of reach for now. However, this thesis made some significant steps towards achieving it. The Decomposition Theorem reveals a lot of structure of Hoffman colorings, being composed of a collection of compatible bipartite parts. Using the Decomposition Theorem, we were able to completely characterize Hoffman colorability in some smaller situations, like cone graphs and line graphs. Using the tensor product and the Composition Theorem, we were able to find various new infinite families of Hoffman colorable graphs, and to study the structure of these families with respect to the Decomposition Theorem. Also, the Decomposition and Composition Theorem enabled us to write an algorithm computing all Hoffman colorable graphs given a number of vertices and a number of colors.

We also investigated a new application of the Hoffman bound with respect to the characterization of strongly regular graphs. In particular, we showed a new condition for strong regularity using only the Hoffman bound of a regular graph and its complement. In the class of strongly regular graphs, we introduced a new parameterization, the "geometric" parameterization, based on the parameters of partial geometries. The geometric parameters revealed the interaction that the Hoffman bound, pseudo-geometricity and spreadability have.

Further questions

In this thesis significant steps are made to the goal of classifying Hoffman colorability completely. However, there are still a lot of questions to answer. Here are a couple that we came up while writing the thesis but we could not solve.

For one, all of the Hoffman colorable graphs that we have seen have triangles. We might pose the following question.

Question 1. Does there exist a triangle-free Hoffman colorable graph that is not bipartite?

One can restrict this question to the 3-chromatic case by the Decomposition Theorem. If a Hoffman colorable graph contains a complete bipartite part, then we get a triangle by Corollary 6.5. The class of tensor graphs might be interesting for this question. The tensor $K_3 \times H$ is triangle-free if H is triangle-free and has no self loops. A triangle-free simple graph H with Hoffman bound at least 3 is a source for a triangle-free Hoffman colorable graphs this way. The Mycielksi operation, a source of triangle-free graphs with large chromatic number might give such a graph.

We can generalize the question of triangle-free Hoffman colorable graphs in this way.

Question 2. For which pairs of numbers $\omega < \chi$, does there exist a Hoffman colorable graph with clique number ω and chromatic number χ?

For $\omega = 2$ and $\chi = 3$, we get Question 1. The A-type of strongly regular graphs is interesting in this context, as they have their clique numbers strictly less than their chromatic numbers. For example the Shrikhande graph has clique number 3 and chromatic number 4.

Another graph theoretic object that may have interesting interaction with Hoffman colorings, is homomorphisms and cores. See [16, Chapter 6] for a very nice introduction to this topic. Colorings and cliques can be expressed in terms of homomorphisms. Furthermore, if a graph has equal clique number and chromatic number, then its core is a complete graph. All of the graphs that we have studied that do not have a complete core, are strongly regular, which implies that they are cores themselves by [29, Corollary 4.2]. A graph that is a core or has a complete graph as its core is called a *pseudocore*.

Question 3. Does there exist a Hoffman colorable graph that is not a pseudocore?

A graph parameter p is *homomorphism-monotone* if whenever a homomrphism from G to H exists, then $p(G) \leq p(H)$. The clique number and many of the variations of the chromatic number have this property, due to being expressible in terms of (quantum) homomorphisms, including the quantum and vector chromatic number ([25]). However, the Hoffman bound does not have this property. In all the examples we have seen in this thesis, the core of a graph G (which is homomorphically equivalent to G itself) has a Hoffman bound at least the Hoffman bound of G.

Question 4. What is there to say about the interaction of the Hoffman bound and homomorphisms? Does there exist a graph G whose core has a lower Hoffman bound?

Yet another interesting question is about extending Hoffman colorable graphs. Let G and H be Hoffman colorable graphs, then we say that G *extends* H, if there exists some optimal coloring of G such that H is the induced subgraph of G on a collection of the color classes with respect to this coloring. From the examples we have seen, it seems like for most cases, there are just finitely many extensions of a given graph G. Take for example the tensor graphs. The graph $K_n \times H$ extends $K_m \times H$ if $n \geq m$. However, there are only finitely many n for which $K_n \times H$ is Hoffman colorable, since we need the requirement $n \leq h(H)$. There is one exception, and this is when H is the fully complete graph (the complete graph with all possible self loops added) that has an infinite Hoffman bound. This case is the case of regular complete multipartite graphs.

Furthermore, Proposition 7.7 ensures Hoffman colorability of regular graphs in the case where ν, the valency of the bipartite parts is at least $m \cdot \frac{\chi-1}{\chi}$. If we want a graph to have infinitely many extensions, we have to take χ to infinity, and then we get that ν is at least m, the class size. This automatically gives that we have a regular complete multipartite graph as well. However, these are two very restricted settings so maybe in general something else might happen.

Question 5. Does there exist a Hoffman colorable graph G that is not regular complete multipartite, and has infinitely many extensions? Does there exist a non-complete bipartite part that is part of infinitely many Hoffman colorable graphs?

Bibliography

[1] A. Abiad. A characterization and an application of weight-regular partitions of graphs. *Linear Algebra Appl.*, 569:162–174, 2019.

[2] A. Abiad, B. De Bruyn, J. D'haeseleer, and J. H. Koolen. Neumaier graphs with few eigenvalues. *Des. Codes Cryptography*, 90(9):2003–2019, 2022.

[3] L. W. Beineke, R. J. Wilson, and Peter J. Cameron, editors. *Topics in algebraic graph theory*, volume 102 of *Encycl. Math. Appl.* Cambridge: Cambridge University Press, 2004.

[4] T. Beth, D. Jungnickel, and H. Lenz. *Design theory. Vol. I.*, volume 69 of *Encycl. Math. Appl.* Cambridge: Cambridge University Press, 2nd ed. edition, 1999.

[5] A. Blokhuis, A. E. Brouwer, and W. H. Haemers. On 3-chromatic distance-regular graphs. *Des. Codes Cryptography*, 44(1-3):293–305, 2007.

[6] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[7] A. E. Brouwer and W. H. Haemers. *Spectra of graphs.* Universitext. Berlin: Springer, 2012.

[8] S. M. Cioabă, K. Guo, and W. H. Haemers. The chromatic index of strongly regular graphs. *Ars Math. Contemp.*, 20(2):187–194, 2021.

[9] D. M. Cvetkovic. Graphs and their spectra. *Publ. Fac. Électrotech. Univ. Belgrade, Sér. Math. Phys.*, 354-356:1–50, 1971.

[10] P. Delsarte. *An algebraic approach to the association schemes of coding theory.* Ann Arbor, MI: Historical Jrl., 1973.

[11] N. C. Fiala and W. H. Haemers. 5-chromatic strongly regular graphs. *Discrete Math.*, 306(23):3083–3096, 2006.

[12] M. A. Fiol. Eigenvalue interlacing and weight parameters of graphs. *Linear Algebra Appl.*, 290(1-3):275–301, 1999.

[13] A. Galtman. Spectral characterizations of the Lovász number and the Delsarte number of a graph. *J. Algebr. Comb.*, 12(2):131–143, 2000.

[14] C. Godsil, D. E. Roberson, B. Rooney, R. Šámal, and A. Varvitsiotis. Graph homomorphisms via vector colorings. *Eur. J. Comb.*, 79:244–261, 2019.

[15] C. Godsil, D. E. Roberson, R. Šámal, and S. Severini. Sabidussi versus Hedetniemi for three variations of the chromatic number. *Combinatorica*, 36(4):395–415, 2016.

[16] C. Godsil and G. Royle. *Algebraic graph theory*, volume 207 of *Grad. Texts Math.* New York, NY: Springer, 2001.

[17] W. H. Haemers. Eigenvalue techniques in design and graph theory. Mathematical Centre Tracts 121. Amsterdam: Mathematisch Centrum. V, 102 p. Dfl. 13.00 (1980)., 1980.

[18] W. H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra Appl.*, 226-228:593–616, 1995.

[19] W. H. Haemers and V. D. Tonchev. Spreads in strongly regular graphs. *Des. Codes Cryptography*, 8(1-2):145–157, 1996.

[20] Willem H. Haemers. Hoffman's ratio bound. *Linear Algebra Appl.*, 617:215–219, 2021. doi:10.1016/j.laa.2021.02.010.

[21] A. J. Hoffman. On eigenvalues and colorings of graphs. Graph Theory Appl., Proc. advanced Sem. Wisconsin, Madison 1969, 79-91 (1970)., 1970.

[22] D. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. *J. ACM*, 45(2):246–265, 1998.

[23] M. Kwan and Y. Wigderson. The inertia bound is far from tight, 2023.

[24] L. Lovász. On the Shannon capacity of a graph. *Kibern. Sb., Nov. Ser.*, 19:5–22, 1983.

[25] L. Mančinska and D. E. Roberson. Quantum homomorphisms. *J. Comb. Theory, Ser. B*, 118:228–267, 2016.

[26] R. J. McEliece, E. R. Rodemich, and H. C. jun. Rumsey. The Lovasz bound and some generalizations. *J. Comb. Inf. Syst. Sci.*, 3:134–152, 1978.

[27] B. D. McKay. Database of strongly regular graphs (referenced july 2024). URL: https://users.cecs.anu.edu.au/~bdm/data/graphs.html.

[28] B. D. McKay. Isomorph-free exhaustive generation. *J. Algorithms*, 26(2):306–324, 1998.

[29] D. E. Roberson. Homomorphisms of strongly regular graphs. *Algebr. Comb.*, 2(4):481–497, 2019.

[30] G. Sabidussi. Graphs with given group and given graph-theoretical properties. *Can. J. Math.*, 9:515–525, 1957.

[31] A. Schrijver. A comparison of the Delsarte and Lovász bounds. *IEEE Trans. Inf. Theory*, 25:425–429, 1979.

[32] Y. Shitov. Counterexamples to Hedetniemi's conjecture. *Ann. Math. (2)*, 190(2):663–667, 2019.

[33] J. A. Thas and S. E. Payne. Spreads and ovoids in finite generalized quadrangles. *Geom. Dedicata*, 52(3):227–253, 1994.

[34] V. G. Vizing. Kritische Graphen mit gegebener chromatischer Klassen. *Diskret. Analiz, Novosibirsk*, 5:9–17, 1965.

[35] P. Wocjan and C. Elphick. Spectral lower bounds for the orthogonal and projective ranks of a graph. *Electron. J. Comb.*, 26(3):research paper p3.45, 14, 2019.